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$(2,k)$-Distance Fibonacci Polynomials

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Abstract: In this paper we introduce and study $(2,k)$-distance Fibonacci polynomials which are natural extensions of $(2,k)$-Fibonacci numbers. We give some properties of these polynomials—among others, a graph interpretation and matrix generators. Moreover, we present some connections of $(2,k)$-distance Fibonacci polynomials with Pascal's triangle.

Keywords: Fibonacci numbers; Fibonacci polynomials; matrix generators; Pascal's triangle

MSC: 11B39; 11B37; 11C20

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1. Introduction

Fibonacci numbers F_n are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0, F_1 = 1$.

In the literature there are many generalisations of Fibonacci numbers. They have been generalised in many ways, some by preserving initial conditions and other by preserving the recurrence relation. For example, in [1] the authors introduced k-Fibonacci numbers defined recurrently for any integer $k \geq 1$ by $F(k,n) = kF(k,n-1) + F(k,n-2)$ for $n \geq 2$ with $F(k,0) = 0$, $F(k,1) = 1$.

Among one-parameter generalisations of Fibonacci numbers, one can find generalisations in the distance sense, such that for an arbitrary integer k the n-th generalised Fibonacci number is obtained by adding two previous terms: $(n-k)$-th and the second chosen in such a way that the obtained recurrence generalises Fibonacci numbers. Such generalisations have many graph interpretations closely related to the concept of k-independent sets. We recall some such generalisations.

1. Reference [2]. $F_p(n) = F_p(n-1) + F_p(n-p-1)$ for $p \geq 1$ and $n \geq p+1$ with $F_p(0) = F_p(1) = \ldots = F_p(p) = F_p(p+1) = 1$
2. Reference [3]. $F(k,n) = F(k,n-1) + F(k,n-k)$ for $n \geq k+1$ with $F(k,n) = n+1$ for $n \leq k$
3. Reference [4]. $Fd^{(1)}(k,n) = Fd^{(1)}(k,n-k+1) + Fd^{(1)}(k,n-k)$ for $n \geq k$ with $Fd^{(1)}(k,n) = 1$ for $n \leq k-1$
4. Reference [5]. $F_2^{(1)}(k,n) = F_2^{(1)}(k,n-2) + F_2^{(1)}(k,n-k)$ for $n \geq k+1$ with

$$F_2^{(1)}(k,n) = \begin{cases} 1 & \text{if } n \leq k-1 \quad \text{or } n = k = 1, \\ 2 & \text{if } n = k \geq 2. \end{cases} \tag{1}$$

Generalisations of Fibonacci numbers of another type are Fibonacci polynomials. Fibonacci polynomials were studied in 1883 by E. Ch. Catalan and E. Jacobsthal. Catalan studied polynomials $\mathcal{F}_n(x)$ defined in the following way. $\mathcal{F}_n(x) = x\mathcal{F}_{n-1}(x) + \mathcal{F}_{n-2}(x)$ for $n \geq 3$, where $\mathcal{F}_1(x) = 1$, $\mathcal{F}_2(x) = x$. Polynomials studied by Jacobsthal were defined by the recurrence $J_n(x) = J_{n-1}(x) + xJ_{n-2}(x)$ for $n \geq 3$ with $J_1(x) = J_2(x) = 2$. In 1970 M. Bicknell studied Lucas polynomials $L_n(x)$ defined in the following way. $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$ for $n \geq 2$, where $L_0(x) = 2$, $L_1(x) = x$.

In the literature we can find interesting results related to Fibonacci polynomials and their generalisations; see, for example, [6–11].

In this paper we use a concept of $(2,k)$-distance Fibonacci numbers $F_2^{(1)}(k,n)$ defined by (1) and we introduce $(2,k)$-distance Fibonacci polynomials. We show some properties of these polynomials. We present connections of $(2,k)$-distance Fibonacci polynomials with Pascal's 2-triangle. Moreover, we give matrix generators and a graph interpretation of $(2,k)$-distance Fibonacci polynomials.

2. (2, *k*)-Distance Fibonacci Polynomials and Their Graph Interpretation

In this section we introduce $(2,k)$-distance Fibonacci polynomials and we present a graph interpretation of these polynomials.

Let $k \geq 1$ for some integer, $(2,k)$-distance Fibonacci polynomials we define recursively by

$$F_n^{(k)}(x) = F_{n-2}^{(k)}(x) + xF_{n-k}^{(k)}(x) \text{ for } n \geq k \tag{2}$$

with $F_n^{(k)}(x) = 1$ for $n = 0, 1, \ldots, k-1$.

Note that for $k = 1$ we obtain polynomials $\mathcal{F}_n(x)$ studied by Catalan and for $k = 2$ we obtain $(2,2)$-distance Fibonacci polynomials given by the direct formula $F_n^{(2)}(x) = (1+x)^{\lfloor \frac{n}{2} \rfloor}$.

For example, if $k = 3$, initial $(2,3)$-distance Fibonacci polynomials are

$$\begin{aligned}
F_0^{(3)}(x) &= 1\\
F_1^{(3)}(x) &= 1\\
F_2^{(3)}(x) &= 1\\
F_3^{(3)}(x) &= x+1\\
F_4^{(3)}(x) &= x+1\\
F_5^{(3)}(x) &= 2x+1\\
F_6^{(3)}(x) &= x^2+2x+1\\
F_7^{(3)}(x) &= x^2+3x+1\\
F_8^{(3)}(x) &= 3x^2+3x+1\\
F_9^{(3)}(x) &= x^3+3x^2+4x+1\\
&\vdots
\end{aligned}$$

First we give a graph interpretation of $F_n^{(k)}(x)$ for integer $x \geq 1$. Let P_n, $n \geq 2$ be a path without loops and multiple edges with the vertex set $V(P_n) = \{v_1, ..., v_n\}$, and vertices of P_n are numbered in the natural fashion. Let us consider a colouring c of vertices of P_n such that the vertex v_n can be uncoloured. Let $\{0, 1, ..., x\}$ be a set of colours and $c(v_j) = i$ denote that the vertex v_j has a colour i, $0 \leq i \leq x$.

The following recurrent procedure defines the colouring c of P_n. Denote by A a set of uncoloured vertices of a path P_n.

Until $|A| \leq 1$, repeat following operations:

Let $v_j \in A$ be a vertex with the smallest index.

If $|A| < k$, then $c(v_j) = 0$; otherwise $c(v_j) = i$, $0 \leq i \leq x$.

If $c(v_j) = 0$, then $c(v_{j+1}) = 0$.

If $c(v_j) = i$ and $1 \leq i \leq x$, then $c(v_{j+t}) = i$ for $t = 1, ..., k-1$.

We can interpret obtained monochromatic paths of the length k or 2 as monochromatic scraps, so with the above colouring c of P_n we will define a $\{P_2, P_k\}$-scrap colouring of P_n. Denote by P_2^0 a scrap of two vertices coloured 0 and by P_k^s a scrap of k vertices coloured s, where $s \in \{0, \ldots, k\}$. In other words the $\{P_2, P_k\}$-scrap colouring of P_n is a covering of P_n by scraps from the set $\{P_2^0, P_k^1, ..., P_k^x\}$.

Consider, for example, the path P_8 and $k = 3$. Then we have the following possibilities of a $\{P_2, P_3\}$-scrap colouring:

(a) $P_3^s P_3^s P_2^0$, $P_3^s P_2^0 P_3^s$, $P_2^0 P_3^s P_3^s$. Each scrap P_3^s has x colour variants, so there are $3x^2$ such colourings.

(b) $P_2^0 P_2^0 P_2^0 P_2^0$. Each scrap has two vertices, so we have exactly one colouring.

(c) Colourings when the last vertex is uncoloured are $P_3^s P_2^0 P_2^0 v_8$, $P_2^0 P_3^s P_2^0 v_8$, $P_2^0 P_2^0 P_3^s v_8$, which gives $3x$.

Summing up, we have that there exist $3x^2 + 3x + 1$ $\{P_2, P_k\}$-scrap colourings of P_8.

Denote by $\alpha(n)$ the total number of $\{P_2, P_k\}$-scrap colourings of P_n.

Theorem 1. *Let $k \geq 3$, $n \geq 2$ be integers. The number of all $\{P_2, P_k\}$-scrap colourings of P_n is equal to $F_n^{(k)}(x)$.*

Proof. Denote by $\alpha_k(n)$ a $\{P_2, P_k\}$-scrap colouring starting from P_k and by $\alpha_2(n)$ a $\{P_2, P_k\}$-scrap colouring starting from P_2. If $n \leq k-1$, then there is exactly one $\{P_2, P_k\}$-scrap colouring only by scraps P_2^0. Thus $\alpha(n) = \alpha_2(n) = 1 = F_n^{(k)}(x)$.

Assume that $n \geq k$ and $\alpha(n) = F_n^{(k)}(x)$. Then

$$\begin{aligned} \alpha(n+1) &= \alpha_2(n+1) + \alpha_k(n+1) = \alpha(n+1-2) + x\alpha(n+1-k) \\ &= F_{n+1-2}^{(k)}(x) + xF_{n+1-k}^{(k)}(x) = F_{n+1}^{(k)}(x), \end{aligned}$$

which ends the proof. □

Using the above graph interpretation we give direct formulas for $F_n^{(k)}(x)$.

Theorem 2. *Let $k \geq 3$, $n \geq 0$ be integers. Then*

$$F_n^{(k)}(x) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i + \lfloor \frac{n-ik}{2} \rfloor}{i} x^i.$$

Proof. If $n \leq k-1$, then $\lfloor \frac{n}{k} \rfloor = 0$ and

$$F_n^{(k)}(x) = \sum_{i=0}^{0} \binom{i + \lfloor \frac{n-ik}{2} \rfloor}{i} x^i = \binom{0 + \lfloor \frac{n}{2} \rfloor}{0} x^0 = 1.$$

Assume that $n \geq k$. By Theorem 1, the number $F_n^{(k)}(x)$ is equal to the number of $\{P_2, P_k\}$-scrap colourings of P_n. Each $\{P_2, P_k\}$-scrap colouring consists of i monochromatic paths P_k^x with x colours and j monochromatic paths P_2^0, where $0 \leq i \leq \lfloor \frac{n}{k} \rfloor$, $0 \leq j \leq \lfloor \frac{n}{k} \rfloor$. Moreover, for a fixed i we have $j = \lfloor \frac{n-ik}{2} \rfloor$ and the number of $\{P_2, P_k\}$-scrap colourings is equal to $\binom{i+j}{i} x^i = \binom{i + \lfloor \frac{n-ik}{2} \rfloor}{i} x^i$. Thus $F_n^{(k)}(x) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i + \lfloor \frac{n-ik}{2} \rfloor}{i} x^i$. □

Theorem 3. *Let $k \geq 3$, $n \geq 2$ be integers. Then*

$$F_n^{(k)}(x) = \sum_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=\lceil \frac{n-2t-1}{k-2} \rceil}^{\lfloor \frac{n-2t}{k-2} \rfloor} \binom{t}{i} x^i.$$

Proof. By Theorem 1, the number $F_n^{(k)}(x)$ is equal to the number of $\{P_2, P_k\}$-scrap colourings of P_n. A $\{P_2, P_k\}$-scrap colouring of P_n corresponds to a sum of numbers counting vertices with the same colour, so $n = n_1 + n_2 + ... + n_t + n_r$ where $n_i \in \{2, k\}$ for $i \in \{1, ..., t\}$ and $n_r \in \{0, 1\}$. Instead of such sum we can consider a t-tuple $(n_1, ..., n_t)$ where $\lfloor \frac{n}{k} \rfloor \leq t \leq \lfloor \frac{n}{2} \rfloor$. There is 2^t such t-tuples. Let a tuple have i components equal

to k and $t-i$ components equal to 2. If $\sum_{i=0}^{t} n_i$ is greater than n or less then $n-1$, then the t-tuple does not correspond to any $\{P_2, P_k\}$-scrap colouring of P_n. Thus we have $2^t - \sum_{i=0}^{\lceil \frac{n-2t-1}{k-2} \rceil - 1} \binom{t}{i} - \sum_{i=\lfloor \frac{n-2t}{k-2} \rfloor + 1}^{t} \binom{t}{i}$ such t-tuples, with sum n or $n-1$. By simple calculations we have that there are $\sum_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=\lceil \frac{n-2t-i}{k-2} \rceil}^{\lfloor \frac{n-2t}{k-2} \rfloor} \binom{t}{i}$ such tuples. Note that each component equal to k in a t–tuple corresponds to k consecutive vertices coloured with the same colour (a scrap P_k). They can be coloured on x ways. Thus we have $F_n^{(k)}(x) = \sum_{t=\lfloor \frac{n}{k} \rfloor}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=\lceil \frac{n-2t-i}{k-2} \rceil}^{\lfloor \frac{n-2t}{k-2} \rfloor} \binom{t}{i} x^i$. □

In the next theorem we present some identities for $(2,k)$-distance Fibonacci polynomials.

Theorem 4. *Let $n \geq 0$, $k > 2$ be integers. Then*

$$\sum_{i=0}^{n} F_{2i}^{(k)}(x) = \frac{F_{2n+k}^{(k)}(x) - 1}{x}, \tag{3}$$

$$\sum_{i=0}^{n} F_{2i+1}^{(k)}(x) = \frac{F_{2n+k+1}^{(k)}(x) - 1}{x}, \tag{4}$$

$$\sum_{i=0}^{2n} F_{i}^{(k)}(x) = \frac{F_{2n+k}^{(k)}(x) + F_{2n+k-1}^{(k)}(x) - 2}{x}, \tag{5}$$

$$\sum_{i=0}^{2n+1} F_{i}^{(k)}(x) = \frac{F_{2n+k}^{(k)}(x) + F_{2n+k+1}^{(k)}(x) - 2}{x}. \tag{6}$$

Proof. (by induction on n). If $n = 0$, then $F_0^{(k)}(x) = \frac{F_k^{(k)}(x)-1}{x} = \frac{x+1-1}{x} = 1$.
If $n = 1$, then $F_0^{(k)}(x) + F_2^{(k)}(x) = \frac{F_{2+k}^{(k)}(x)-1}{x} = \frac{2x+1-1}{x} = 2$.
Assume that the formula (3) holds for n; we will prove it for $n+1$. By the induction hypothesis we have

$$\begin{aligned}\sum_{i=0}^{n+1} F_{2i}^{(k)}(x) &= \sum_{i=0}^{n} F_{2i}^{(k)}(x) + F_{2n+2}^{(k)} = \frac{F_{2n+k}^{(k)}(x) - 1}{x} + \frac{xF_{2n+2}^{(k)}(x)}{x} \\ &= \frac{F_{2n+k}^{(k)}(x) + xF_{2n+2}^{(k)} - 1}{x} = \frac{F_{2n+2+k}^{(k)} - 1}{x}.\end{aligned}$$

The formula (5) is the sum of (3) and (4).

$$\begin{aligned}\sum_{i=0}^{2n} F_{i}^{(k)}(x) &= (F_0^{(k)} + \cdots + F_{2n}^{(k)}) + (F_1^{(k)} + \cdots + F_{2n-1}^{(k)}) = \frac{F_{2n+k}^{(k)}(x) - 1}{x} + \frac{F_{2(n-1)+k+1}^{(k)}(x) - 1}{x} \\ &= \frac{F_{2n+k}^{(k)}(x) + F_{2n+k-1}^{(k)}(x) - 2}{x}.\end{aligned}$$

Analogously, we prove the formula (6). □

Theorem 5. *Let $n \geq 0$, $k \geq 1, k \neq 2$ be integers. Then*

$$F_n^{(k)}(x) = F_{n-2}^{(k)}(x) + xF_{n-k+2}^{(k)}(x) - x^2 F_{n-2k+2}^{(k)}(x) \text{ for } n > 2k-2.$$

Proof. Using the definition (2) of $(2,k)$-distance Fibonacci polynomials we have

$$\begin{aligned}
&F^{(k)}_{n-2}(x) + xF^{(k)}_{n-k+2}(x) - x^2F^{(k)}_{n-2k+2}(x) = \\
&F^{(k)}_{n-2}(x) + x(F^{(k)}_{n-k} + xF^{(k)}_{n-2k+2}) - x^2F^{(k)}_{n-2k+2}(x) = \\
&= F^{(k)}_{n-2}(x) + xF^{(k)}_{n-k}(x) + x^2F^{(k)}_{n-2k+2}(x) - x^2F^{(k)}_{n-2k+2}(x) \\
&= F^{(k)}_{n-2}(x) + xF^{(k)}_{n-k}(x) = F^{(k)}_{n}(x).
\end{aligned}$$

□

3. Matrix Generators

Let $Q_k = [q_{ij}]_{k\times k}$ be a square matrix. For a fixed $1 \le i \le k$ an element q_{i1} is equal to the coefficient at $F^{(k)}_{n-i}(x)$ of the right hand side of the formula (2). For $j > 1$ and an arbitrary i we have $q_{ij} = 1$ if $j = i+1$, and $q_{ij} = 0$ otherwise.
The above definition gives matrices

$$Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ x & 0 & 0 \end{bmatrix},\ Q_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 \end{bmatrix}, \ldots, Q_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Moreover, we define a square matrix A_k of order k as the matrix of initial conditions

$$A_k = \begin{bmatrix} F^{(k)}_{2k-2}(x) & F^{(k)}_{2k-3}(x) & \cdots & F^{(k)}_{k}(x) & F^{(k)}_{k-1}(x) \\ F^{(k)}_{2k-3}(x) & F^{(k)}_{2k-4}(x) & \cdots & F^{(k)}_{k-1}(x) & F^{(k)}_{k-2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{(k)}_{k}(x) & F^{(k)}_{k-1}(x) & \cdots & F^{(k)}_{2}(x) & F^{(k)}_{1}(x) \\ F^{(k)}_{k-1}(x) & F^{(k)}_{k-2}(x) & \cdots & F^{(k)}_{1}(x) & F^{(k)}_{0}(x) \end{bmatrix}.$$

Using Laplace's Theorem and some properties of determinants, we get the following results.

Theorem 6. *Let $k \ge 3$ be an integer. Then*

$$\det Q_k = (-1)^{k+1}x,$$

$$\det A_k = (-1)^{\frac{(k-1)(k+2)}{2}}x^{k-1}.$$

Theorem 7. *Let $k \ge 3$, $n \ge 1$ be integers. Then*

$$A_kQ^n_k = \begin{bmatrix} F^{(k)}_{n+2k-2}(x) & F^{(k)}_{n+2k-3}(x) & \cdots & F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) \\ F^{(k)}_{n+2k-3}(x) & F^{(k)}_{n+2k-4}(x) & \cdots & F^{(k)}_{n+k-1}(x) & F^{(k)}_{n+k-2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) & \cdots & F^{(k)}_{n+2}(x) & F^{(k)}_{n+1}(x) \\ F^{(k)}_{n+k-1}(x) & F^{(k)}_{n+k-2}(x) & \cdots & F^{(k)}_{n+1}(x) & F^{(k)}_{n}(x) \end{bmatrix}. \tag{7}$$

Proof. If $n = 1$, then by (2) and simple calculations the result immediately follows. Assume the formula (7) holds for n, we will prove it for $n+1$. Since $A_kQ^{n+1}_k = (A_kQ^n_k)Q_k$, by our assumption and by the recurrence (2) we obtain $A_kQ^{n+1}_k =$

$$\begin{bmatrix} F^{(k)}_{n+2k-2}(x) & F^{(k)}_{n+2k-3}(x) & \cdots & F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) \\ F^{(k)}_{n+2k-3}(x) & F^{(k)}_{n+2k-4}(x) & \cdots & F^{(k)}_{n+k-1}(x) & F^{(k)}_{n+k-2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) & \cdots & F^{(k)}_{n+2}(x) & F^{(k)}_{n+1}(x) \\ F^{(k)}_{n+k-1}(x) & F^{(k)}_{n+k-2}(x) & \cdots & F^{(k)}_{n+1}(x) & F^{(k)}_{n}(x) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x & 0 & 0 & \cdots & 0 \end{bmatrix} =$$

$$\begin{bmatrix} F^{(k)}_{n+2k-1}(x) & F^{(k)}_{n+2k-2}(x) & \cdots & F^{(k)}_{n+k+1}(x) & F^{(k)}_{n+k}(x) \\ F^{(k)}_{n+2k-2}(x) & F^{(k)}_{n+2k-3}(x) & \cdots & F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{(k)}_{n+k+1}(x) & F^{(k)}_{n+k}(x) & \cdots & F^{(k)}_{n+3}(x) & F^{(k)}_{n+2}(x) \\ F^{(k)}_{n+k}(x) & F^{(k)}_{n+k-1}(x) & \cdots & F^{(k)}_{n+2}(x) & F^{(k)}_{n+1}(x) \end{bmatrix}, \text{ which ends the proof. } \square$$

By Theorem 6 we get the following result.

Corollary 1. *Let $k \geq 3$, $n \geq 2$ be integers. Then*

$$\det A_k Q_k^n = (-1)^{\frac{(k-1)(k+2)+2n(k+1)}{2}} x^{n+k-1}.$$

4. Connections with Pascal's Triangle

In [12] it was proved that all Fibonacci type sequences have binomial formulas. Moreover, a graphical rule for calculating elements of these sequences from Pascal's triangle was presented. It is natural to find similar properties of $(2,k)$-distance Fibonacci polynomials.

In this section we inspect infinite matrices generated by coefficients of $(2,k)$-distance Fibonacci polynomials. Note that similar considerations were done in [13] for sequence $\mathcal{F}_n(x) = x\mathcal{F}_{n-1}(x) + \mathcal{F}_{n-2}(x)$ with $\mathcal{F}_0(x) = 1$ and $\mathcal{F}_1(x) = x$.

For a convenience we will write Pascal's triangle in such a form.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}$$

Consider $(2,2)$-distance Fibonacci polynomials. Initial elements of this sequence are 1, 1, $1+x$, $1+x$, $1+2x+x^2$, $1+2x+x^2$, $1+3x+3x^2+x^3$, $1+3x+3x^2+x^3$, $1+4x+6x^2+4x^3+x^4$, $1+4x+6x^2+4x^3+x^4$, $1+5x+10x^2+10x^3+5x^4+x^5$, $1+5x+10x^2+10x^3+5x^4+x^5$, $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$, ...

We obtain a double sequence of polynomials. If we write coefficients of consecutive polynomials in rows of a matrix, then an element $a_{i,j}$ is the coefficient at x^i. The obtained matrix

M_2 has the following form.

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 4 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 4 & 1 & 0 & 0 & 0 \\ 1 & 6 & 10 & 10 & 1 & 0 & 0 & 0 \\ 1 & 6 & 15 & 10 & 5 & 0 & 0 & 0 \end{bmatrix}$$

Coefficients arising in M_2 can be written in a triangular position, in such a way that every row of the triangle is double. Such a triangle is called Pascal's 2-triangle.

For $k = 3$, the initial terms of the sequence of polynomials are as follows: $1, 1, 1, 1 + x$, $1 + x, 1 + 2x, 1 + 2x + x^2, 1 + 3x + x^2, 1 + 3x + 3x^2, 1 + 4x + 3x^2 + x^3, 1 + 4x + 6x^2 + x^3$, $1 + 5x + 6x^2 + 4x^3, 1 + 5x + 10x^2 + 4x^3 + x^4, 1 + 6x + 10x^2 + 10x^3 + x^4, 1 + 6x + 15x^2 + 10x^3 + 5x^4, \ldots$ By Theorems 2 and 3, for an arbitrary k, the coefficients of polynomials are binomial coefficients. Matrix M_3 discloses a recurrence rule of getting coefficients of polynomials from M_2.

Generally, starting from the first column of M_2, to obtain M_k from M_2, we shift all of the following columns $k - 2$ rows downward and repeat this operation for all consecutive columns.

See matrices M_4 and M_5 for $k = 4$ and $k = 5$.

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 10 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Analogously to the classical Pascal's triangle, we can give a general rule of calculating entries of a matrix M_k. For convenience, in the next theorem, rows and columns of a matrix are numbered starting from the number zero.

Theorem 8. *Let $k \geq 1$, $n \geq 0$, $i \geq 1$ be integers. Then $M_k = [a_{n,i}]$ where $a_{n,i} = a_{n-2,i} + a_{n-k,i-1}$ with $a_{n,1} = 1$ for $n = 0, \ldots, k-1$ and $a_{n,i} = 0$ for $n < 0$ or $i < 0$.*

Proof. For $i = 1$ and $i = 2$ the result is obvious.
Let $i \geq 2$ and $F_n^{(k)}(x) = \sum_{i=0}^{\infty} a_{n,i} x^i$. Coefficients $a_{n,i}$ are elements of Pascal's 2-triangle. Then

$$\begin{aligned} F_n^{(k)}(x) &= F_{n-2}^{(k)}(x) + xF_{n-k}^{(k)}(x) = \sum_{i=0}^{\infty} a_{n-2,i} x^i + \sum_{i=0}^{\infty} a_{n-k,i} x^{i+1} \\ &= \sum_{i=0}^{\infty} (a_{n-2,i} x^i + a_{n-k,i} x^{i+1}) = \sum_{i=0}^{\infty} (a_{n-2,i} x^i + a_{n-k,i-1} x^i) \\ &= \sum_{i=0}^{\infty} (a_{n-2,i} + a_{n-k,i-1}) x^i. \end{aligned}$$

Comparing coefficients at x^n, we have $a_{n,i} = a_{n-2,i} + a_{n-k,i-1}$. □

We can observe that sums in rows or in diagonals form known sequences (see [14]) dependences are presented in Table 1. Note that for a given sequence $\{s_n\}$ with $n = 0, 1, 2, \ldots$, a sequence $\{s_{\lfloor \frac{n}{2} \rfloor}\}$ we will call a double sequence.

Table 1. Sums in rows and sums on diagonals of M_k matrices.

k	M_k	Sums in Rows of M_k	Sums on Diagonals of M_k
2	M_2	double 2^n sequence	Padovan sequence
3	M_3	Padovan sequence	double Fibonacci sequence
4	M_4	double Fibonacci sequence	A005686 [14]
5	M_5	A005686 [14]	double Narayan sequence
6	M_6	double Narayan sequence	unknown sequence

The matrix below of coefficients leads to a method of calculating polynomials from Pascal's triangle. This method we will call a staircase method.

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{bmatrix}$$

For example, for $k = 3$ we obtain stairs with steps of the height equal to 1 and the length equal to 1 (green colour). We extend the stairs presented above up to infinity in both directions. By moving such infinite stairs one row downward, we obtain coefficients of consecutive polynomials. For $k = 4$ the sequence of polynomials is double. In this case steps have the height 2 and the length 1 (red colour). For $k = 5$ the height is 3 (blue colour). Generally for an arbitrary $k \geq 3$ the length of the step is 1 and the height of the step is $k - 2$.

The above considerations lead to the following theorem.

Theorem 9. *Let $k \geq 1$, $n \geq 0$ be integers. Then*

$$F_n^{(k)}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n-1-i(k-2)}{2} \rfloor}{i} x^i.$$

Proof. Let $a_{n,i}^k$ denote an element of a matrix M_k. We know that $F_n^{(k)} = \sum_{i=0}^{n} a_{n,i}^k x^i$. To prove this Theorem it is enough to perform transformations M_k to M_2 and transformations from M_2 to Pascal's triangle P. A reverse operation to shifting columns downwards gives $a_{n,i}^k = a_{n-i(k-2),i}^2$. Next we delete the first row (it has the number 0) in the matrix M_2, so $a_{n,i}^k = a_{n-i(k-2)-1,i}^2$. Finally, we contract double rows $a_{n,i}^k = a_{\lfloor \frac{n-1-i(k-2)}{2} \rfloor,i}$ and we obtain a corresponding entry in Pascal's triangle. In the sum we omit zero terms, because all entries above the main diagonal of Pascal's triangle are zeros. Thus we have

$$F_n^{(k)}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n-1-i(k-2)}{2} \rfloor}{i} x^i,$$

which ends the proof. □

5. Conclusions

In this paper we studied $(2,k)$-distance Fibonacci polynomials which generalise $(2,k)$-distance Fibonacci numbers. To prove some properties of the sequences of such polynomials, we used a special graph interpretation and matrix generators. The connections of $(2,k)$-distance Fibonacci polynomials with Pascal's 2-triangle seem interesting. They lead to binomial formulas and new triangles corresponding to sequences of these polynomials. All such triangles, generating coefficients of polynomials, are obtained from Pascal's 2-triangle by shifting. It is natural to consider similar generalisations for other distance Fibonacci numbers.

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