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Growth Analysis of Meromorphic Solutions of Linear Difference Equations with Entire or Meromorphic Coefficients of Finite φ-Order

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Abstract: Many researchers' attentions have been attracted to various growth properties of meromorphic solution f (of finite φ-order) of the following higher order linear difference equation $A_n(z)f(z+n)+\ldots+A_1(z)f(z+1)+A_0(z)f(z)=0$, where $A_n(z),\ldots,A_0(z)$ are entire or meromorphic coefficients (of finite φ-order) in the complex plane ($\varphi:[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function). In this paper, by introducing a constant $\mathbf{b}$ (depending on φ) defined by $\varliminf\limits_{r\to\infty}\frac{\log r}{\log\varphi(r)}=\mathbf{b}<\infty$, and we show how nicely diverse known results for the meromorphic solution f of finite φ-order of the above difference equation can be modified.

Keywords: linear difference equations; nevanlinna's theory; meromorphic solutions; φ-order

MSC: 30D30; 30D35; 39A10; 39A13

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1. Introduction and Preliminaries

Throughout this paper, a meromorphic function is meant to be analytic in the whole complex plane $\mathbb{C}$ except possibly for poles. In the following, let $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ and $\mathbb{N}$ denote the extended complex plane and the set of positive integers, respectively. The readers are assumed to be familiar with the basic results and standard notations of Nevanlinna's value distribution theory of meromorphic functions (see, e.g., Reference [1–4]). Yet, here, some fundamental notations for Nevanlinna theory of meromorphic functions are recalled. Let f be a meromorphic function and $r>0$. For $0\le t\le r$ let $n(t,f)$ denote the number of poles of f in the closed disk $\overline{D}(0,t):=\{z\in\mathbb{C}\,:\,|z|\le t\}$, counting multiplicities. Then,

$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\,\log r$$

is called the (Nevanlinna) counting function of the poles of f. Let $\mathbb{R}$ be the set of real numbers. Define $\log^+:\mathbb{R}\to\mathbb{R}$ by

$$\log^+x:=\begin{cases}\log x & (x\ge 1),\\ 0 & (x\le 1).\end{cases}$$

Let $r>0$ and f be meromorphic in $\overline{D}(0,r)$. Then,

$$m(r,f):=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta$$

is called the proximity function, and

$$T(r,f) := N(r,f) + m(r,f)$$

is called the (Nevanlinna) characteristic of f.

Consider the following higher order linear difference (discrete) equation

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{1}$$

where $A_n(z), \ldots, A_0(z)$ are meromorphic (or entire) functions with $A_n(z) \cdot A_0(z) \not\equiv 0$. A lot of interests in such a difference equation as the Equation (1) have recently been renewed, in particular, together with Nevanlinna theory [2,4] (see, e.g., Reference [5–14] and the references cited therein). For a later use, from the Equation (1), we find that, for $f(\not\equiv 0)$,

$$-A_\ell(z) = \sum_{\substack{0\le j\le n \\ j\ne \ell}} A_j(z) \frac{f(z+j)}{f(z+\ell)};$$

hence,

$$|A_\ell(z)| \le \sum_{\substack{0\le j\le n \\ j\ne \ell}} |A_j(z)| \left| \frac{f(z+j)}{f(z+\ell)} \right|. \tag{2}$$

Yet, some notations and results are recalled. The linear measure for a set $E \subset [0,\infty)$ and the logarithmic measure for a set $E \subset [1,\infty)$ are defined and denoted by $m(E) = \int_E dt$ and $m_\ell(E) = \int_E \frac{dt}{t}$, respectively. The upper density $\overline{\text{dens}}\, E$ of a set $E \subset [0,\infty)$ and the upper logarithmic density $\overline{\log \text{dens}}\, E$ of a set $E \subset (1,\infty)$ are defined as

$$\overline{\text{dens}}\, E = \varlimsup_{r\to\infty} \frac{m(E\cap[0,r])}{r}$$

and

$$\overline{\log \text{dens}}\, E = \varlimsup_{r\to\infty} \frac{m_\ell(E\cap[1,r])}{\log r}.$$

Then, some easily-derivable implications among measure, logarithmic measure, upper density, and upper logarithmic density are given in the following remark.

Remark 1 (Reference [15], Proposition 1)**.** *Let $E \subset [1,\infty)$. Then,*

(i) $m_\ell(E) = \infty$ *implies* $m(E) = \infty$;

(ii) $\overline{\text{dens}}\, E > 0$ *implies* $m(E) = \infty$;

(iii) $\overline{\log \text{dens}}\, E > 0$ *implies* $m_\ell(E) = \infty$.

For a more refined growth of meromorphic solutions of the Equation (1), the following (modified) definitions are recalled. Here, and in the following, let $\varphi : [0,\infty) \to (0,\infty)$ be a non-decreasing unbounded function.

Definition 1 (Reference [13,15–19])**.** *The φ-order and the φ-lower order of a meromorphic function f are defined, respectively, as*

$$\sigma(f,\varphi) = \varlimsup_{r\to\infty} \frac{\log T(r,f)}{\log \varphi(r)}, \quad \textit{and} \quad \mu(f,\varphi) = \varliminf_{r\to\infty} \frac{\log T(r,f)}{\log \varphi(r)}.$$

For f an entire function, the corresponding orders are

$$\sigma(f,\varphi) = \varlimsup_{r\to\infty} \frac{\log\log M(r,f)}{\log \varphi(r)}, \quad \textit{and} \quad \mu(f,\varphi) = \varliminf_{r\to\infty} \frac{\log\log M(r,f)}{\log \varphi(r)}.$$

Definition 2 (Reference [13,17,18]). *If f is a meromorphic function (or an entire function) satisfying $0 < \sigma(f, \varphi) = \sigma < \infty$, then φ-type of f is defined, respectively, as*

$$\tau(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{T(r, f)}{\varphi(r)^{\sigma}},$$

and

$$\tau(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{\varphi(r)^{\sigma}}.$$

In addition, the φ-lower type of an entire function f with $0 < \mu(f, \varphi) = \mu < \infty$ is defined by

$$\underline{\tau}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log M(r, f)}{\varphi(r)^{\mu}}.$$

It is noted that Definitions 1 and 2, where $\varphi(r) = r$ may become the standard definitions of order, lower order, type, and lower type, respectively.

Definition 3 (Reference [18], and Reference [2], Section 2.4)). *For $a \in \overline{\mathbb{C}}$, the deficiency of a with respect to a meromorphic function f is defined as*

$$\delta(a, f) = \underline{\lim_{r\to\infty}} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} \quad (a \neq \infty),$$

and

$$\delta(\infty, f) = \underline{\lim_{r\to\infty}} \frac{m(r, f)}{T(r, f)} = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, f)}{T(r, f)}.$$

Remark 2 (Reference [19], p. 4). *In the following, the non-decreasing unbounded function $\varphi : [0, \infty) \to (0, \infty)$ is assumed to satisfy the following two conditions:*

(i) $\lim_{r\to\infty} \frac{\log\log r}{\log \varphi(r)} = 0$;

(ii) $\lim_{r\to\infty} \frac{\log \varphi(\alpha r)}{\log \varphi(r)} = 1$ *for some* $\alpha > 1$.

Several interesting and important results about (1) are recalled in the following theorems.

Theorem 1 (Reference [8], Theorem 9.2). *Assume that there exists an integer p $(0 \leq p \leq n)$ such that*

$$\sigma(A_p) > \max_{\substack{0\leq j\leq n \\ j\neq p}} \sigma(A_j) =: \sigma, \tag{3}$$

where $A_j(z)$ $(j = 0, \ldots, n)$ are entire functions. If $f(z)$ is a meromorphic solution of the Equation (1), then $\sigma(f) \geq \sigma(A_p) + 1$.

Instead of the restriction (3), assuming that among the maximal order σ, exactly one has its type strictly greater than the others, Laine and Yang (Reference [11], Theorem 5.2) obtained the following conclusion for any meromorphic solution of the Equation (1):

$$\sigma(f) \geq \sigma + 1. \tag{4}$$

In Theorem 1, the Equation (1) has only one dominating coefficient A_p. The following two theorems are concerned with the case when there are at least two coefficients which have the maximal order.

Theorem 2 (Reference [14], Theorem 1.1)**.** *Let $A_j(z)$ $(j = 0, 1, \dots, n)$ be entire functions such that there exists an integer p $(0 \le p \le n)$ satisfying*

$$\max\{\sigma(A_j) \mid j = 0, 1, \dots, n,\ j \ne p\} \le \mu(A_p) < \infty,$$

and

$$\max\{\tau(A_j) \mid \sigma(A_j) = \mu(A_p),\ j = 0, 1, \dots, n,\ j \ne p\} < \underline{\tau}(A_p).$$

Then, every meromorphic solution $f(\not\equiv 0)$ of the Equation (1) satisfies $\mu(f) \ge \mu(A_p) + 1$.

Theorem 3 (Reference [20], Theorem 1.3)**.** *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_j(z)$ $(j = 0, 1, \dots, n)$ be entire functions satisfying $\max\limits_{0 \le j \le n} \sigma(A_j)$ $\le \sigma$. In addition, assume that there exists an integer p $(0 \le p \le n)$ such that, for some constants, $0 \le \alpha < \beta$ and $\varepsilon > 0$ sufficiently small,*

$$|A_p(z)| \ge \exp(\beta r^{\sigma - \varepsilon})$$

and

$$|A_j(z)| \le \exp(\alpha r^{\sigma - \varepsilon}) \quad (j = 0, \dots, n,\ j \ne p),$$

as $|z| = r \to \infty$ for $z \in H$. Then, every meromorphic solution $f(\not\equiv 0)$ of the Equation (1) satisfies $\sigma(f) \ge \sigma(A_p) + 1$.

When the coefficients $A_j(z)$ $(j = 0, 1, \dots, n)$ in (1) are meromorphic, Chen and Shon [21] extended Theorem 1 as in the following theorem.

Theorem 4 (Reference [21], Theorem 11)**.** *Let $A_j(z)$ $(j = 0, 1, \dots, n)$ be meromorphic functions such that there exists an integer p $(0 \le p \le n)$ such that $\sigma(A_p) > \max\{\sigma(A_j) \mid 0 \le j \le n,\ j \ne p\}$, $\delta(\infty, A_p) > 0$. Then, for every meromorphic solution $f(\not\equiv 0)$ of the Equation (1), one has $\sigma(f) \ge \sigma(A_p) + 1$.*

Here, the following natural question is occurred: When the coefficients of the Equation (1) are entire or meromorphic functions *of finite φ-order*, what would the growth properties of solutions of the linear difference Equation (1) be like? In this paper, for an answer to this question, by introducing a constant $\mathbf{b}$, which depends on φ, defined by

$$\varliminf_{r \to \infty} \frac{\log r}{\log \varphi(r)} = \mathbf{b} < \infty, \tag{5}$$

we show how nicely diverse known results for the meromorphic solution f of finite φ-order of the difference Equation (1) can be amended.

2. Main Results

In this section, main theorems are provided.

Theorem 5. *Let $A_j(z)$ $(j = 0, 1, \dots, n)$ be entire functions such that there exists an integer ℓ $(0 \le \ell \le n)$ satisfying*

$$\max_{\substack{0 \le j \le n \\ j \ne \ell}} \sigma(A_j, \varphi) < \sigma(A_\ell, \varphi). \tag{6}$$

Then, every transcendental meromorphic solution $f(\not\equiv 0)$ of the Equation (1) satisfies $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$.

Theorem 6. *Let $A_j(z)$ $(j = 0, 1, \dots, n)$ be entire functions such that there exists an integer ℓ $(0 \le \ell \le n)$ satisfying*

$$\max\{\sigma(A_j, \varphi) \mid j = 0, 1, \dots, n,\ j \ne \ell\} \le \mu(A_\ell, , \varphi) < \infty \tag{7}$$

and

$$\max\{\tau(A_j,\varphi)\,|\,\sigma(A_j,\varphi)=\mu(A_\ell,\varphi),\ j=0,1,\ldots,n,\ j\neq\ell\}<\underline{\tau}(A_\ell,\varphi). \tag{8}$$

Then, for every transcendental meromorphic solution $f(\not\equiv 0)$ *of the Equation* (1)*, we have* $\mu(f,\varphi)\geq\mu(A_\ell,\varphi)+\mathbf{b}$.

Theorem 7. *Let* H *be a set of complex numbers satisfying* $\overline{\log\mathrm{dens}}\,\{|z|:z\in H\}>0$. *In addition, let* $A_j(z)$ $(j=0,1,\ldots,n)$ *be entire functions satisfying* $\max\limits_{0\leq j\leq n}\sigma(A_j,\varphi)\leq\sigma$. *Further assume that there exists an integer* ℓ $(0\leq\ell\leq n)$ *such that, for some constants,* $0\leq\alpha<\beta$ *and* $\delta>0$ *sufficiently small,*

$$|A_\ell(z)|\geq\exp\left\{\beta(\varphi(r))^{\sigma-\delta}\right\} \tag{9}$$

and

$$|A_j(z)|\leq\exp\left\{\alpha(\varphi(r))^{\sigma-\delta}\right\}\quad(0\leq j\leq n,\ j\neq\ell), \tag{10}$$

as $|z|=r\to\infty$ *for* $z\in H$. *Then, every transcendental meromorphic solution* $f(\not\equiv 0)$ *of the Equation* (1) *satisfies* $\sigma(f,\varphi)\geq\sigma(A_\ell,\varphi)+\mathbf{b}$.

Remark 3. *Under the assumptions of Theorem 7, we find* $\sigma(A_\ell,\varphi)=\sigma$. *Indeed, obviously* $\sigma(A_\ell,\varphi)\leq\sigma$. *Suppose that* $\sigma(A_\ell,\varphi)=\eta<\sigma$. *Let* ε $\left(0<\varepsilon<\frac{\sigma-\eta}{2}\right)$ *be given. From Definition 1 and* (9)*, we obtain*

$$\exp\{\beta(\varphi(r))^{\sigma-\varepsilon}\}\leq|A_\ell(z)|<\exp\{(\varphi(r))^{\eta+\varepsilon}\}, \tag{11}$$

as $|z|=r\to\infty$ *for* $z\in H$ *(see (iii) in Remark 1). Any* ε $\left(0<\varepsilon<\frac{\sigma-\eta}{2}\right)$ *in* (11) *can be taken. For example, take* $\frac{\sigma-\eta}{4}$ *in* (11) *to yield*

$$\exp\left\{\beta(\varphi(r))^{\frac{\sigma-\eta}{2}}\cdot(\varphi(r))^{\frac{\sigma+3\eta}{4}}\right\}\leq|A_\ell(z)|<\exp\left\{(\varphi(r))^{\frac{\sigma+3\eta}{4}}\right\}, \tag{12}$$

as $|z|=r\to\infty$ *for* $z\in H$. *Since* $\varphi:[0,\infty)\to(0,\infty)$ *is a non-decreasing unbounded function, and* $\beta>0$ *is fixed, we can choose a sufficiently large* r *so that* $\beta(\varphi(r))^{\frac{\sigma-\eta}{2}}\geq 1$ *in* (12)*. This leads to a contradiction. Therefore,* $\sigma(A_\ell,\varphi)=\sigma$.

Theorem 8. *Let* H *be a set of complex numbers satisfying* $\overline{\log\mathrm{dens}}\,\{|z|:z\in H\}>0$. *In addition, let* $A_j(z)$ $(j=0,1,\ldots,n)$ *be entire functions of finite* φ*-order such that there exists an integer* ℓ $(0\leq\ell\leq n)$ *satisfying*

$$\varlimsup_{r\to\infty}\sum_{\substack{0\leq j\leq n\\ j\neq\ell}}\frac{m(r,A_j)}{m(r,A_\ell)}<1 \tag{13}$$

for $|z|=r$ $(z\in H)$. *Then, every transcendental meromorphic solution* $f(\not\equiv 0)$ *of the Equation* (1) *satisfies* $\sigma(f,\varphi)\geq\sigma(A_\ell,\varphi)+\mathbf{b}$.

Theorem 9. *Let* $A_j(z)$ $(j=0,1,\ldots,n)$ *be meromorphic functions such that there exists an integer* ℓ $(0\leq\ell\leq n)$ *satisfying*

$$\max_{\substack{0\leq j\leq n\\ j\neq\ell}}\sigma(A_j,\varphi)<\sigma(A_\ell,\varphi), \tag{14}$$

and $\delta(\infty,A_\ell)>0$. *Then, every transcendental meromorphic solution* $f(\not\equiv 0)$ *of the Equation* (1) *satisfies* $\sigma(f,\varphi)\geq\sigma(A_\ell,\varphi)+\mathbf{b}$.

Theorem 10. *Let $A_j(z)$ $(j = 0, 1, \ldots, n)$ be meromorphic functions of finite φ-order such that there exists an integer ℓ $(0 \le \ell \le n)$ satisfying*

$$\varlimsup_{r\to\infty} \sum_{\substack{0\le j\le n \\ j\ne \ell}} \frac{m(r, A_j)}{m(r, A_\ell)} < 1, \tag{15}$$

and $\delta(\infty, A_\ell) > 0$. *Then, every transcendental meromorphic solution $f(\not\equiv 0)$ of the Equation (1) satisfies $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$.*

3. Preliminary Lemmas

For proof of the main results in Section 2, here, diverse estimations regarding meromorphic functions are recalled and established in the following lemmas. We begin with an elementary fact for the upper and lower limits.

Lemma 1. *Let $g : (0, \infty) \to (1, \infty)$ be a function. In addition, let $\{u_n\}_{n=1}^{\infty}$ be a sequence with $u_n > 1$ $(n \in \mathbb{N})$. Assume that*

$$\varlimsup_{x\to\infty} f(x) = \sigma < \infty \quad \text{and} \quad \varliminf_{x\to\infty} f(x) = \mu < \infty.$$

Then, there exist strictly increasing sequences $\{x_n\}$ and $\{y_n\}$ in $(1, \infty)$ such that $u_n x_n < x_{n+1}$ and $u_n y_n < y_{n+1}$ for each $n \in \mathbb{N}$, and $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$, and

$$\lim_{x_n\to\infty} f(x_n) = \sigma \quad \text{and} \quad \lim_{y_n\to\infty} f(y_n) = \mu.$$

Proof. We prove only the upper limit case. Let $\varepsilon > 0$ be given. Then, there exists $x(\varepsilon) \in (1, \infty)$ such that $f(x) < \sigma + \varepsilon$ for all $x \in (1, \infty)$ with $x > x(\varepsilon)$, and $\sigma - \varepsilon < f(x)$ for infinitely many $x \in (1, \infty)$ with $x > x(\varepsilon)$. In particular, let $\varepsilon = \frac{1}{n}$ $(n \in \mathbb{N})$. Start to choose $x_1 \in (1, \infty)$. Choose $x_2 \in (1, \infty)$ such that $x_2 > u_1 x_1$, $x_2 > 2$, and $\sigma - \frac{1}{2} < f(x_2) < \sigma + \frac{1}{2}$. Continuing in the way, we have chosen $x_n \in (1, \infty)$ for some $n \in \mathbb{N}$. Then, we can choose $x_{n+1} \in (1, \infty)$ such that $x_{n+1} > u_n x_n$, $x_{n+1} > n + 1$, and $\sigma - \frac{1}{n+1} < f(x_{n+1}) < \sigma + \frac{1}{n+1}$. By induction on n, we can choose $x_n \in (1, \infty)$ which satisfies the above statement for the upper limit.

For the lower limit case, we consider the following fact: Let $\varepsilon > 0$ be given. Then, there exists $x(\varepsilon) \in (1, \infty)$ such that $\mu - \varepsilon < f(x)$ for all $x \in (1, \infty)$ with $x > x(\varepsilon)$, and $f(x) < \mu + \varepsilon$ for infinitely many $x \in (1, \infty)$ with $x > x(\varepsilon)$. $\square$

Lemma 2 (Reference [8], Theorem 8.2)**.** *Let f be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$ be a given real constant. Then, there exist a subset $E_1 \subset (1, \infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that, for all $|z| = r \notin E_1 \cup [0, 1]$,*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right),$$

where $n(t) = n(t, \infty, f) + n\left(t, \infty, \frac{1}{f}\right)$ denotes the sum of zeros and poles, respectively, of f, counting multiplicities, which lie in the disk $|z| \le t$.

Lemma 3 (Reference [22], Lemma 7)**.** *Let f be a transcendental meromorphic function. In addition, let $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $a \in \overline{\mathbb{C}}$, and $\alpha > 1$ be a real constant. Then, there exists a constant $R > 0$ such that, for all $r \ge R$,*

$$n\left(r, a, f^{(j)}\right) \le \frac{2j+6}{\log \alpha} T(\alpha r, f).$$

Lemma 4 (Reference [8], Theorem 2.4)**.** *Let α, R, R' be real numbers such that $0<\alpha<1$, and $0<R<R'$. In addition, let η be a non-zero complex number. Then, there is a positive constant C_α depending only on α such that, for a given meromorphic function $f(z)$, when $|z|=r$ and $\max\{1, r+|\eta|\}<R<R'$, we have the estimate*

$$\begin{aligned} & m\left(r, \frac{f(z+\eta)}{f(z)}\right)+m\left(r, \frac{f(z)}{f(z+\eta)}\right) \\ & \leq \frac{2|\eta| R}{(R-r-|\eta|)^2}\left(m(R, f)+m\left(R, \frac{1}{f}\right)\right) \\ & \quad+\frac{2R'}{R'-R}\left(\frac{|\eta|}{R-r-|\eta|}+\frac{C_\alpha|\eta|^\alpha}{(1-\alpha)\, r^\alpha}\right)\left(N(R', f)+N\left(R', \frac{1}{f}\right)\right). \end{aligned}$$

Lemma 5 (Reference [18], Lemma 3.2)**.** *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$. In addition, let f be a finite φ-order meromorphic function whose order is $\sigma=\sigma(f,\varphi)$. Then, for each $\varepsilon>0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=O\left((\varphi(r))^{\sigma-1+\varepsilon}\right).$$

Lemma 6 (Reference [17], p = q = 1, Lemma 2.4)**.** *Let f be a meromorphic function satisfying $\mu(f,\varphi)=\mu<\infty$. Then, there exists a set $E_2 \subset (1,\infty)$ of infinite logarithmic measure such that, for all $r \in E_2$, we have*

$$\mu=\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log T(r,f)}{\log \varphi(r)},$$

and for any given $\varepsilon>0$ and sufficiently large $|z|=r\in E_2$,

$$T(r,f)<(\varphi(r))^{\mu+\varepsilon}.$$

Lemma 7. *Let f be a transcendental meromorphic function which has finite φ-order σ. In addition, let η be a non-zero complex number. Then, there exists a subset $E_3 \subset (1,\infty)$ of finite logarithmic measure such that, for any given $\varepsilon>0$ and all $|z|=r\notin E_3\cup[0,1]$,*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}<\left|\frac{f(z+\eta)}{f(z)}\right|<\exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

Proof. We begin by recalling that

(a) the Nevanlinna characteristic $T(r,f)$ is non-decreasing on $r>0$,
(b) and, furthermore, if f is a transcendental meromorphic function, then

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

By Lemma 2, for $\gamma>1$ any given real constant, there exist a subset $E_3\subset(1,\infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that, for all $|z|=r\notin E_3\cup[0,1]$,

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r,f)}{r}+\frac{n(\gamma r)}{r}\log^\gamma r\log^+ n(\gamma r)\right), \tag{16}$$

where $n(t)=n(t,\infty,f)+n\left(t,\infty,\frac{1}{f}\right)$. For $\alpha>1$ any constant, applying Lemma 3 to the right member of the inequality in (16), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r,f)}{r}+\frac{12}{\log\alpha}\frac{T(\alpha\gamma r,f)}{r}\log^\gamma r\log^+\left(\frac{12}{\log\alpha}T(\alpha\gamma r,f)\right)\right) \tag{17}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. We may choose $\alpha = e^{12}$ in (17) to get

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le \frac{A}{r}\left(T(\gamma r, f) + T\left(e^{12}\gamma r, f\right) \log^{\gamma} r \log^{+}\left(T\left(e^{12}\gamma r, f\right)\right)\right) \tag{18}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. Since $T(r, f)$ is non-decreasing on $r > 0$ and $\log r$ is positive and increasing for $x > 1$, it follows from (18) that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le \frac{B}{e^{12}\gamma\, r} T\left(e^{12}\gamma r, f\right)\left(1 + \log^{\gamma}\left(e^{12}\gamma r\right) \log^{+}\left(T\left(e^{12}\gamma r, f\right)\right)\right) \tag{19}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r and $B := e^{12}\gamma\, A$. Setting $e^{12}\gamma r = r'$ and taking r so large that $\frac{|z|}{e^{12}\gamma} = \frac{r}{e^{12}\gamma} \notin E_3 \cup [0,1]$ in (19) and dropping the prime on r, we find that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le \frac{B}{r} T(r, f)\left(1 + \log^{\gamma} r \, \log^{+}(T(r, f))\right) \tag{20}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. Since $T(r, f)$ is non-decreasing on $r > 0$, $T(r, f) > 1$ for sufficiently large r. In view of (a) and (b), we find from (20) that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le \frac{T(r, f)}{r} \log^{\gamma+1} r \, \log T(r, f) \tag{21}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r.

Let $\varepsilon > 0$ be given. From Definition 1, we obtain

$$T(r, f) < (\varphi(r))^{\sigma + \frac{\varepsilon}{2}} \tag{22}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r.

For $\eta > 0$ small enough that $\eta(\gamma + 1) < \frac{\varepsilon}{4}$, we find from (i) of Remark 2 that

$$\log r < (\varphi(r))^{\eta} \tag{23}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r.

Let $\mu > 0$ be given. From Definition 1, we have

$$\log T(r, f) < \log(\varphi(r))^{\sigma+\mu} \tag{24}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. Let $\nu > 0$ be so small that $(\sigma + \mu)\nu < \frac{\varepsilon}{4}$. Since $\varphi(r) \uparrow \infty$ as $r \to \infty$,

$$\log \varphi(r) < (\varphi(r))^{\nu} \tag{25}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. Therefore, from (24) and (25) we get

$$\log T(r, f) < (\varphi(r))^{(\sigma+\mu)\nu} \le (\varphi(r))^{\frac{\varepsilon}{4}} \tag{26}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r.

Finally, employing (22), (23), and (26) in the right member of the inequality (21), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| < \frac{(\varphi(r))^{\sigma+\varepsilon}}{r} \tag{27}$$

for $|z| = r \notin E_3 \cup [0,1]$ and sufficiently large r. Hence, if necessary, we can make the subset $E_3 \subset (1, \infty)$ larger by including the large r's which may not satisfy the inequalities in the process of proof. With this enlarged set $E_3 \subset (1, \infty)$, the inequality (27) is equivalent to that in this lemma. This completes the proof. □

Lemma 8. *Let f be a transcendental meromorphic function of finite φ-order $\sigma := \sigma(f,\varphi)$. Then, for any pair of distinct complex numbers η_1, η_2, and any given $\varepsilon > 0$, there exists a subset $E_4 \subset (1,\infty)$ of finite logarithmic measure such that, for all $|z| - |\eta_2| \notin [0,1] \cup E_4$, we have*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} < \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| < \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}, \tag{28}$$

and

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\Big((\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+\varepsilon}\Big) \tag{29}$$

for sufficiently large $|z| = r \notin [0,1] \cup E_4 \cup \{|\eta_2|\}$.

Proof. Let $\eta := \eta_1 - \eta_2 \neq 0$ and $r' := |z+\eta_2| \geq |z| - |\eta_2|$. We observe $r' \notin [0,1] \cup E_4$. Now, applying Lemma 7 to

$$\frac{f(z+\eta_1)}{f(z+\eta_2)} = \frac{f(z+\eta_2+\eta)}{f(z+\eta_2)}$$

gives

$$\exp\left\{-\frac{(\varphi(r'))^{\sigma+\varepsilon}}{r'}\right\} < \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| < \exp\left\{\frac{(\varphi(r'))^{\sigma+\varepsilon}}{r'}\right\}, \tag{30}$$

for any given $\varepsilon > 0$ and all $r' \notin [0,1] \cup E_4$ with some $E_4 \subset (1,\infty)$ of a positive finite logarithmic measure. Then, dropping the prime on r from the chain of the inequalities in (30) proves (28).

$$(\varphi(r))^{\mathbf{b}-\varepsilon} < r \tag{31}$$

for sufficiently large $|z| = r$. Using (31) in the second inequality of (28), we obtain that, for sufficiently such large $|z| = r$,

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| < \exp\Big\{(\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+2\varepsilon}\Big\}, \tag{32}$$

which yields (29). Since $\varepsilon > 0$ is arbitrary, 2ε in (32) can be replaced by ε. □

By using Lemma 6, as well as Lemmas 2 and 3, we may give an analogue of Lemma 7, and hence Lemma 8, for finite φ-lower order, which is stated in the following lemma without proof.

Lemma 9. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let f be a transcendental meromorphic function of finite φ-lower order μ. Then, there exists a subset $E_5 \subset (1,\infty)$ of infinite logarithmic measure such that, for any given $\varepsilon > 0$,*

$$\exp\left\{-\frac{(\varphi(r))^{\mu+\varepsilon}}{r}\right\} < \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| < \exp\left\{\frac{(\varphi(r))^{\mu+\varepsilon}}{r}\right\}.$$

for sufficiently large $|z| = r \in E_5$.

Lemma 10 (Reference [17], p = q = 1, Lemma 2.4)**.** *Let f be a meromorphic function with $\sigma(f,\varphi) = \sigma < \infty$. Then, there exists a set $E_6 \subset (1,\infty)$ of infinite logarithmic measure such that*

$$\lim_{\substack{r\to\infty \\ r\in E_6}} \frac{\log T(r,f)}{\log \varphi(r)} = \sigma$$

and for any given $\varepsilon > 0$ and sufficiently large $|z| = r \in E_6$,

$$T(r,f) > (\varphi(r))^{\sigma-\varepsilon}.$$

Proof. We first note that only the proof of the assertions in Lemma 6 was given in Reference [17], p = q = 1, Lemma 2.4, as that of this lemma remains to be showed in the same way. It seems meaningful for the authors and the interested reader to copy and modify the proof in Reference [17], p = q = 1, Lemma 2.4, in a little more detailed manner.

Indeed, employing Lemma 1 in Definition 1, there exists a sequence $\{r_n\}$ in $(1, \infty)$ such that $r_n \to \infty$ as $n \to \infty$, $\left(1+\frac{1}{n}\right)r_n < r_{n+1}$ $(n \in \mathbb{N})$, and

$$\lim_{r_n \to \infty} \frac{\log T(r_n, f)}{\log \varphi(r_n)} = \sigma.$$

For given ε with $0 < \varepsilon < 1$, there exists $n_1 \in \mathbb{N}$ such that

$$\sigma - \varepsilon < \frac{\log T(r_n, f)}{\log \varphi(r_n)} < \sigma + \varepsilon \tag{33}$$

for all $n \geq n_1$. Then, for all $n \geq n_1$ and any $r \in \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, since $T(r, f)$ and $\varphi(r)$ are non-decreasing on $(0, \infty)$ and $(1, \infty)$, respectively, we find

$$\begin{aligned} \frac{\log T(r_n, f)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} &= \frac{\log T(r_n, f)}{\log \varphi(r_n)} \frac{\log \varphi(r_n)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} \\ &\leq \frac{\log T(r, f)}{\log \varphi(r)} \leq \frac{\log T\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} \frac{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)}{\log \varphi(r_n)}. \end{aligned} \tag{34}$$

It follows from (ii), Remark 2, that

$$\lim_{n \to \infty} \frac{\log \varphi(r_n)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} = 1 = \lim_{n \to \infty} \frac{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)}{\log \varphi(r_n)}.$$

For any such given ε with $0 < \varepsilon < 1$, there exists $n_2 \in \mathbb{N}$ such that, for all $n \geq n_2$,

$$1 - \varepsilon < \frac{\log \varphi(r_n)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} \quad \text{and} \quad \frac{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)}{\log \varphi(r_n)} < 1 + \varepsilon. \tag{35}$$

Now, let

$$n_0 := \max\{n_1, n_2\} \quad \text{and} \quad E_6 := \bigcup_{n=n_0}^{\infty} \left[r_n, \left(1+\frac{1}{n}\right)r_n\right].$$

Then, combining the inequalities (33)–(35) gives that for all $n \geq n_0$ and $r \in E_6$,

$$(\sigma - \varepsilon)(1 - \varepsilon) < \frac{\log T(r, f)}{\log \varphi(r)} < (\sigma + \varepsilon)(1 + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{\substack{r \to \infty \\ r \in E_6}} \frac{\log T(r, f)}{\log \varphi(r)} = \sigma.$$

Obviously, sets $\left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$ $(n \in \mathbb{N})$ are mutually disjoint. Therefore, we have

$$m_\ell(E_6) = \sum_{n=n_0}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_0}^{\infty} \log\left(1+1/n\right). \tag{36}$$

Let $a_n := \log\left(1+1/n\right)$ $(n \in \mathbb{N})$. Clearly $a_n > 0$ $(n \in \mathbb{N})$. We find that $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$ and, therefore, the root test cannot be employed whether the series $\sum_{n=n_0}^{\infty} a_n$ is convergent or not. Let $f(x) = \log\left(1+1/x\right)$ $(x \geq 1)$. Then, $f(x) \downarrow 0$ as $x \uparrow \infty$ and $\int_{n_0}^{\infty} f(x)\,dx = \infty$. By the integral test, the last series in (36) diverges to ∞. Hence, $m_\ell(E_6) = \infty$. This completes the proof. □

Lemma 11 (Reference [17], p = q = 1, Lemma 2.5)**.** *Let f_1 and f_2 be meromorphic functions satisfying $\sigma(f_1, \varphi) > \sigma(f_2, \varphi)$. Then, there exists a set $E_7 \subset (1, \infty)$ of infinite logarithmic measure such that, for all $r \in E_7$, we have*

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Lemma 12. *Let f be an entire function with $0 < \mu(f, \varphi) = \mu < \infty$. Then, there exists a set $E_8 \subset (1, \infty)$ of infinite logarithmic measure such that, for all $r \in E_8$, we have*

$$\underline{\tau} = \underline{\tau}(f, \varphi) = \lim_{\substack{r\to\infty \\ r\in E_8}} \frac{\log M(r, f)}{\varphi(r)^{\mu}}.$$

Proof. The proof would run parallel to that of Lemma 10. As in Lemma 1, in view of Definition 2, there exists a strictly increasing sequence $\{r_n\}$ in $(1, \infty)$ such that $r_n \to \infty$ as $n \to \infty$, $\left(1+\frac{1}{n}\right)r_n < r_{n+1}$ $(n \in \mathbb{N})$, and

$$\underline{\tau} = \lim_{r_n\to\infty} \frac{\log M(r_n, f)}{\varphi(r_n)^{\mu}}.$$

We omit the remaining details. □

4. Proof of Main Results

Proof of Theorem 5. The proof here would proceeded in line with that of (Reference [8], Theorem 9.2) which is modified in a little detailed manner (see, in particular, (39) and (41)).

Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). If $\sigma(f, \varphi) = \infty$, then the result is obvious. So we assume that $\sigma(f, \varphi) = \sigma < \infty$. Suppose to the contrary that

$$\sigma(f, \varphi) < \sigma(A_\ell, \varphi) + \mathbf{b}. \tag{37}$$

From (6), a positive real number η can be chosen such that

$$\max_{\substack{0\leq j\leq n \\ j\neq \ell}} \sigma(A_j, \varphi) < \eta < \sigma(A_\ell, \varphi). \tag{38}$$

From (37) and (38), we may choose $\varepsilon > 0$ so small that

$$\sigma(f, \varphi) + 2\varepsilon < \sigma(A_\ell, \varphi) + \mathbf{b} \quad \text{and} \quad \eta + 2\varepsilon < \sigma(A_\ell, \varphi). \tag{39}$$

From (2), we find

$$m(r, A_\ell) \le \sum_{\substack{0\le j\le n\\ j\ne \ell}} m(r, A_j) + \sum_{\substack{0\le j\le n\\ j\ne \ell}} m\left(r, \frac{f(z+j)}{f(z+\ell)}\right) + n\log 2 + \log n. \tag{40}$$

For such an $\varepsilon > 0$ in (39), using in (29) in Lemma 8, and (39), we find from (40) that, for sufficiently large $|z| = r$,

$$\begin{aligned} m(r, A_\ell) &\le O\Big((\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+\varepsilon}\Big) + O\big((\varphi(r))^{\eta+\varepsilon}\big) + O(1) \\ &\le O\Big((\varphi(r))^{\sigma(A_\ell,\varphi)-\varepsilon}\Big) + O(1). \end{aligned} \tag{41}$$

Finally, taking logarithm on both sides of the inequality composed by the first and last terms in (41), and dividing each side of the resulting inequality by $\log \varphi(r)$, and taking the upper limit as $r \to \infty$ on both sides of the last resultant inequality, we obtain $\sigma(A_\ell, \varphi) \le \sigma(A_\ell, \varphi) - \varepsilon$, which is a contradiction. Hence, we have $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$. □

Proof of Theorem 6. Here, the proof would run parallel to that of (Reference [14], Theorem 1.1) which is modified in a little detailed manner (see Theorem 2) (see, in particular, (47) and (50)).

Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). Suppose to the contrary that

$$\mu(f, \varphi) < \mu(A_\ell, \varphi) + \mathbf{b} < \infty. \tag{42}$$

Let

$$\mathsf{N}_1 := \{j \,|\, \sigma(A_j, \varphi) < \mu(A_\ell, \varphi),\, j = 0, 1, \ldots, n,\, j \ne \ell\},$$

and

$$\mathsf{N}_2 := \{j \,|\, \sigma(A_j, \varphi) = \mu(A_\ell, \varphi),\, j = 0, 1, \ldots, n,\, j \ne \ell\}.$$

From (7), we see that $\mathsf{N}_1 \cup \mathsf{N}_2 = \{0, 1, \ldots, n\} \setminus \{\ell\}$ and, clearly, $\mathsf{N}_1 \cap \mathsf{N}_2 = \emptyset$. In addition, let

$$\sigma := \max_{j\in \mathsf{N}_1} \sigma(A_j, \varphi) < \infty \qquad \text{and} \qquad \tau := \max_{j\in \mathsf{N}_2} \tau(A_j, \varphi).$$

Then, obviously and from (8), we have

$$\sigma < (A_\ell, \varphi) \qquad \text{and} \qquad \tau < \underline{\tau}(A_\ell, \varphi). \tag{43}$$

From Definitions 1 and 2, for any given $\varepsilon > 0$ and sufficiently large $|z| = r$, we obtain that

$$|A_j(z)| < \exp\big((\varphi(r))^{\sigma+\varepsilon}\big) \quad (j \in \mathsf{N}_1), \tag{44}$$

and

$$|A_j(z)| < \exp\Big\{(\tau+\varepsilon)(\varphi(r))^{\mu(A_\ell,\varphi)}\Big\} \quad (j \in \mathsf{N}_2). \tag{45}$$

Moreover, by Lemma 9, for any given $\varepsilon > 0$, there exists a subset $E_5 \subset (1, \infty)$ of infinite logarithmic measure such that, for all $|z| = r \in E_5$, we have

$$\left|\frac{f(z+j)}{f(z+\ell)}\right| < \exp\left\{\frac{(\varphi(r))^{\mu(f,\varphi)+\varepsilon}}{r}\right\} \quad (j \in \mathsf{N}_1 \cup \mathsf{N}_2). \tag{46}$$

From (42) and (43), we can choose $\varepsilon(> 0)$ so small that

$$\max\{\sigma, \mu(f, \varphi) - \mathbf{b}\} + 2\varepsilon < \mu(A_\ell, \varphi), \quad \text{and} \quad \tau + 2\varepsilon < \underline{\tau}(A_\ell, \varphi). \tag{47}$$

From Equation (1), we get

$$-A_\ell(z) = \sum_{j\in \mathsf{N}_1\cup\mathsf{N}_2} A_j(z)\,\frac{f(z+j)}{f(z+\ell)};$$

hence,

$$|A_\ell(z)| \le \sum_{j\in \mathsf{N}_1\cup\mathsf{N}_2} |A_j(z)|\left|\frac{f(z+j)}{f(z+\ell)}\right|. \tag{48}$$

Using the inequalities (44)–(46) on the right-hand side of (48), we get that, for any given $\varepsilon > 0$ and sufficiently large $|z| = r \in E_5$,

$$|A_\ell| < \exp\left\{\frac{(\varphi(r))^{\mu(f,\varphi)+\varepsilon}}{r}\right\}\Big(\sum_{j\in\mathsf{N}_1}\exp\{(\varphi(r))^{\sigma+\varepsilon}\} + \sum_{j\in\mathsf{N}_2}\exp\Big\{(\tau+\varepsilon)(\varphi(r))^{\mu(A_\ell,\varphi)}\Big\}\Big).$$

We, thus, find that, for any given $\varepsilon > 0$ and sufficiently large $|z| = r \in E_5$,

$$\begin{aligned} M(r, A_\ell(z)) \le & n \exp\left\{\frac{(\varphi(r))^{\mu(f,\varphi)+\varepsilon}}{r}\right\} \\ & \times \Big(\exp\{(\varphi(r))^{\sigma+\varepsilon}\} + \exp\Big\{(\tau+\varepsilon)(\varphi(r))^{\mu(A_\ell,\varphi)}\Big\}\Big). \end{aligned} \tag{49}$$

Taking logarithm on both sides of the inequality (48) and using (5), we obtain that for any sufficiently small $\varepsilon > 0$ and sufficiently large $|z| = r \in E_5$,

$$\begin{aligned} \log M(r, A_\ell(z)) \le & \log n + (\varphi(r))^{\mu(f,\varphi)-\mathbf{b}+\frac{3}{2}\varepsilon} \\ & + \log\Big(\exp\{(\varphi(r))^{\sigma+\varepsilon}\} + \exp\Big\{(\tau+\varepsilon)(\varphi(r))^{\mu(A_\ell,\varphi)}\Big\}\Big). \end{aligned} \tag{50}$$

Recalling the following inequality

$$\begin{gathered} \log\left(\sum_{k=1}^{n} x_k\right) \le \sum_{k=1}^{n} \log\, x_k + \log m \\ (m \in \mathbb{N},\ x_k \ge 1,\ k = 1, \dots, m) \end{gathered} \tag{51}$$

to use in (50), we get that, for any sufficiently small $\varepsilon > 0$ and sufficiently large $|z| = r \in E_5$,

$$\begin{aligned} \log M(r, A_\ell) \le & \log n + (\varphi(r))^{\mu(f,\varphi)-\mathbf{b}+\frac{3}{2}\varepsilon} \\ & + (\varphi(r))^{\sigma+\varepsilon} + (\tau+\varepsilon)(\varphi(r))^{\mu(A_\ell,\varphi)} + \log 2. \end{aligned} \tag{52}$$

From Definition (2), using (47) and (52), we obtain

$$\underline{\tau}(A_\ell,\varphi) = \varliminf_{r\to\infty}\frac{\log M(r,A_\ell)}{\varphi(r)^{\mu(A_\ell,\varphi)}} \le \varliminf_{\substack{r\to\infty\\ r\in E_5}}\frac{\log M(r,A_\ell)}{\varphi(r)^{\mu(A_\ell,\varphi)}} = \tau+\varepsilon < \underline{\tau}(A_\ell,\varphi)-\varepsilon,$$

which leads to a contradiction. Hence, we conclude $\mu(f,\varphi) \ge \mu(A_\ell,\varphi) + \mathbf{b}$. □

Proof of Theorem 7. The proof would be proceeded by modifying that of (Reference [20], Theorem 1.3) in a little detailed manner (also see Reference [14], Theorem 1.3) (see, in particular, (56) and (57)).

Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). Under the given assumptions, by Remark 3, we have $\sigma(A_\ell,\varphi) = \sigma$. Suppose to the contrary that

$$\sigma(f,\varphi) < \sigma(A_\ell,\varphi) + \mathbf{b} = \sigma + \mathbf{b} < \infty. \tag{53}$$

Then, for sufficiently small $\delta > 0$, we still get $\sigma + \mathbf{b} - \sigma(f,\varphi) - 2\delta > 0$. Take any given $\varepsilon > 0$ so small that

$$0 < \varepsilon < \sigma + \mathbf{b} - \sigma(f,\varphi) - 2\delta. \tag{54}$$

By Lemma 8, there exists a set $E \subset (1,\infty)$ of finite logarithmic measure such that for all $|z| = r \notin E \cup [0,1] \cup \{\ell\}$, we obtain

$$\left|\frac{f(z+j)}{f(z+\ell)}\right| < \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\} \quad (j = 0, 1, \ldots, n,\ j \neq \ell). \tag{55}$$

Using (54) in (55) gives

$$\left|\frac{f(z+j)}{f(z+\ell)}\right| < \exp\left\{\frac{(\varphi(r))^{\sigma+\mathbf{b}-2\delta}}{r}\right\} \quad (j = 0, 1, \ldots, n,\ j \neq \ell), \tag{56}$$

for sufficiently large $|z| = r \notin E \cup [0,1] \cup \{\ell\}$. For the $\frac{\delta}{2} > 0$, from (5), we find

$$((\varphi(r))^{\mathbf{b}-\frac{\delta}{2}} < r \tag{57}$$

for sufficiently large r. Employing (57) in the inequality (56) provides

$$\left|\frac{f(z+j)}{f(z+\ell)}\right| < \exp\left\{(\varphi(r))^{\sigma-\frac{3}{2}\delta}\right\} \quad (j = 0, 1, \ldots, n,\ j \neq \ell), \tag{58}$$

for sufficiently large $|z| = r \notin E \cup [0,1] \cup \{\ell\}$.

Here, let $H := (0,\infty) \setminus E \cup [0,1] \cup \{\ell\}$. By Remark 1, we find that $\overline{\log \mathrm{dens}}\,\{E \cup [0,1] \cup \{\ell\}\} = 0$ and so $\overline{\log \mathrm{dens}}\, H > 0$, which implies $m(H) = \infty$. Using the inequalities (9), (10), and (58) in (2), we obtain

$$\exp\left\{\beta(\varphi(r))^{\sigma-\delta}\right\} < n\left\{\alpha(\varphi(r))^{\sigma-\delta}\right\} \exp\left\{(\varphi(r))^{\sigma-\frac{3}{2}\delta}\right\},$$

or, equivalently,

$$1 < n\,\exp\left\{(\alpha-\beta)(\varphi(r))^{\sigma-\delta}\right\} \exp\left\{(\varphi(r))^{\sigma-\frac{3}{2}\delta}\right\}, \tag{59}$$

for sufficiently large $|z| = r \in H$. Since $\varphi(r) \uparrow \infty$ as $r \to \infty$, $(\varphi(r))^{-\frac{\delta}{2}} < \nu < \eta := \beta - \alpha$ for sufficiently large $|z| = r \in H$ and some $\nu > 0$. Therefore, we find that, as $|z| = r \in H$ increases to ∞, the right-hand side of the inequality (59) becomes smaller than $n\exp\{(\nu-\eta)(\varphi(r))^{\sigma-\delta}\}$, which, due to $\nu - \eta < 0$, may approach to 0. In view of (59), this leads to contradiction. Hence, $\sigma(f,\varphi) \geq \sigma(A_\ell,\varphi) + \mathbf{b}$. □

Proof of Theorem 8. Here, the proof would run parallel with that of (Reference [14], Theorem 1.4) which is modified in a little detailed manner (see, in particular, (62) and (62)).

Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). If $\sigma(f,\varphi) = \infty$, then the result is obvious. So we assume that $\sigma(f,\varphi) = \sigma < \infty$. From (40), we find

$$m(r, A_\ell) \leq \sum_{\substack{0\leq j\leq n \\ j\neq \ell}} m(r, A_j) + \sum_{\substack{0\leq j\leq n \\ j\neq \ell}} m\left(r, \frac{f(z+j)}{f(z+\ell)}\right) + O(1) \tag{60}$$

for $|z| = r$ $(z \in H)$. From (13), consider any $\eta \in (0,1)$ such that

$$\overline{\lim_{r\to\infty}} \sum_{\substack{0\leq j\leq n \\ j\neq \ell}} \frac{m(r, A_j)}{m(r, A_\ell)} < \eta < 1$$

for $|z| = r$ $(z \in H)$. Then, for sufficiently large $|z| = r$ $(z \in H)$, we have

$$\sum_{\substack{0 \le j \le n \\ j \ne \ell}} \frac{m(r, A_j)}{m(r, A_\ell)} < \eta \quad \text{if and only if} \quad \sum_{\substack{0 \le j \le n \\ j \ne \ell}} m(r, A_j) < \eta\, m(r, A_\ell). \tag{61}$$

In view of (29) in Lemma 8,

$$m\left(r, \left|\frac{f(z+j)}{f(z+\ell)}\right|\right) = O\Big((\varphi(r))^{\sigma(f,\varphi) - \mathbf{b} + \varepsilon}\Big) \quad (j = 0, 1, \ldots, n,\ j \ne \ell) \tag{62}$$

for sufficiently large $|z| = r$ $(z \in H)$. Employing (61) and (62) in (60), for given $\varepsilon > 0$, we get that

$$(1 - \eta)\, m(r, A_\ell) \le O\Big((\varphi(r))^{\sigma(f,\varphi) - \mathbf{b} + \varepsilon}\Big) + O(1) \tag{63}$$

for sufficiently large $|z| = r$ $(z \in H)$. From Definition 1, we have

$$\sigma(A_\ell, \varphi) \le \sigma(f, \varphi) - \mathbf{b} + \varepsilon,$$

which, upon $\varepsilon > 0$ being arbitrary, leads to $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$. $\square$

Proof of Theorem 9. The process of the proof would be flowed as in that of Theorem II in Reference [21], which is modified in a little detailed manner (see, in particular, (66) and (69)).

Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). If $\sigma(f, \varphi) = \infty$, then the result is clear. So we suppose that $\sigma(f, \varphi) = \sigma < \infty$. From Definition 3,

$$\delta(\infty, A_\ell) = \varliminf_{r \to \infty} \frac{m(r, A_\ell)}{T(r, A_\ell)} = \delta > 0,$$

which gives that, for sufficiently large r,

$$\frac{\delta}{2} T(r, A_\ell) < m(r, A_\ell). \tag{64}$$

Combining (64) and (40), we obtain

$$\begin{aligned} &\frac{\delta}{2} T(r, A_\ell) < m(r, A_\ell) \\ &\quad \le \sum_{\substack{0 \le j \le n \\ j \ne \ell}} m(r, A_j) + \sum_{\substack{0 \le j \le n \\ j \ne \ell}} m\left(r, \frac{f(z+j)}{f(z+\ell)}\right) + O(1) \end{aligned} \tag{65}$$

for sufficiently large r. By using (29) in Lemma 8 and the relation between $T(r, f)$ and $m(r, f)$ in (65), we get

$$\begin{aligned} &\frac{\delta}{2} T(r, A_\ell) < m(r, A_\ell) \\ &\quad \le \sum_{\substack{0 \le j \le n \\ j \ne \ell}} T(r, A_j) + O\Big((\varphi(r))^{\sigma(f,\varphi) - \mathbf{b} + \varepsilon}\Big) + O(1) \end{aligned} \tag{66}$$

for sufficiently large r. In view of (14), by Lemma 11, there exists a set $E_7 \subset (1, \infty)$ of infinite logarithmic measure such that

$$\sum_{\substack{0 \le j \le n \\ j \ne \ell}} \frac{T(r, A_j)}{T(r, A_\ell)} \to 0 \tag{67}$$

as $r(\in E_7) \to \infty$. Considering the given $\frac{\delta}{4} > 0$ in (67), we have that, for sufficiently large $r \in E_7$,

$$\sum_{\substack{0 \le j \le n \\ j \ne \ell}} \frac{T(r, A_j)}{T(r, A_\ell)} < \frac{\delta}{4}. \tag{68}$$

Applying (68) to (66), we get that, for sufficiently large $r \in E_7$,

$$\frac{\delta}{4} T(r, A_\ell) < O\Big((\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+\varepsilon}\Big) + O(1). \tag{69}$$

Taking logarithm on both sides of the inequality (69), and dividing the resulting inequality by $\log \varphi(r)$, and taking the upper limit as $r(\in E_7) \to \infty$ on both sides of the last resultant inequality, we finally obtain

$$\sigma(A_\ell, \varphi) \le \sigma(f, \varphi) - \mathbf{b} + \varepsilon,$$

which, upon $\varepsilon > 0$ being arbitrary, leads to the desired inequality $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$. □

Proof of Theorem 10. Let $f(\not\equiv 0)$ be a transcendental meromorphic solution of the Equation (1). If $\sigma(f, \varphi) = \infty$, then the result is trivial. So we consider that $\sigma(f, \varphi) = \sigma < \infty$. As in the process of the proof of Theorem 8, we find from (63) that, for given $\varepsilon > 0$,

$$(1-\eta)\, m(r, A_\ell) \le O\Big((\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+\varepsilon}\Big) + O(1) \tag{70}$$

for some η $(0 < \eta < 1)$ and sufficiently large $|z| = r$ $(z \in H)$. Since $\delta(\infty, A_\ell) = \delta > 0$, we can use the same inequality (64) in (70) to obtain

$$(1-\eta)\, \frac{\delta}{2}\, T(r, A_\ell) \le O\Big((\varphi(r))^{\sigma(f,\varphi)-\mathbf{b}+\varepsilon}\Big) + O(1) \tag{71}$$

for sufficiently large r. Finally, employing the same process in the last paragraph of the proof of Theorem 9, we may have the desired inequality $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + \mathbf{b}$. □

5. Concluding Remarks

In this paper, in order to answer the following natural question: When the coefficients of the Equation (1) are entire or meromorphic functions of finite φ-order, what would the growth properties of solutions of the linear difference Equation (1) be like?, we introduced the constant $\mathbf{b}$ in (5), depending on φ. Then we showed how nicely diverse known results for the meromorphic solution f of finite φ-order of the difference Equation (1) can be amended.

When $\varphi(x) = x$ $(x \in [0, \infty))$ is chosen in (5), we have $\mathbf{b} = 1$. Accordingly, all conclusions of Theorems 5–10 become $\sigma(f, \varphi) \ge \sigma(A_\ell, \varphi) + 1$. In this case, transcendental meromorphic solution may be replaced by meromorphic solution. Therefore, Theorems 5–9 are found to reduce to some known corresponding results. For example,

- Theorem 5 may yield (Reference [8], Theorem 9.2) (Theorem 1) (also see (Reference [11], Theorem 5.2), (Reference [6], Theorem 1));
- Theorem 6 may give (Reference [14], Theorem 1.1) (Theorem 2);
- Theorem 7 may provide (Reference [20], Theorem 1.3) (Theorem 3);
- Theorem 8 may afford (Reference [14], Theorem 1.4);
- Theorem 9 may produce (Reference [21], Theorem 11) (Theorem 4).

In addition, it may be interesting to compare Theorem 10 and (Reference [15], Theorem 10).

Next, setting $\varphi(x) = x^\mu$ $(x \in [0, \infty),\ \mu \in (0, \infty))$ in (5), we have $\mathbf{b} = \frac{1}{\mu}$. In addition, it is obvious that $\varphi(x) = x^\mu$ satisfies (i) and (ii) in Remark 2. Therefore, all conclusions of Theorems 5–10 become $\sigma(f,\varphi) \geq \sigma(A_\ell,\varphi) + \frac{1}{\mu}$. In this case, *transcendental meromorphic* solution may be replaced by *meromorphic* solution. Further, $\sigma(A_\ell,\varphi) + \frac{1}{\mu} \downarrow \sigma(A_\ell,\varphi)$ as $\mu \uparrow \infty$, while $\sigma(A_\ell,\varphi) + \frac{1}{\mu} \uparrow \infty$ as $\mu \downarrow 0$.

Posing a Problem

Considering the results presented in this paper, by using the constant **b** in (5), some known other results for this subject are supposed to be amendable as those in Theorems 5–10, which are left to the interested readers for future investigation.

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