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Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function

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Abstract: Both the theory of differential subordination and its dual, the theory of differential superordination, introduced by Professors Miller and Mocanu are based on reinterpreting certain inequalities for real-valued functions for the case of complex-valued functions. Studying subordination and superordination properties using different types of operators is a technique that is still widely used, some studies resulting in sandwich-type theorems as is the case in the present paper. The fractional integral of confluent hypergeometric function is introduced in the paper and certain subordination and superordination results are stated in theorems and corollaries, the study being completed by the statement of a sandwich-type theorem connecting the results obtained by using the two theories.

Keywords: differential operator; differential subordination; differential superordination; analytic function; univalent function; dominant; best dominant; subordinant; best subordinant

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1. Introduction

The theory of differential subordination emerged from the remark that using a real-valued function f twice continuously differentiable on an interval $I = (-1, 1)$ and assuming that the differential operator

$$D[f](t) = t^2 f''(t) + 4tf'(t) + 2f(t) + 6t = [t^2 f(t) + t^3]''$$

satisfies

$$0 < D[f](t) < 2, \tag{1}$$

for $t \in I$, it is obvious that such a function has the property that $-1 < f(t) < 2$ for $t \in I$ which can be written in an equivalent form using containment relations related to intervals as

$$D[f](t) \subset (0, 2) \Rightarrow f(I) \subset (-1, 2). \tag{2}$$

Relation (1) cannot be reinterpreted using a complex-valued function instead of the real-valued function $f(t)$, but the first containment from (2) can be stated for the complex-valued function $f(z)$ as

$$D[f](U) \subset \Omega, \tag{3}$$

where

$$D[f](z) = z^2 f''(z) + 4zf'(z) + 2f(z) + 6z,$$

U denotes the unit disc of the complex plane and $\Omega \subset \mathbb{C}$.

If a function $f : U \to \mathbb{C}$ satisfies inclusion (3), then the problem that appears is whether there exists a "smallest" set $\Delta \subset \mathbb{C}$ such that

$$D[f](U) \subset \Omega \Rightarrow f(U) \subset \Delta.$$

Solving this problem led to the introduction of the notion of differential subordination related to complex valued functions in two papers published by Miller and Mocanu in 1978 [1] and 1981 [2].

Later, in 2003 [3], the dual notion of differential superordination was introduced by the same authors answering the question related to the existence of a "smallest" set $\Omega \subset \mathbb{C}$ for which

$$\Omega \subset D[f](U) \Rightarrow \Delta \subset f(U).$$

Many interesting outcomes of the study done using the theories of differential subordination and superordination are due to the use of operators. A very interesting function which helps in defining such operators, used by many researchers, was introduced by N.E.Cho and A.M.K. Aouf [4] named the fractional integral of order λ. It is defined as follows:

Definition 1. *([4]) The fractional integral of order λ ($\lambda > 0$) is defined for a function f by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

Interesting results were obtained and published recently using this function as it can be seen in [5,6]. Inspired by the results obtained by applying fractional integral on different hypergeometric functions seen in papers [7–9], we have chosen the confluent (or Kummer) hypergeometric function to extend the study done on it in [10]. The univalence of confluent Kummer function was also studied in [11].

The confluent (Kummer) hypergeometric function of the first kind is given in the following definition:

Definition 2. *([12] p. 5) Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$ and consider*

$$\phi(a,c;z) = {}_1F_1(a,c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \ldots, z \in U. \tag{4}$$

This function is called confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$ and satisfies Kummer's differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\ldots(d+k-1) \text{ and } (d)_0 = 1,$$

then (4) can be written in the form

$$\phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k!}. \tag{5}$$

The interest in the study of hypergeometric functions and their connection to the theory of univalent functions reappeared when L. de Branges used hypergeometric functions in the proof of the famous Bieberbach conjecture [13]. Confluent (Kummer) hypergeometric function was studied lately from many points of view. Conditions related to its univalence were stated in [10], its applications on certain classes of univalent functions are shown in [14] and an analytical study on Mittag-Leffler–confluent hypergeometric functions was conducted in [15] using fractional integral operator. An operator defined using fractional integral is introduced and studied using the theories of differential subordination and superordination in the next section of this paper.

In order to achieve the study, the usual definitions are used.

$\mathcal{H}(U)$ denotes the class of analytic functions in the unit disc of the complex plane. For n a positive integer and $a \in \mathbb{C}$, $\mathcal{H}[a,n]$ denotes the subclass of $\mathcal{H}(U)$ gathering the functions written in the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$.

Next, the definitions of the notions from the theories of differential subordination and superordination used in the present paper are given.

Definition 3. *[12] Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.*

Definition 4. *[12] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be a univalent function in U. If p is analytic in U and satisfies the second order differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U, \tag{6}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (6). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (6) is said to be the best dominant of (6). The best dominant is unique up to a rotation of U.

Definition 5. *[3] Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi\big(p(z), zp'(z), z^2p''(z); z\big)$ are univalent in U satisfy the (second-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U, \tag{7}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination or more simply a subordinant, if $q \prec p$ for all p satisfying (7). A subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (7) is said to be the best subordinant of (7). Note that the best subordinant is unique up to a rotation of U.

Definition 6. *[12] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Two lemmas are used in the next section in the proofs of the original results.

Lemma 1. *[12] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$. If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 2. *[16] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ for $z \in U$ and $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and $\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.*

Using the definitions already known, a new operator is introduced connecting the fractional integral of order λ and the confluent (Kummer) hypergeometric function. By using this operator, the methods of the theory of differential subordination and those of the theory of differential superordination are implemented in order to conduct a study obtaining interesting new differential subordinations and superordinations for which the best dominant and the best subordinant are found, respectively. The Lemmas listed above are part of those classical methods used for obtaining original results related to operators. The most interesting part is the form that the results have due to the operator used. Combining the results of the study done using both theories, a sandwich-type theorem is stated which also generates two corollaries for particular functions involved.

2. Results

The new operator introduced in this paper is defined using Definitions 1 and 2.

Definition 7. *Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$ and $\lambda > 0$. We define the fractional integral of confluent hypergeometric function*

$$D_z^{-\lambda}\phi(a,c;z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{\phi(a,c;t)}{(z-t)^{1-\lambda}}dt \tag{8}$$

$$= \frac{1}{\Gamma(\lambda)}\frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)\Gamma(k+1)}\int_0^z \frac{t^k}{(z-t)^{1-\lambda}}dt.$$

After a simple calculation, the fractional integral of confluent hypergeometric function has the following form

$$D_z^{-\lambda}\phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)\Gamma(\lambda+k+1)}z^{k+\lambda}. \tag{9}$$

We note that $D_z^{-\lambda}\phi(a,c;z) \in \mathcal{H}[0,\lambda]$.

The first subordination result obtained using the operator given by (8) is the following theorem:

Theorem 1. *Let $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}(U)$ and consider a function q analytic and univalent in U with $q(z) \neq 0$, when $z \in U$, $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike and univalent function in U. Let*

$$Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{10}$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ and

$$\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z) := \alpha + \xi\left[\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right]^{\delta} + \tag{11}$$

$$\mu\left[\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right]^{2\delta} + \beta\delta\left[\frac{z\left(D_z^{-\lambda}\phi(a,c;z)\right)'}{D_z^{-\lambda}\phi(a,c;z)} - 1\right].$$

If function q satisfies

$$\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z) \prec \alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{12}$$

considering $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, then

$$\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \prec q(z), \quad z \in U, \tag{13}$$

and q is the best dominant.

Proof. Take function p of the form

$$p(z) := \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}, z \in U, z \neq 0.$$

We compute

$$p'(z) = \delta\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta-1}\left[\frac{\left(D_z^{-\lambda}\phi(a,c;z)\right)'}{z} - \frac{D_z^{-\lambda}\phi(a,c;z)}{z^2}\right]$$

$$= \delta\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta-1}\frac{\left(D_z^{-\lambda}\phi(a,c;z)\right)'}{z} - \frac{\delta}{z}p(z).$$

Then

$$\frac{zp'(z)}{p(z)} = \delta\left[\frac{z\left(D_z^{-\lambda}\phi(a,c;z)\right)'}{D_z^{-\lambda}\phi(a,c;z)} - 1\right].$$

By setting

$$\theta(w) := \alpha + \xi w + \mu w^2$$

and

$$\phi(w) := \frac{\beta}{w}.$$

it is evidently that θ and ϕ are analytic in $\mathbb{C}$, respectively $\mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, considering

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

$Q(z)$ is starlike univalent in U.

We have

$$h'(z) = \xi + q'(z) + 2\mu q(z)q'(z) + \beta\frac{(q'(z) + zq''(z))q(z) - z(q'(z))^2}{(q(z))^2}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{zh'(z)}{\beta\frac{zq'(z)}{q(z)}} = 1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}.$$

We obtain

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0.$$

We get

$$\alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)} =$$

$$\alpha + \xi\left[\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right]^{\delta} + \mu\left[\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right]^{2\delta} + \beta\delta\left[\frac{z\left(D_z^{-\lambda}\phi(a,c;z)\right)'}{D_z^{-\lambda}\phi(a,c;z)} - 1\right].$$

By using (12), we get

$$\alpha+\xi p(z)+\mu(p(z))^2+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\beta q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}.$$

Applying Lemma 1 we obtain $p(z)\prec q(z)$, $z\in U$, that is,

$$\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}\prec q(z),\ \ z\in U$$

and q is the best dominant. □

Corollary 1. *Let $a,c\in\mathbb{C}$, $c\neq 0,-1,-2,\dots$ and $\lambda,\delta>0$. Assume that (10) holds. If*

$$\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^2+\beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}\prec\frac{1+Az}{1+Bz},\ \ z\in U,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. The results stated in this corollary are obtained using $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 1. □

Corollary 2. *Let $a,c\in\mathbb{C}$, $c\neq 0,-1,-2,\dots$ and $\lambda,\delta>0$. Assume that (10) holds. If*

$$\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\beta\frac{2\gamma z}{1-z^2},$$

for $\alpha,\xi,\mu,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},\ z\in U,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.

Proof. The conclusion of the corollary follows from Theorem 1 by taking $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. □

Theorem 2. *Consider q an analytic function in U with $q(z)\neq 0$ and let $\frac{zq'(z)}{q(z)}$ be starlike and univalent in U. Assume that*

$$Re\left(\frac{2\mu}{\beta}(q(z))^2+\frac{\xi}{\beta}q(z)\right)>0,\ \text{for}\ \xi,\mu,\beta\in\mathbb{C},\ \beta\neq 0. \tag{14}$$

Let $a,c\in\mathbb{C}$, $c\neq 0,-1,-2,\dots$ and $\lambda,\delta>0$. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}\in\mathcal{H}[0,(\lambda-1)\delta]\cap Q$ and $\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z)$ is univalent in U, where $\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z)$ is as defined in (11), then

$$\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}\prec\psi_{\lambda}^{a,c}(\delta,\alpha,\xi,\mu,\beta;z) \tag{15}$$

implies

$$q(z) \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta},\ z \in U, \tag{16}$$

and q is the best subordinant.

Proof. Define function p as

$$p(z) := \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta},\ \ z \in U, z \neq 0.$$

By setting

$$\nu(w) := \alpha + \xi w + \mu w^2$$

and

$$\phi(w) := \frac{\beta}{w},$$

it is evidently that ν and ϕ are analytic in $\mathbb{C}$, respectively $\mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$.

Since

$$\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)[\xi + 2\mu q(z)]q(z)}{\beta},$$

it follows that

$$Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{2\mu}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right) > 0,$$

for $\mu, \xi, \beta \in \mathbb{C}, \beta \neq 0$.

We get

$$\alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)}.$$

Using Lemma 2, we obtain

$$q(z) \prec p(z) = \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta},\ \ z \in U,$$

and q is the best subordinant. □

Corollary 3. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. Assume that (14) holds. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}[0,(\lambda-1)\delta] \cap Q$ and*

$$\alpha + \xi\frac{1+Az}{1+Bz} + \mu\left(\frac{1+Az}{1+Bz}\right)^2 + \beta\frac{(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z),$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta},\ \ z \in U,$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. Using Theorem 2, the conclusion of the corollary derives from setting $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$. □

Corollary 4. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. Assume that (14) holds. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}[0, (\lambda-1)\delta] \cap Q$ and*

$$\alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma} + \beta\frac{2\gamma z}{1-z^2} \prec \psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z),$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta}, \; z \in U,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.

Proof. By using Theorem 2 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$ we obtain the result. □

Using the conclusions of Theorem 1 and Theorem 2 combined, a sandwich-type theorem can be stated as it follows:

Theorem 3. *Consider two analytic and univalent functions q_1 and q_2 satiasfying $q_1(z) \neq 0$ and $q_2(z) \neq 0$, when $z \in U$, such that $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ be starlike and univalent functions. Suppose that q_1 satisfies (10) and q_2 satisfies (14). Let a, c be complex numbers with $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}[0, (\lambda-1)\delta] \cap Q$ and $\psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z)$ is as defined in (11) univalent in U, then*

$$\alpha + \xi q_1(z) + \mu(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi q_2(z) + \mu(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)},$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z) \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \prec q_2(z), \quad z \in U,$$

and q_1 and q_2 are the best subordinant and the best dominant, respectively.

For $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 5. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. Assume that (10) and (14) hold. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}[0, (\lambda-1)\delta] \cap Q$ and*

$$\alpha + \xi\frac{1+A_1z}{1+B_1z} + \mu\left(\frac{1+A_1z}{1+B_1z}\right)^2 + \beta\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)} \prec \psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi\frac{1+A_2z}{1+B_2z} + \mu\left(\frac{1+A_2z}{1+B_2z}\right)^2 + \frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)},$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \prec \frac{1+A_2z}{1+B_2z},$$

hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant, respectively.

Corollary 6. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $\lambda, \delta > 0$. Assume that (10) and (14) hold. If $\left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \in \mathcal{H}[0, (\lambda-1)\delta] \cap Q$ and*

$$\alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_1} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \beta\frac{2\gamma_1 z}{1-z^2} \prec \psi_{\lambda}^{a,c}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_2} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \beta\frac{2\gamma_2 z}{1-z^2},$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $0 < \gamma_1, \gamma_2 \leq 1$, $\beta \neq 0$,, where $\psi_{\lambda}^{a,c}$ is defined in (11), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \left(\frac{D_z^{-\lambda}\phi(a,c;z)}{z}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.

3. Discussion

Having as inspiration the results obtained by applying fractional integral on certain hypergeometric functions, the confluent hypergeometric function is considered in the present study and the fractional integral of confluent hypergeometric function is introduced. Using it, a new operator is defined and the theory of differential subordination is applied in order to obtain interesting subordinations related to it for which best dominants are also given. Some nice corollaries are stated for specific well-known functions used as best dominants of the investigated subordinations. Following the theory of differential superordination, some differential superordinations are also obtained for the operator involving fractional integral of confluent hypergeometric function and the best subordinants are found. Using Theorems 1 and 2, a sandwich-type outcome connects the subordination and superordination results. Interesting corollaries follow for particular functions used as best subordinant and best dominant. The study done in the present paper can inspire the use of other hypergeometric functions connected with fractional integral. Also, with the best subordinants of the differential subordinations given, conditions for univalence of the operator introduced here could be investigated. Further studies related to introducing new classes of functions using the operator given in (8) could be conducted. Since the classes obtained using this operator should be interesting enough and different from any other classes previously obtained by using different operators, relations to other known classes could be investigated and coefficient estimates could be established. The results contained in the corollaries could inspire ideas for continuing the study done considering particular functions.

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