

## Article

# Coefficient Bounds for Certain Classes of Analytic Functions Associated with Faber Polynomial

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**Abstract:** In this paper, we introduce a family of analytic functions in the open unit disk which is bi-univalent. By the virtue of the Faber polynomial expansions, the estimation of  $n$ -th ( $n \geq 3$ ) Taylor–Maclaurin coefficients  $|a_n|$  is obtained. Furthermore, the bounds value of the first two coefficients of such functions is established.

**Keywords:** faber polynomial; coefficient bounds; uniformly convex; uniformly starlike; univalent functions; bi-univalent functions



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## 1. Introduction

Faber polynomials, which were introduced by Faber in 1903 [1], play an important role in the theory of functions of a complex variable and different areas of mathematics and there is a rich literature [2–7] describing their properties and their applications. Given a function  $h(z)$  of the form

$$h(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots,$$

consider the expansion

$$\frac{\zeta h'(\zeta)}{h(\zeta) - w} = \sum_{n=0}^{\infty} \Psi_n(w) \zeta^{-n},$$

valid for all  $\zeta$  in some neighborhood of  $\infty$ . The function  $\Psi_n(w) = w^n + \sum_{k=1}^n a_{nk} w^{n-k}$  is a polynomial of degree  $n$ , called the  $n$ -th Faber polynomial with respect to the function  $h(z)$ . In particular,

$$\Psi_0(w) = 1, \quad \Psi_1(w) = w - b_0,$$

$$\Psi_2(w) = w^2 - 2b_0 w + (b_0^2 - 2b_1),$$

$$\Psi_3(w) = w^3 - 3b_0 w^2 + (3b_0^2 - 3b_1)w + (b_0^3 + 3b_1 b_0 - 3b_2).$$

Let  $\Psi_n(0) = F_n(b_0, b_1, \dots, b_n)$ ,  $n \geq 0$ , see ([8], p. 118). Let  $A$  denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc  $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and let  $S$  be the class of all functions in  $A$  which are univalent in  $U$ . By using the Faber polynomial expansion of functions of the form (1), Airault and Bouali [9], p. 184 showed that

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{j=2}^{\infty} F_{j-1}(a_2, a_3, \dots, a_j) z^{j-1}, \quad (2)$$

where  $F_{j-1}(a_2, a_3, \dots, a_j)$  is the Faber polynomial given by:

$$F_{j-1}(a_2, a_3, \dots, a_j) = \sum_{i_1+2i_2+\dots+(j-1)i_{j-1}=j-1} A(i_1, i_2, \dots, i_{j-1})(a_2^{i_1}, a_3^{i_2}, \dots, a_j^{i_{j-1}})$$

and

$$A(i_1, i_2, \dots, i_{j-1}) := (-1)^{(j-1)+2i_1+\dots+ji_{j-1}} \frac{(i_1 + i_2 + \dots + i_{j-1} - 1)!(j-1)}{(i_1)!(i_2)! \dots (i_{j-1})!}.$$

The first few terms of the Faber polynomials  $F_{j-1}, j \geq 2$ , are given by (e.g., see [10], p. 52)

$$\begin{aligned} F_1 &= -a_2, \\ F_2 &= a_2^2 - 2a_3, \\ F_3 &= -a_2^3 + 3a_2a_3 - 3a_4, \\ F_4 &= a_2^4 - 4a_2^2a_3 + 4a_2a_4 + 2a_3^2 - 4a_5 \\ F_5 &= -a_2^5 + 5a_2^3a_3 + 5a_2^2a_4 - 5a_2(a_3^2 - a_5) + 5a_3a_4 - 5a_6. \end{aligned}$$

The Koebe one-quarter theorem [8], p. 31 ensures the range of every function of the class  $S$  contains the disc  $\{w : |w| < \frac{1}{4}\}$ . Thus every univalent function  $f \in S$  has an inverse  $f^{-1}$ , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega \quad (|\omega| < \frac{1}{4}).$$

The inverse map  $g := f^{-1}$  of the function  $f \in A$  has Taylor expansion given by (see [9], p. 185)

$$\begin{aligned} g(\omega) &= f^{-1}(\omega) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) \omega^n \\ &= w - a_2 \omega^2 + (2a_2^2 - a_3) \omega^3 - (5a_2^2 - 5a_2a_3 + a_4) \omega^4 + \dots, \end{aligned}$$

where the coefficients  $K_n^p(a_2, a_3, \dots, a_n)$  are given by

$$\begin{aligned}
 K_1^p &= pa_2, & K_2^p &= \frac{p(p-1)}{2}a_2^2 + pa_3, \\
 K_3^p &= p(p-1)a_2a_3 + pa_4 + \frac{p(p-1)(p-2)}{3!}a_2^3, \\
 K_4^p &= p(p-1)a_2a_4 + pa_5 + \frac{p(p-1)}{2}a_3^2 + \frac{p(p-1)(p-2)}{2}a_2^2a_3 + \frac{p!}{(p-4)!4!}a_2^4, \\
 &\vdots \\
 K_n^p &= \frac{p!}{(p-n)!n!}a_2^n + \frac{p!}{(p-n+1)!(n-2)!}a_2^{n-2}a_3 + \frac{p!}{(p-n+2)!(n-3)!}a_2^{n-3}a_4 \\
 &\quad + \frac{p!}{(p-n+3)!(n-4)!}a_2^{n-4}\left[a_5 + \frac{p-n+3}{2}a_3^2\right] \\
 &\quad + \frac{p!}{(p-n+4)!(n-5)!}a_2^{n-4}[a_6 + (p-n+3)a_3a_4] + \sum_{j \geq 6}^{\infty} a_2^{n-j}V_j
 \end{aligned} \tag{3}$$

and  $V_j$  is homogeneous polynomial of degree  $j$  in the variables  $a_3, \dots, a_n$ , see ([11], p. 349 and [9], p. 183 and p. 205).

**Lemma 1.** (Schwarz lemma [8], p. 3) Let  $\omega(z)$  be analytic in the unit disc  $U$ , with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  in  $U$ . Then  $|\omega(z)| < |z|$  and  $|\omega'(0)| < 1$  in  $U$ .

If  $f$  and  $g$  are analytic functions in  $U$ , we say that  $f$  is subordinate to  $g$ , written  $f(z) \prec g(z)$  if there exists a Schwarz function  $\omega(z)$  such that  $f(z) = g(\omega(z))$ . Let  $\phi$  be an analytic function with positive real part in  $U$ , satisfying  $\phi(0) = 1$ ,  $\phi'(0) > 0$ , and  $\phi(U)$  is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots \quad (B_1 > 0). \tag{4}$$

Using this  $\phi$ , Ma and Minda [12] considered the classes

$$S(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \right\}$$

and

$$K(\phi) = \left\{ f \in A : zf'(z) \in S(\phi), z \in U \right\}$$

Several well-known classes can be obtained by specializing of the function  $\phi$ , for instance

1. By taking  $\phi(z) = \frac{1+Az}{1+Bz}$ ,  $-1 \leq B < A \leq 1$ , we obtain the classes  $S[A, B]$  and  $K[A, B]$  of the well-known Janowski starlike and convex functions.
2. If we set  $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ , we obtain the classes  $S^*(\alpha)$  and  $K(\alpha)$  of starlike and convex functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ).
3. The class  $S_L^* := S(\sqrt{1+z})$  was considered by Sokol and Stankiewicz [13], consisting of functions  $f$  such that  $\frac{zf'(z)}{f(z)}$  lies in the region bounded by the right half of the Bernoulli lemniscate given by  $|w^2 - 1| < 1$ .
4. Taking  $\phi(z) = \left(\frac{1+z}{1-z}\right)^\delta$  ( $0 < \delta \leq 1$ ) yields the classes of strongly starlike and convex functions.

5. The function class  $S_{\nabla}^* := S(z + \sqrt{1+z^2})$  was considered by Raina and Sokol [14], consisting of normalized starlike functions  $f$  satisfying the inequality

$$\left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|.$$

6. Kanas et al. [15] considered the family of analytic functions  $S(\frac{1}{(1-z)^s})$  and  $K(\frac{1}{(1-z)^s})$  with the property that  $\frac{zf'(z)}{f(z)}$  and  $1 + \frac{zf''(z)}{f'(z)}$  lie in a domain bounded by the right branch of a hyperbola  $\rho = \rho(s) = (2 \cos \frac{\varphi}{s})^{-1} (0 < s \leq 1, |\varphi| < \frac{\pi s}{2})$ .
7. The function class  $S_e^* := S(e^z)$  was introduced and studied by Mendiratta et al. [16]. The exponential function  $\phi(z) = e^z$  has positive real part in  $U$ , maps  $U$  onto a domain  $\phi(U) := \{w \in \mathbb{C} : |\log w| < 1\}$  is symmetric with respect to the real axis and starlike with respect to 1 and  $\phi'(0) > 0$ .
8. The classes  $S(\frac{2}{1+e^{-z}})$  and  $K(\frac{2}{1+e^{-z}})$  were introduced and studied by Goel and Kumar [17]. The modified sigmoid function  $\phi(z) = \frac{2}{1+e^{-z}}$  maps  $U$  onto a domain  $S_G := \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$ , which is symmetric about the real axis. Moreover,  $G$  is a convex and hence starlike function with respect to  $G(0) = 1$ .

An interesting families of the domains that are bounded by a conic sections were introduced and studied by Shams et al. [18], they introduced the class  $SD(\alpha, \beta)$  of  $\beta$ -uniformly starlike functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $U$  consisting of functions  $f \in A$  which satisfy the following inequality

$$\Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (\beta \geq 0; 0 \leq \alpha < 1; z \in U). \quad (5)$$

and class  $KD(\alpha, \beta)$  of  $\beta$ -uniformly convex of order  $\alpha$  ( $0 \leq \alpha < 1$ ), defined by

$$f \in KD(\alpha, \beta) \Leftrightarrow zf'(z) \in SD(\alpha, \beta).$$

Since  $\operatorname{Re} w > \alpha|w-1| + \gamma$  if and only if  $\operatorname{Re}\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma$  (see [19]), then the condition (5) is equivalent to

$$\Re \left\{ (1 + \beta e^{i\theta}) \frac{zf'(z)}{f(z)} - \beta e^{i\theta} \right\} > \alpha.$$

Motivated by the classes  $SD(\alpha, \beta)$  and  $KD(\alpha, \beta)$  we now introduce and investigate the following subclasses of  $A$ , and obtain some interesting results.

**Definition 1.** A function  $f(z) \in A$  is said to be in the class  $M(\lambda, \beta, \gamma, \phi)$  if it satisfies

$$(1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \prec \phi(z) \quad (z \in U),$$

where  $\beta \geq 0, 0 \leq \lambda \leq 1$  and  $-\pi \leq \gamma < \pi$ .

We note that:

1. The class  $M(0, 0, \gamma, \phi) = S(\phi)$  and the class  $M(1, 0, \gamma, \phi) = K(\phi)$ .
2. The class  $M(0, \beta, \gamma, \frac{1+(1-2\alpha)z}{1-z}) = SD(\alpha, \beta)$  and the class  $M(1, \beta, \gamma, \frac{1+(1-2\alpha)z}{1-z}) = KD(\alpha, \beta)$ .
3. The class  $M(\lambda, 0, \gamma, \frac{1+(1-2\alpha)z}{1-z})$  was introduced and studied by Aouf et al. [20].

**Definition 2.** A function  $f(z) \in A$  is said to be in the class  $S(\lambda, \beta, \gamma, \phi)$  if it satisfies

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \prec \phi(z) \quad (z \in U),$$

where  $\beta \geq 0, 0 \leq \lambda \leq 1$  and  $-\pi \leq \gamma < \pi$ .

We note that:

1. The class  $S(0, 0, \gamma, \phi) = S(\phi)$  and  $S(1, 0, \gamma, \phi) = K(\phi)$ .
2.  $M(1, \beta, \gamma, \frac{1+(1-2\alpha)z}{1-z}) = SD(\alpha, \beta)$  and  $S(0, \beta, \gamma, \frac{1+(1-2\alpha)z}{1-z}) = KD(\alpha, \beta)$ .

A single-valued function  $f$  analytic in a domain  $D \subset \mathbb{C}$  is said to be univalent there if it never take the same value twice; that is, if  $f(z_1) \neq f(z_2)$  for all points  $z_1$  and  $z_2$  in  $D$  with  $z_1 \neq z_2$  (see [8], p. 26). A function  $f \in A$  is said to be bi-univalent in  $U$  if  $f$  and its inverse map  $f^{-1}$  are univalent in  $U$ . Let  $\sigma$  denote the class of bi-univalent functions in  $U$  given by (1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [21] and showed that  $|a_2| < 1.51$ . Recently, many authors found non-sharp estimates on the first two Taylor–Maclaurin coefficients  $|a_2|$  and  $|a_3|$  for various subclasses of bi-univalent functions, see for example, ([22–43]). For other related topics see also, ([44–47]).

**Definition 3.** A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma(\lambda, \beta, \gamma, \phi)$  if both  $f$  and its inverse map  $g = f^{-1}$  are in  $M(\lambda, \beta, \gamma, \phi)$ .

We note that:

1. The class  $M_\sigma(0, 1, \gamma, \phi) = \mathfrak{R}_\sigma(\phi)$  was introduced and studied by Darwish et al. [48].
2. The class  $M_\sigma(0, 0, \gamma, \frac{1+Az}{1+Bz}) = S[A, B]$  was introduced and studied by Hamidi and Jahangiri [49].

**Definition 4.** A function  $f \in \sigma$  given by (1) is said to be in the class  $S_\sigma(\lambda, \beta, \gamma, \phi)$  if both  $f$  and its inverse map  $g = f^{-1}$  are in  $S(\lambda, \beta, \gamma, \phi)$ .

We note that:

1. The class  $S_\sigma(0, 1, \gamma, \phi) = \mathfrak{R}_\sigma(\phi)$ .
2. The class  $S_\sigma(0, 0, \gamma, \frac{1+Az}{1+Bz}) = S[A, B]$ .
3. The class  $S_\sigma(\lambda, 0, \gamma, \phi) = M_q^\sigma(\lambda, \phi)$  was introduced and studied by Goyal and Kumar [50], see also Zireh et al. [51].

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients  $|a_n|$  of bi-univalent functions in  $M_\sigma(\lambda, \beta, \gamma, \phi)$  and  $S_\sigma(\lambda, \beta, \gamma, \phi)$  as well as we provide estimates for the initial coefficients of these functions.

## 2. Coefficient Estimates for the Class $M_\sigma(p, \lambda, \tau, \phi)$

**Theorem 1.** Let the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ . If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{B_1}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Proof.** If we set  $F(z) := (1 - \lambda)f(z) + \lambda zf'(z) = z + \sum_{n=2}^{\infty} [1 + \lambda(n-1)]a_n z^n := z + \sum_{n=2}^{\infty} \delta_n z^n$ , then

$$f \in M(\lambda, \beta, \gamma, \phi) \Leftrightarrow (1 + \beta e^{i\gamma}) \frac{zF'(z)}{F(z)} - \beta e^{i\gamma} \prec \phi(z).$$

Since, both functions  $f$  and its inverse map  $g = f^{-1}$  are in  $M(\lambda, \beta, \gamma, \phi)$ , by the definition of subordination, there are analytic functions  $u, v : U \rightarrow U$  with  $u(0) = v(0) = 0$ ,  $|u(z)| < 1$  and  $|v(z)| < 1$ , such that

$$(1 + \beta e^{i\gamma}) \frac{zF'(z)}{F(z)} - \beta e^{i\gamma} = \phi(u(z)) \quad (z \in U) \quad (6)$$

and

$$(1 + \beta e^{i\gamma}) \frac{wG'(w)}{G(w)} - \beta e^{i\gamma} = \phi(v(w)) \quad (z \in U), \quad (7)$$

where  $G(z) := (1 - \lambda)g(z) + \lambda zg'(z) = z + \sum_{n=2}^{\infty} [1 + \lambda(n-1)]d_n z^n := z + \sum_{n=2}^{\infty} \zeta_n z^n$  and  $d_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n)$ . Define the functions  $u(z)$  and  $v(z)$  by

$$u(z) = \sum_{n=1}^{\infty} b_n z^n, \quad v(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in U). \quad (8)$$

It is well known that (see Duren [8], p. 265)

$$|b_n| \leq 1, |c_n| \leq 1 \quad n = 2, 3, \dots \quad (9)$$

By a simple calculation, we have

$$\begin{aligned} \phi(u(z)) &= 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(b_1, b_2, \dots, b_n, B_1, B_2, B_3, \dots, B_n) z^n \\ &= 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \dots \quad (z \in U), \end{aligned} \quad (10)$$

and

$$\begin{aligned} \phi(v(\omega)) &= 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(c_1, c_2, \dots, c_n, B_1, B_2, B_3, \dots, B_n) \omega^n \\ &= 1 + B_1 c_1 \omega + (B_1 c_2 + B_2 c_1^2) \omega^2 + \dots \quad (\omega \in U), \end{aligned} \quad (11)$$

In general (see [52], p. 649), the coefficients  $K_n^p(k_1, k_2, \dots, k_n, B_1, B_2, B_3, \dots, B_n)$  are given by

$$\begin{aligned} &K_n^p(k_1, k_2, \dots, k_n, B_1, B_2, B_3, \dots, B_n) \\ &= \frac{p!}{(p-n)!n!} k_1^n \frac{(-1)^{n+1} B_n}{B_1} + \frac{p!}{(p-n+1)!(n-2)!} k_1^{n-2} k_2 \frac{(-1)^n B_{n-1}}{B_1} \\ &+ \frac{p!}{(p-n+2)!(n-3)!} k_1^{n-3} k_3 \frac{(-1)^{n-1} B_{n-2}}{B_1} \\ &+ \frac{p!}{(p-n+3)!(n-4)!} k_1^{n-4} \left[ k_4 \frac{(-1)^{n-2} B_{n-3}}{B_1} + \frac{p-n+3}{2} k_2^2 k_3 \frac{(-1)^{n-1} B_{n-2}}{B_1} \right] \\ &+ \sum_{j \geq 5} k_1^{n-j} X_j, \end{aligned}$$

where  $X_j$  is a homogeneous polynomial of degree  $j$  in the variables  $k_2, \dots, k_n$ .

Using the Faber polynomial expansion (2) yield the following identities

$$(1 + \beta e^{i\gamma}) \frac{zF'(z)}{F(z)} - \beta e^{i\gamma} = (1 + \beta e^{i\gamma}) \left[ 1 - \sum_{j=2}^{\infty} F_{j-1}(\delta_2, \delta_3, \dots, \delta_j) z^{j-1} \right] - \beta e^{i\gamma}, \quad (12)$$

and

$$(1 + \beta e^{i\gamma}) \frac{wG'(w)}{G(w)} - \beta e^{i\gamma} = (1 + \beta e^{i\gamma}) \left[ 1 - \sum_{j=2}^{\infty} F_{j-1}(\zeta_2, \zeta_3, \dots, \zeta_j) w^{j-1} \right] - \beta e^{i\gamma}. \quad (13)$$

Comparing the corresponding coefficients of (10) and (12) yields

$$(1 + \beta e^{i\gamma}) F_{n-1}(\delta_2, \delta_3, \dots, \delta_n) = B_1 K_{n-1}^{-1}(b_1, b_2, \dots, b_{n-1}, B_1, B_2, B_3, \dots, B_{n-1}) \quad (14)$$

and similarly, from (11) and (13), we have

$$(1 + \beta e^{i\gamma}) F_{n-1}(\zeta_2, \zeta_3, \dots, \zeta_n) = B_1 K_{n-1}^{-1}(c_1, c_2, \dots, c_{n-1}, B_1, B_2, B_3, \dots, B_{n-1}). \quad (15)$$

Since  $a_k = 0$  for  $2 \leq k \leq n-1$ , by substituting  $\delta_n = [1 + \lambda(n-1)]a_n$ ,  $\zeta_n = [1 + \lambda(n-1)]d_n$  and  $d_n = -a_n$  in (14) and (15), we have

$$(1 + \beta e^{i\gamma})(n-1)[1 + \lambda(n-1)]a_n = B_1 b_{n-1}$$

and

$$-(1 + \beta e^{i\gamma})(n-1)[1 + \lambda(n-1)]a_n = B_1 c_{n-1}.$$

By using (9), we conclude that

$$|a_n| \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(n-1)[1 + \lambda(n-1)]},$$

this completes the proof.  $\square$

To prove our next theorem, we shall need the following lemma.

**Lemma 2.** Ref. [52] Let the function  $\Phi(z) = \sum_{n=1}^{\infty} \Phi_n z^n$  be a Schwarz function with  $|\Phi(z)| < 1$ ,  $z \in U$ . Then for  $-\infty < \rho < \infty$ .

$$|\Phi_2 + \rho \Phi_1^2| \leq \begin{cases} 1 - (1 - \rho)|\Phi_1^2| & \rho > 0 \\ 1 - (1 + \rho)|\Phi_1^2| & \rho \leq 0 \end{cases}$$

**Theorem 2.** Let the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , then

$$|a_2| \leq \begin{cases} \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 + B_2)}} & (B_2 \leq 0, B_1 + B_2 \geq 0) \\ \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 - B_2)}} & (B_2 > 0, B_1 - B_2 \geq 0) \end{cases} \quad (16)$$

and

$$|a_3 - a_2^2| \leq \begin{cases} \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} & (B_1 \geq |B_2|) \\ \frac{|B_2|}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} & (B_1 < |B_2|). \end{cases} \quad (17)$$

**Proof.** Replacing  $n = 2$  and  $3$  in (14) and (15), respectively, we find that

$$(1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1 b_1, \quad (18)$$

$$(1 + \beta e^{i\gamma})[2(1 + 2\lambda)a_3 - (1 + \lambda)^2 a_2^2] = [B_1 b_2 + B_2 b_1^2], \quad (19)$$

$$(-1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1 c_1, \quad (20)$$

$$(1 + \beta e^{i\gamma})\{-2(1 + 2\lambda)a_3 + [4(1 + 2\lambda) - (1 + \lambda)^2]a_2^2\} = [B_1 c_2 + B_2 c_1^2]. \quad (21)$$

It follows from (18) and (20) that

$$b_1 = -c_1. \quad (22)$$

Adding (19) to (21) implies

$$2(1 + \beta e^{i\gamma})[2(1 + 2\lambda) - (1 + \lambda)^2]a_2^2 = B_1(b_2 + c_2) + B_2(b_1^2 + c_1^2). \quad (23)$$

Taking the absolute values of both sides of the above equation, we get

$$|a_2|^2 \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)} \left( \left| b_2 + \frac{B_2}{B_1} b_1^2 \right| + \left| c_2 + \frac{B_2}{B_1} c_1^2 \right| \right). \quad (24)$$

Case 1. Let  $B_2 \leq 0$ . Applying Lemma 2 with  $\rho = \frac{B_2}{B_1} \leq 0$  and using (22) we obtain

$$|a_2|^2 \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)} \left( 1 - \left[ \frac{B_1 + B_2}{B_1} \right] |b_1|^2 \right).$$

If  $B_1 + B_2 \geq 0$ , then (18) yields

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 + B_2)}}. \quad (25)$$

Case 2. Let  $B_2 > 0$ . Applying Lemma 2 with  $\rho = \frac{B_2}{B_1} > 0$  and using (22), we obtain

$$|a_2|^2 \leq \frac{B_1}{|1 + \beta e^{i\gamma}|[2(1 + 2\lambda) - (1 + \lambda)^2]} \left( 1 - \left[ \frac{B_1 - B_2}{B_1} \right] |b_1|^2 \right).$$

If  $B_1 - B_2 \geq 0$ , then (18) gives

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2]B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 - B_2)}}. \quad (26)$$

From (25) and (26) we obtain the desired estimate of  $|a_2|$  given by (16). Next, from (19) and (21), we have

$$|a_3 - a_2^2| \leq \frac{B_1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|} \left( \left| b_2 + \frac{B_2}{B_1} b_1^2 \right| + \left| c_2 + \frac{B_2}{B_1} c_1^2 \right| \right). \quad (27)$$

Let  $B_2 \leq 0$ . Applying Lemma 2 for  $\rho = \frac{B_2}{B_1} \leq 0$ , we get

$$|a_3 - a_2^2| \leq \frac{B_1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|} \left( \left[ 1 - \frac{B_1 + B_2}{B_1} |b_1|^2 \right] + \left[ 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right] \right). \quad (28)$$

If  $B_1 + B_2 \geq 0$ , then (28) gives

$$|a_3 - a_2^2| \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If  $B_1 + B_2 < 0$ , then (9) and (28) lead to

$$|a_3 - a_2^2| \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} \left[ 1 - \frac{B_1 + B_2}{B_1} \right] = -\frac{B_2}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

Let  $B_2 > 0$ . Applying Lemma 2 for  $\rho = \frac{B_2}{B_1} > 0$ , (27) gives

$$|a_3 - a_2^2| \leq \frac{B_1}{4(1+2\lambda)|1+\beta e^{i\gamma}|} \left( \left[ 1 - \frac{B_1 - B_2}{B_1} |b_1|^2 \right] + \left[ 1 - \frac{B_1 - B_2}{B_1} |c_1|^2 \right] \right). \quad (29)$$

If  $B_1 - B_2 \geq 0$ , then (29) gives

$$|a_3 - a_2^2| \leq \frac{B_1}{2(1+2\lambda)|1+\beta e^{i\gamma}|}.$$

If  $B_1 - B_2 < 0$ , then from (9) and (29) we have

$$|a_3 - a_2^2| \leq \frac{B_1}{2(1+2\lambda)|1+\beta e^{i\gamma}|} \left[ 1 - \frac{B_1 - B_2}{B_1} \right] = \frac{B_2}{2(1+2\lambda)|1+\beta e^{i\gamma}|}.$$

Which is the second part of assertion (17). This completes the proof of Theorem 2.  $\square$

**Remark 1.** If we take  $\beta = 0$  in Theorem 2 we obtain that the bounds on  $|a_3 - a_2^2|$  given by Deniz et al. [52] when  $\gamma = 1$ .

If we set

$$\phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 \dots$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_\sigma(\lambda, \beta, \gamma, A, B)$  given by Definition 5 below.

**Definition 5.** A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma(\lambda, \beta, \gamma, A, B)$ ,  $-1 \leq B < A \leq 1$ , if the following conditions are satisfied:

$$(1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U)$$

and

$$(1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \prec \frac{1 + A\omega}{1 + B\omega} \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 5 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 1.** Let the function  $f \in M_\sigma(\lambda, \beta, \gamma, A, B)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{(A - B)}{(n - 1)[1 + \lambda(n - 1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 2.** If the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, A, B)$ , then

$$|a_2| \leq \begin{cases} \frac{(A - B)}{\sqrt{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2](A - B) + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - B)}} & (B \geq 0) \\ \frac{(A - B)}{\sqrt{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2](A - B) + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 + B)}} & (-1 \leq B < 0) \end{cases}$$

and

$$|a_3 - a_2^2| \leq \frac{A - B}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set

$$\phi(z) = \left( \frac{1+z}{1-z} \right)^\delta = 1 + 2\delta z + 2\delta^2 z^2 + \dots \quad (0 < \delta \leq 1, z \in U)$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_\sigma(\lambda, \beta, \gamma, \delta)$  given by Definition 6 below.

**Definition 6.** Let  $0 < \delta \leq 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma(\lambda, \beta, \gamma, \delta)$ , if the following conditions are satisfied:

$$\left| \arg \left( (1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \right) \right| \leq \frac{\pi}{2} \delta \quad (z \in U)$$

and

$$\left| \arg \left( (1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \right) \right| \leq \frac{\pi}{2} \delta \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 6 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 3.** Let the function  $f \in M_\sigma(\lambda, \beta, \gamma, \delta)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{2\delta}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 4.** Let  $0 < \delta \leq 1$ . If the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, \delta)$ , then

$$|a_2| \leq \frac{2\delta}{\sqrt{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2]2\delta + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - \delta)}}$$

and

$$|a_3 - a_2^2| \leq \frac{\delta}{(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set

$$\phi(z) = \frac{1 + (1 - 2v)z}{1 - z} = 1 + 2(1 - v)z + 2(1 - v)z^2 + \dots \quad (0 \leq v < 1, z \in U)$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_\sigma^v(\lambda, \beta, \gamma)$  given by Definition 7 below.

**Definition 7.** Let  $0 \leq v < 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma^v(\lambda, \beta, \gamma)$  if the following conditions hold true:

$$\Re \left( (1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \right) > v \quad (z \in U)$$

and

$$\Re \left( (1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \right) > v \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 7 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 5.** Let the function  $f \in M_{\sigma}^v(\lambda, \beta, \gamma)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{2(1-v)}{(n-1)[1+\lambda(n-1)]|1+\beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 6.** Let the function  $f \in M_{\sigma}^v(\lambda, \beta, \gamma)$  be given by (1). Then

$$|a_2| \leq \sqrt{\frac{2(1-v)}{(1+2\lambda-\lambda^2)|1+\beta e^{i\gamma}|}}$$

and

$$|a_3 - a_2^2| \leq \frac{(1-v)}{(1+2\lambda)|1+\beta e^{i\gamma}|}.$$

If we set

$$\phi(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \dots (z \in U)$$

in Definition 3 of the bi-univalent function class  $M_{\sigma}(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_{L\sigma}(\lambda, \beta, \gamma)$  given by Definition 8 below.

**Definition 8.** A function  $f \in \sigma$  given by (1) is said to be in the class  $M_{L\sigma}(\lambda, \beta, \gamma)$ , if the following conditions are satisfied:

$$\left| \left( (1+\beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \right)^2 - 1 \right| < 1 \quad (z \in U)$$

and

$$\left| \left( (1+\beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1-\lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \right)^2 - 1 \right| < 1 \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 8 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 7.** Let the function  $f \in M_{L\sigma}(\lambda, \beta, \gamma)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{1}{2(n-1)[1+\lambda(n-1)]|1+\beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 8.** If the function  $f \in \sigma$  given by (1) be in the class  $M_{L\sigma}(\lambda, \beta, \gamma)$ , then

$$|a_2| \leq \frac{1}{\sqrt{2|1+\beta e^{i\gamma}||1+2\lambda-\lambda^2|+3|1+\beta e^{i\gamma}|^2(1+\lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{4(1+2\lambda)|1+\beta e^{i\gamma}|}.$$

If we set

$$\phi(z) = z + \sqrt{1+z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 \dots (z \in U),$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_\sigma^\Delta(\lambda, \beta, \gamma)$  given by Definition 9 below.

**Definition 9.** A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma^\Delta(\lambda, \beta, \gamma)$  if the following conditions are satisfied:

$$\left| \left( (1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \right)^2 - 1 \right| < 2 \left| (1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \right| \quad (z \in U)$$

and

$$\left| \left( (1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \right)^2 - 1 \right| < 2 \left| (1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma} \right| \quad (\omega \in U)$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 9 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 9.** Let the function  $f \in M_\sigma^\Delta(\lambda, \beta, \gamma)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{1}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 10.** If the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma^\Delta(\lambda, \beta, \gamma)$ , then

$$|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2] + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set

$$\begin{aligned} \phi(z) &= \frac{1}{(1-z)^s} = 1 + sz + \frac{s(s+1)}{2}z^2 + \frac{s(s+1)(s+2)}{6}z^3 \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{s(s+1) \dots (s+n-1)}{n!} z^n \quad (z \in U), \end{aligned}$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_\sigma(\lambda, \beta, \gamma, s)$  given by Definition 10 below.

**Definition 10.** Let  $0 < s \leq 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $M_\sigma(\lambda, \beta, \gamma, s)$ , if the following conditions are satisfied:

$$(1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} \prec \frac{1}{(1-z)^s} \quad (z \in U)$$

and

$$(1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} - \beta e^{i\gamma} \prec \frac{1}{(1 - \omega)^s} \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 10 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 11.** Let the function  $f \in M_\sigma(\lambda, \beta, \gamma, s)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{s}{(n - 1)[1 + \lambda(n - 1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 12.** If the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, s)$ , then

$$|a_2| \leq \frac{\sqrt{2}s}{\sqrt{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2]2s + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - s)}}$$

and

$$|a_3 - a_2^2| \leq \frac{s}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set

$$\phi(z) = e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots \quad (z \in U),$$

in Definition 3 of the bi-univalent function class  $M_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $M_{\sigma e}(\lambda, \beta, \gamma)$  given by Definition 11 below.

**Definition 11.** A function  $f \in \sigma$  given by (1) is said to be in the class  $M_{\sigma e}(\lambda, \beta, \gamma)$  if the following conditions are satisfied:

$$\left| \log \left( (1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - \beta e^{i\gamma} \right) \right| < 1 \quad (z \in U)$$

and

$$\left| \log \left( (1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} - \beta e^{i\gamma} \right) \right| < 1 \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 11 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 13.** Let the function  $f \in M_\sigma(\lambda, \beta, \gamma, s)$ , be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{1}{(n - 1)[1 + \lambda(n - 1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 14.** If the function  $f \in \sigma$  given by (1) be in the class  $M_\sigma(\lambda, \beta, \gamma, s)$ , then

$$|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2] + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\lambda)|1 + \beta e^{i\gamma}|}.$$

### 3. Coefficient Estimates for the Class $S_\sigma(\lambda, \beta, \gamma, \phi)$

**Theorem 3.** Let the function  $f \in \sigma$  given by (1) be in the class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ . If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(n-1)[1 + (n-1)\lambda]}, \quad n \geq 3.$$

**Proof.** Since, both functions  $f$  and its inverse map  $g = f^{-1}$  are in  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , by the definition of subordination, there are analytic functions  $u, v : U \rightarrow U$  given by (8) such that

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \frac{(zf'(z))'}{f'(z)} \right] - \beta e^{i\gamma} = \phi(u(z))$$

and

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \frac{(wg'(w))'}{g'(w)} \right] - \beta e^{i\gamma} = \phi(v(w)).$$

Now, from (2), we get that

$$\begin{aligned} & (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \frac{(zf'(z))'}{f'(z)} \right] - \beta e^{i\gamma} \\ &= 1 - (1 + \beta e^{i\gamma}) \sum_{j=2}^{\infty} [(1 - \lambda) F_{j-1}(a_2, a_3, \dots, a_j) \\ &+ \lambda F_{j-1}(2a_2, 3a_3, \dots, ja_j)] z^{j-1}, \end{aligned} \quad (30)$$

and

$$\begin{aligned} & (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \frac{(wg'(w))'}{g'(w)} \right] - \beta e^{i\gamma} \\ &= 1 - (1 + \beta e^{i\gamma}) \sum_{j=2}^{\infty} [(1 - \lambda) F_{j-1}(d_2, d_3, \dots, d_j) \\ &+ \lambda F_{j-1}(2d_2, 3d_3, \dots, jd_j)] w^{j-1}, \end{aligned} \quad (31)$$

where  $d_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n)$ . Now, upon comparing the corresponding coefficients in (10) and (30), we get

$$\begin{aligned} & (1 + \beta e^{i\gamma}) [(1 - \lambda) F_{n-1}(a_2, a_3, \dots, a_n) + \lambda F_{n-1}(2a_2, 3a_3, \dots, na_n)] \\ &= B_1 K_{n-1}^{-1}(b_1, b_2, \dots, b_{n-1}, B_1, B_1, B_2, B_3, \dots, B_{n-1}) \end{aligned} \quad (32)$$

and similarly, from (11) and (31), we have

$$\begin{aligned} & (1 + \beta e^{i\gamma}) [(1 - \lambda) F_{n-1}(d_2, d_3, \dots, d_n) + \lambda F_{n-1}(2d_2, 3d_3, \dots, nd_n)] \\ &= B_1 K_{n-1}^{-1}(c_1, c_2, \dots, c_{n-1}, B_1, B_2, B_3, \dots, B_{n-1}). \end{aligned} \quad (33)$$

Since  $a_k = 0$  for  $2 \leq k \leq n-1$ , by using  $d_n = -a_n$  and  $F_{n-1}(a_2, a_3, \dots, a_n) = -(n-1)a_n$ , we have

$$(1 + \beta e^{i\gamma})(n-1)[1 + (n-1)\lambda]a_n = B_1 b_{n-1} \quad (34)$$

and

$$-(1 + \beta e^{i\gamma})(n-1)[1 + (n-1)\lambda]a_n = B_1 c_{n-1}. \quad (35)$$

By using (9), we conclude that

$$|a_n| \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(n-1)[1 + (n-1)\lambda]}.$$

□

**Remark 2.** If we take  $\beta = 0$  in Theorem 3, then we have the results which were given by Zireh et al. [51] when  $\varphi(z) = 1$ .

**Theorem 4.** If the function  $f \in \sigma$  given by (1) be in the class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , then

$$|a_2| \leq \begin{cases} \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1+\lambda)B_1^2 + |1 + \beta e^{i\gamma}|^2(1+\lambda)^2(B_1+B_2)}} & (B_2 \leq 0, B_1 + B_2 \geq 0) \\ \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1+\lambda)B_1^2 + |1 + \beta e^{i\gamma}|^2(1+\lambda)^2(B_1-B_2)}} & (B_2 > 0, B_1 - B_2 \geq 0) \end{cases}, \quad (36)$$

and

$$|a_3 - a_2^2| \leq \begin{cases} \frac{B_1}{2|1 + \beta e^{i\gamma}|(1+2\lambda)} & (B_1 \geq |B_2|) \\ \frac{|B_2|}{2|1 + \beta e^{i\gamma}|(1+2\lambda)} & (B_1 < |B_2|). \end{cases} \quad (37)$$

**Proof.** Letting  $n = 2$  and  $3$  in (32) and (33), respectively, we find that

$$(1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1 b_1, \quad (38)$$

$$(1 + \beta e^{i\gamma})[2(1 + 2\lambda)a_3 - (1 + 3\lambda)a_2^2] = B_1 b_2 + B_2 b_1^2, \quad (39)$$

$$-(1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1 c_1, \quad (40)$$

$$(1 + \beta e^{i\gamma})\{-2(1 + 2\lambda)a_3 + [4(1 + 2\lambda) - (1 + 3\lambda)]a_2^2\} = B_1 c_2 + B_2 c_1^2. \quad (41)$$

Equations (38) and (40) lead to

$$b_1 = -c_1. \quad (42)$$

Adding (39) and (41) yields

$$2(1 + \beta e^{i\gamma})(1 + \lambda)a_2^2 = B_1(b_2 + c_2) + B_2(b_1^2 + c_1^2) \quad (43)$$

or

$$|a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + \lambda)} \left( \left| b_2 + \frac{B_2}{B_1} b_1^2 \right| + \left| c_2 + \frac{B_2}{B_1} c_1^2 \right| \right).$$

First, let  $B_2 \leq 0$ . Applying Lemma 2 with  $\rho = \frac{B_2}{B_1} \leq 0$  and using (42), we get

$$|a_2^2| \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(1 + \lambda)} \left( 1 - \left[ \frac{B_1 + B_2}{B_1} \right] |b_1^2| \right) \quad (44)$$

If  $B_1 + B_2 \geq 0$ , then (38) yields

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1 + \lambda)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 + B_2)}} \quad (45)$$

Similarly, for  $B_2 > 0$  ( $\rho = \frac{B_2}{B_1} > 0, B_1 - B_2 \geq 0$ ), we have

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|1 + \beta e^{i\gamma}|(1 + \lambda)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 - B_2)}} \quad (46)$$

From (45) and (46) we obtain the desired estimate of  $|a_2|$  given by (36).

Next, in order to find the bound on  $|a_3 - a_2^2|$ , by subtracting (41) from (39), we have

$$|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( \left| b_2 + \frac{B_2}{B_1} b_1^2 \right| + \left| c_2 + \frac{B_2}{B_1} c_1^2 \right| \right). \quad (47)$$

Let  $B_2 \leq 0$ . Applying Lemma 2 with  $\rho = \frac{B_2}{B_1} \leq 0$ , we get

$$|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( \left[ 1 - \frac{B_1 + B_2}{B_1} |b_1|^2 \right] + \left[ 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right] \right). \quad (48)$$

If  $B_1 + B_2 \geq 0$ , then (48) gives  $|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}$ .

If  $B_1 + B_2 < 0$ , then (9) and (48) give

$$|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left[ 1 - \frac{B_1 + B_2}{B_1} \right] = -\frac{B_2}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.$$

Let  $B_2 > 0$ . Applying Lemma 2 with  $\rho = \frac{B_2}{B_1} > 0$ , (47) gives

$$|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( \left[ 1 - \frac{B_1 - B_2}{B_1} |b_1|^2 \right] + \left[ 1 - \frac{B_1 - B_2}{B_1} |c_1|^2 \right] \right). \quad (49)$$

If  $B_1 - B_2 \geq 0$ , then (49) gives

$$|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.$$

If  $B_1 - B_2 < 0$ , then from (9) and (49) we get

$$|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left[ 1 - \frac{B_1 - B_2}{B_1} \right] = \frac{B_2}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.$$

This completes the proof of Theorem 3.  $\square$

**Remark 3.** If we set  $\beta = 0$  in Theorem 4, then we obtain the results of Goyal and Kumar [50] when  $\varphi(z) = 1$ .

If we set  $\phi(z) = \left( \frac{1+z}{1-z} \right)^\delta$  ( $0 < \delta \leq 1, z \in U$ ) in Definition 4 of the bi-univalent function class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_\sigma^\delta(\lambda, \beta, \gamma)$  given by Definition 12 below.

**Definition 12.** Let  $0 < \delta \leq 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $S_\sigma^\delta(\lambda, \beta, \gamma)$  if the following subordinations hold:

$$\left| \arg \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right) \right| \leq \frac{\pi}{2} \delta \quad (z \in U)$$

and

$$\left| \arg \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right) \right| \leq \frac{\pi}{2} \delta \quad (w \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 12 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 15.** Let the function  $f \in S_\sigma^\delta(\lambda, \beta, \gamma)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{2\delta}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 16.** Let  $0 < \gamma \leq 1$ . If the function  $f \in \sigma$  given by (1) be in the class  $S_\sigma^\delta(\lambda, \beta, \gamma)$ , then

$$|a_2| \leq \frac{2\delta}{\sqrt{2|1 + \beta e^{i\gamma}|(1 + \lambda)\delta + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - \delta)}}$$

and

$$|a_3 - a_2^2| \leq \frac{\delta}{|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.$$

If we set  $\phi(z) = \frac{1+(1-2v)z}{1-z}$  ( $0 \leq v < 1, z \in U$ ) in Definition 4 of the bi-univalent function class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_\sigma(\lambda, \beta, \gamma, v)$  given by Definition 13 below.

**Definition 13.** Let  $0 \leq v < 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $S_\sigma(\lambda, \beta, \gamma, v)$ , if the following conditions are satisfied:

$$\Re \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right) > v \quad (z \in U)$$

and

$$\Re \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right) > v \quad (w \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 13 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 17.** Let the function  $f \in S_\sigma(\lambda, \beta, \gamma, v)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{2(1-v)}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 18.** Let  $0 \leq v < 1$ . If the function  $f \in S_\sigma(\lambda, \beta, \gamma, v)$  be of the form (1), then

$$|a_2| \leq \sqrt{\frac{2(1-v)}{|1+\beta e^{i\gamma}|(1+\lambda)}}$$

and

$$|a_3 - a_2^2| \leq \frac{(1-v)}{|1+\beta e^{i\gamma}|(1+2\lambda)}.$$

If we set  $\phi(z) = \frac{1+Az}{1+Bz}$  in Definition 4 of the bi-univalent function class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_\sigma(\lambda, \beta, \gamma, A, B)$  given by Definition 14 below.

**Definition 14.** A function  $f \in \sigma$  given by (1) is said to be in the class  $S_\sigma(\lambda, \beta, \gamma, A, B)$ ,  $-1 \leq B < A \leq 1$ , if the following conditions are satisfied:

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U)$$

and

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \prec \frac{1 + A\omega}{1 + B\omega} \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 14 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 19.** Let the function  $f \in S_\sigma(\lambda, \beta, \gamma, A, B)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{(A-B)}{(n-1)[1+\lambda(n-1)]|1+\beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 20.** If the function  $f \in \sigma$  given by (1) be in the class  $S_\sigma(\lambda, \beta, \gamma, A, B)$ , then

$$|a_2| \leq \begin{cases} \frac{(A-B)}{\sqrt{|1+\beta e^{i\gamma}|(1+\lambda)(A-B)+|1+\beta e^{i\gamma}|^2(1-B)}} & (B \geq 0) \\ \frac{(A-B)}{\sqrt{|1+\beta e^{i\gamma}|(1+\lambda)(A-B)+|1+\beta e^{i\gamma}|^2(1+B)}} & (-1 \leq B < 0) \end{cases}$$

and

$$|a_3 - a_2^2| \leq \frac{A-B}{2(1+2\lambda)|1+\beta e^{i\gamma}|}.$$

**Remark 4.** If we put  $\beta = \lambda = 0$  in Corollaries 19 and 20, then we obtain the results of Hamidi and Jahangiri [49]

If we set  $\phi(z) = \sqrt{1+z}$  in Definition 4 of the bi-univalent function class  $S_\sigma(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_{L\sigma}(\lambda, \beta, \gamma)$  given by Definition 15 below.

**Definition 15.** A function  $f \in \sigma$  given by (1) is said to be in the class  $S_{L\sigma}(\lambda, \beta, \gamma)$  if the following conditions are satisfied:

$$\left| \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right)^2 - 1 \right| < 1 \quad (z \in U)$$

and

$$\left| \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right)^2 - 1 \right| < 1 \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 15 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 21.** Let the function  $f \in S_{L\sigma}(\lambda, \beta, \gamma)$ , be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{1}{2(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 22.** If the function  $f \in \sigma$  given by (1) be in the class  $S_{L\sigma}(\lambda, \beta, \gamma)$ , then

$$|a_2| \leq \frac{1}{\sqrt{2|1 + \beta e^{i\gamma}|(1 + \lambda) + 3|1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set  $\phi(z) = z + \sqrt{1 + z^2}$  in Definition 4 of the bi-univalent function class  $S_{\sigma}(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_{\sigma}^{\Delta}(\lambda, \beta, \gamma)$  given by Definition 16 below.

**Definition 16.** A function  $f \in \sigma$  given by (1) is said to be in the class  $S_{\sigma}^{\Delta}(\lambda, \beta, \gamma)$  if the following conditions are satisfied:

$$\begin{aligned} & \left| \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right)^2 - 1 \right| \\ & < 2 \left| (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right| \quad (z \in U) \end{aligned}$$

and

$$\begin{aligned} & \left| \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right)^2 - 1 \right| \\ & < 2 \left| (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right| \quad (\omega \in U) \end{aligned}$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 9 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 23.** Let the function  $f \in S_{\sigma}^{\Delta}(\lambda, \beta, \gamma)$ , be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{1}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 24.** If the function  $f \in \sigma$  given by (1) be in the class  $S_{\sigma}^{\Delta}(\lambda, \beta, \gamma)$ , then

$$|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}|(1 + \lambda) + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set  $\phi(z) = \frac{1}{(1-z)^s}$  in Definition 4 of the bi-univalent function class  $S_{\sigma}(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_{\sigma}(\lambda, \beta, \gamma, s)$  given by Definition 17 below.

**Definition 17.** Let  $0 < s \leq 1$ . A function  $f \in \sigma$  given by (1) is said to be in the class  $S_{\sigma}(\lambda, \beta, \gamma, s)$ , if the following conditions are satisfied:

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \prec \frac{1}{(1-z)^s} \quad (z \in U)$$

and

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \prec \frac{1}{(1-\omega)^s} \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 17 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 25.** Let the function  $f \in S_{\sigma}(\lambda, \beta, \gamma, s)$  be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{s}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 26.** If the function  $f \in \sigma$  given by (1) be in the class  $S_{\sigma}(\lambda, \beta, \gamma, s)$ , then

$$|a_2| \leq \frac{\sqrt{2}s}{\sqrt{|1 + \beta e^{i\gamma}|(1 + \lambda)2s + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - s)}}$$

and

$$|a_3 - a_2^2| \leq \frac{s}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

If we set  $\phi(z) = e^z$  in Definition 4 of the bi-univalent function class  $S_{\sigma}(\lambda, \beta, \gamma, \phi)$ , we obtain a new class  $S_{\sigma e}(\lambda, \beta, \gamma)$  given by Definition 18 below.

**Definition 18.** A function  $f \in \sigma$  given by (1) is said to be in the class  $S_{\sigma e}(\lambda, \beta, \gamma)$ , if the following conditions are satisfied:

$$\left| \log \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] - \beta e^{i\gamma} \right) \right| < 1 \quad (z \in U)$$

and

$$\left| \log \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] - \beta e^{i\gamma} \right) \right| < 1 \quad (\omega \in U),$$

where  $g = f^{-1}$ .

Using the parameter setting of Definition 18 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 27.** Let the function  $f \in S_{\sigma}(\lambda, \beta, \gamma, s)$ , be given by (1). If  $a_k = 0$  for  $2 \leq k \leq n-1$ , then

$$|a_n| \leq \frac{1}{(n-1)[1 + \lambda(n-1)]|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$

**Corollary 28.** If the function  $f \in \sigma$  given by (1) be in the class  $S_{\sigma}(\lambda, \beta, \gamma, s)$ , then

$$|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}|(1 + \lambda) + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$

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