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Complex q-Rung Orthopair Uncertain Linguistic Partitioned Bonferroni Mean Operators with Application in Antivirus Mask Selection

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Abstract: In this paper, complex q-rung orthopair uncertain linguistic sets (CQROULSs) for handling multi-attribute decision making (MADM) issues are proposed so that the assessed estimation of each trait can be presented by CQROULS. Another aggregation operator, called the partitioned Bonferroni mean (PBM) operator, is then considered to manage the circumstances under fuzziness. At that point, the PBM operator is stretched out to CQROULSs in which a complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean (CQROULPBM) operator is then proposed. To wipe out the negative impact of preposterous assessment estimations of characteristics on total outcomes, complex q-rung orthopair uncertain linguistic weighted partitioned Bonferroni mean (CQROULWPBM) operator is further considered. These properties, idempotency, boundedness, and commutativity of the CQROULWPBM operator are obtained. The proposed CQROULSs with the CQROULWPBM operator is novel and important for MADM issues. Finally, an MADM based on CQROULSs is constructed with a numerical case given to delineate the proposed approach and then applied for selecting an antivirus mask for the COVID-19 pandemic. The advantages and comparative analysis with graphical interpretation of the explored operators are also presented to demonstrate the effectiveness and usefulness of the proposed method.

Keywords: complex q-rung orthopair uncertain linguistic sets (CQROULSS); partitioned bonferroni mean (PBM) operators; complex q-rung orthopair uncertain linguistic partitioned bonferroni mean (CQROULPBM); multi-attribute decision making (MADM)

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1. Introduction

Multi-attribute decision making (MADM) issues are inescapable in the field of board science. In numerous functional applications, MADM techniques have a significant effect on the procedure used to take care of such issues. Most existing strategies say that a better way to pick better and appropriate electives depends on decision-makers' (DMs) assessment data. Notwithstanding, due to the intricate outer decision-making condition and the abstract vulnerability of DMs, it is difficult for DMs to clarify their genuine inclination data plainly. In this sense, Zadeh [1] first coined fuzzy sets (FSs) to clarify imprecision and dubiousness during assessment procedures. Until now, FSs have been examined and applied to different fields by an enormous number of scientists. Numerous speculations of FSs had then proposed for different application conditions, for example, intuitionistic FSs (IFSs) [2] with applications [3–5] and Pythagorean FSs [6] with applications [7,8]. As of late, in light of the possibility of the q-rung refutation and the IFS, Yager [9] explored another broad type of orthopair FSs (OFSs), called q-rung orthopair FSs (QROFSs). Like different OFSs, the QROFSs are additionally described by sets of esteems in the unit stretch α and β, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, separately, indicate the supporting degree and the supporting against degree. However, they require the limitation that the aggregate of

the qth intensity of them is not exactly or equivalent to one. Clearly, the requirements of the QROFSs are incredibly discharged. It tends to be effectively seen that as q expands, the scope of conceivable enrollment reviews too increases. Along these lines, the QROFSs are a general and incredible displaying apparatus that can portray DMs' actual feelings in bigger practicable enrollment reviews, and manage the cost of more prominent adaptability for DMs in communicating judgment deftly under complex choice circumstances. In the last few year, many scholars have utilized it in the environment of different fields [10–12].

From the above far-reaching studies and decision-making processes, it is concluded that their entrance is constrained to deal with just vulnerability in the information, yet at the same time neglects management of its changes at a given period of time. To broaden the scope of supporting grade reaching out from a genuine subset to the unit plate of the mind boggling plane, complex FS (CFS) was proposed by Ramot et al. [13]. Additionally, the theory of complex IFS (CIFS) was presented by Alkouri and Salleh [14] to provide a wide range to a decision-maker to make a decision. CIFS composes the grade of supporting and the grade of supporting against in the form of a complex number belonging to a unit disc in a complex plane. The limitations of CIFS is that the sum of the real part (also for the imaginary part) of the both grades cannot exceed from a unit interval. However, there a still a problem, if a decision-maker gives such kinds of values to both grades, whose sum is exceeded from a unit interval. To cope with such kind of issues, the theory of complex PFS (CPFS) was explored by Ullah et al. [15], with a condition that the sum of the squares of the real part (also for imaginary part) of the both grades cannot be exceeded from a unit interval. The theories of CIFS and CPFS again fail, when a decision-maker provides such kind of values, whose sum of the squares of the real part (also for the imaginary part) of the both grades exceeds the unit interval. To cope with such kind of issues, the theory of complex QROFS (CQROFS) was proposed by scholars, such as Liu et al. [16,17] and Garg et al. [18]. The CQROFS is extensive, proficient, and more reliable than existing notions like CIFS and CPFS to cope with uncertain and unpredictable information in realistic decision issues.

In numerous genuine issues, it is difficult for DMs to give their evaluations in quantitative articulations. For instance, when a specialist assesses an up-and-comer's degree of proficient capability, they may feel increasingly helpful or progressively acquainted with utilizing semantic terms, for example, "excellent", "great", or "medium", to communicate their judgment. The theory of linguistic variables were presented by Zadeh [19]; Xu [20] discovered uncertain linguistic sets. Liu and Jin [21] explored some aggregation operators based on intuitionistic fuzzy uncertain linguistic sets, and Lu and Wei [22] considered Pythagorean uncertain linguistic aggregation operators. Liu et al. [23] presented the q-rung orthopair uncertain linguistic aggregation operators and their application in MADM problem. To broad the scope of the supporting grade reaching out from the unit plate in the form of complex number belonging to unit disc in a complex plane, we propose complex q-rung orthopair uncertain linguistic sets (CQROULSs) in this paper. We use CQROULSs for handling MADM issues as the assessed estimation of each trait and then propose the complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean (CQROULPBM) operator. The contributions of the paper are summarized as follows:

1. To propose CQROULSs that can present the assessed estimation of each trait in MADM issues.
2. To give the CQROULPBM and CQROULWPBM operators with properties of idempotency, boundedness, and commutativity.
3. To construct a novel MADM method based on CQROULSs and the CQROULWPBM operator.
4. To apply the created MADM method in antivirus mask selection.

Furthermore, we may use the fuzzy symmetry concept [24,25] for the proposed CQROULSs and operators so that we can have more uncertain linguistic applications in symmetry.

The rest of this article is organized as follows. In Section 2, we review some basic notions, such as linguistic term set (LTS), uncertain linguistic variable (ULV), complex

q-rung orthopair fuzzy sets (CQROFSs), and their operational laws. In Section 3, the notion of complex q-rung orthopair uncertain linguistic sets (CQROULSs) and their fundamental laws are considered. In Section 4, the partitioned Bonferroni mean (PBM) operators based on CQROULSs, called complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean (CQROULPBM) operators, are proposed. We also investigate their special cases of CQROULPBM operators. Based on weighting, we further propose complex q-rung orthopair uncertain linguistic weighted partitioned Bonferroni mean (CQROULWPBM) operator. These properties of idempotency, boundedness, and commutativity for the CQROULWPBM operator are also demonstrated. In Section 5, we construct an MADM based on CQROULSs and then apply it to antivirus mask selection. The advantages, comparative analysis, and graphical interpretation of the explored operator are also presented. We finally make our conclusions in Section 6.

2. Preliminaries

In this section, we review some definitions, such as linguistic term set (LTS), complex q-rung orthopair fuzzy sets (CQROFSs), and their operational laws. Throughout this manuscript, the universal set is expressed by X_{UNI}.

Definition 1. [19] *An LTS is initiated by:*

$$\mathcal{L} = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{k-1}\} \tag{1}$$

where k should be odd and hold the following conditions:

1. If $k > k'$, then $\mathcal{L}_k > \mathcal{L}_{k'}$;
2. The negative operator $neg(\mathcal{L}_k) = \mathcal{L}_{k'}$ with a condition $k + k' = k + 1$;
3. If $k \geq k'$, $\max(\mathcal{L}_k, \mathcal{L}_{k'}) = \mathcal{L}_k$, and if $k \leq k'$, $\max(\mathcal{L}_k, \mathcal{L}_{k'}) = \mathcal{L}_k$.

Additionally, $\hat{\mathcal{L}} = \{\mathcal{L}_i : i \in R\}$ expresses the LTSs.

Definition 2. [20] *Let the set $\mathcal{L} = [\mathcal{L}_i, \mathcal{L}_j], \mathcal{L}_i, \mathcal{L}_j \in \hat{\mathcal{L}}(i \leq j)$, where $\mathcal{L}_i, \mathcal{L}_j$ are the upper and lower limits of $\mathcal{L}$, be a uncertain linguistic variable (ULV). For any two ULVs $\mathcal{L}_1 = [\mathcal{L}_{i_1}, \mathcal{L}_{j_1}]$ and $\mathcal{L}_2 = [\mathcal{L}_{i_2}, \mathcal{L}_{j_2}]$, we have*

1. $\mathcal{L}_1 \oplus \mathcal{L}_2 = [\mathcal{L}_{i_1}, \mathcal{L}_{j_1}] \oplus [\mathcal{L}_{i_2}, \mathcal{L}_{j_2}] = \left[\mathcal{L}_{i_1+i_2-\frac{i_1 i_2}{k}}, \mathcal{L}_{j_1+j_2-\frac{j_1 j_2}{k}}\right]$;
2. $\mathcal{L}_1 \otimes \mathcal{L}_2 = [\mathcal{L}_{i_1}, \mathcal{L}_{j_1}] \otimes [\mathcal{L}_{i_2}, \mathcal{L}_{j_2}] = \left[\mathcal{L}_{\frac{i_1 i_2}{k}}, \mathcal{L}_{\frac{j_1 j_2}{k}}\right]$;
3. $\delta_{SC}\mathcal{L}_1 = \delta_{SC}[\mathcal{L}_{i_1}, \mathcal{L}_{j_1}] = \left[\mathcal{L}_{k-k(1-\frac{i_1}{k})^{\delta_{SC}}}, \mathcal{L}_{k-k(1-\frac{j_1}{k})^{\delta_{SC}}}\right]$;
4. $\mathcal{L}_1^{\delta_{SC}} = \left[\mathcal{L}_{k(\frac{i_1}{k})^{\delta_{SC}}}, \mathcal{L}_{k(\frac{j_1}{k})^{\delta_{SC}}}\right]$.

Definition 3. [16] *A CQROFS is initiated by:*

$$\mathfrak{C}_{CQ} = \left\{\left(x, \left(\mathfrak{M}_{\mathfrak{C}_{CQ}}, \mathfrak{N}_{\mathfrak{C}_{CQ}}\right)\right) : x \in X_{UNI}\right\} \tag{2}$$

where $\mathfrak{M}_{\mathfrak{C}_{CQ}} = \mathfrak{M}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP}})}$ and $\mathfrak{N}_{\mathfrak{C}_{CQ}} = \mathfrak{N}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP}})}$ expresses the grade of complex supporting and complex supporting against with a conditions: $0 \leq \mathfrak{M}_{\mathfrak{C}_{RP}}^{q_{SC}} + \mathfrak{N}_{\mathfrak{C}_{RP}}^{q_{SC}} \leq 1$ and $0 \leq \mathfrak{M}_{\mathfrak{C}_{IP}}^{q_{SC}} + \mathfrak{N}_{\mathfrak{C}_{IP}}^{q_{SC}} \leq 1$. Additionally, the grade of refusal is stated by $\Re_{\mathfrak{C}_{CQ}} = \Re_{\mathfrak{C}_{RP}} e^{i2\pi(\Re_{\mathfrak{C}_{IP}})} = \left(1 - \left(\mathfrak{M}_{\mathfrak{C}_{RP}}^{q_{SC}} + \mathfrak{N}_{\mathfrak{C}_{RP}}^{q_{SC}}\right)\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(\mathfrak{M}_{\mathfrak{C}_{IP}}^{q_{SC}} + \mathfrak{N}_{\mathfrak{C}_{IP}}^{q_{SC}}))^{\frac{1}{q_{SC}}}}$. The complex q-rung orthopair fuzzy number (CQROFN) is expressed by $\mathfrak{C}_{CQ} = \left(\mathfrak{M}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP}})}, \mathfrak{N}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP}})}\right)$.

Definition 4. [17] *For any two CQROFSs* $\mathfrak{C}_{CQ-1} = \left(\mathfrak{M}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}})}, \mathfrak{N}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}})}\right)$ *and* $\mathfrak{C}_{CQ-2} = \left(\mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-2}})}\right)$, *we have*

1. $$\mathfrak{C}_{CQ-1} \oplus_{CQ} \mathfrak{C}_{CQ-2} = \left(\begin{array}{c} \left(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-2}} - \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-2}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-2}} - \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-2}})^{\frac{1}{q_{SC}}}}, \\ \mathfrak{N}_{\mathfrak{C}_{RP-1}} \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}} \mathfrak{N}_{\mathfrak{C}_{IP-2}})} \end{array} \right)$$
2. $$\mathfrak{C}_{CQ-1} \otimes_{CQ} \mathfrak{C}_{CQ-2} = \left(\begin{array}{c} \mathfrak{M}_{\mathfrak{C}_{RP-1}} \mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}} \mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \\ \left(\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-2}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-2}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-2}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}} \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-2}})^{\frac{1}{q_{SC}}}} \end{array} \right)$$
3. $$\delta_{SC}\mathfrak{C}_{CQ-1} = \left(\left(1 - \left(1 - \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}})^{\delta_{SC}})^{\frac{1}{q_{SC}}}}, \mathfrak{N}^{\delta_{SC}}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}^{\delta_{SC}}_{\mathfrak{C}_{IP-1}})} \right), \delta_{SC} > 0$$
4. $$\mathfrak{C}^{\delta_{SC}}_{CQ-1} = \left(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{IP-1}})}, \left(1 - \left(1 - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}})^{\delta_{SC}})^{\frac{1}{q_{SC}}}} \right), \delta_{SC} > 0$$

Definition 5. [18] *For any two CQROFSs* $\mathfrak{C}_{CQ-1} = \left(\mathfrak{M}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}})}, \mathfrak{N}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}})}\right)$ *and* $\mathfrak{C}_{CQ-2} = \left(\mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-2}})}\right)$, *the score and accuracy function are initiated by*

$$\mathbb{S}_{SF}(\mathfrak{C}_{CQ-1}) = \frac{1}{2}\left(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}}\right) \tag{3}$$

$$\mathbb{H}_{AF}(\mathfrak{C}_{CQ-1}) = \frac{1}{2}\left(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}}\right) \tag{4}$$

To examine the interrelationship between any two CQROFSs, the following rules are used:

1. If $\mathbb{S}_{SF}(\mathfrak{C}_{CQ-1}) > \mathbb{S}_{SF}(\mathfrak{C}_{CQ-2})$, then $\mathfrak{C}_{CQ-1} > \mathfrak{C}_{CQ-2}$;
2. If $\mathbb{S}_{SF}(\mathfrak{C}_{CQ-1}) = \mathbb{S}_{SF}(\mathfrak{C}_{CQ-2})$, then
   (1) If $\mathbb{H}_{AF}(\mathfrak{C}_{CQ-1}) > \mathbb{H}_{AF}(\mathfrak{C}_{CQ-2})$, then $\mathfrak{C}_{CQ-1} > \mathfrak{C}_{CQ-2}$;
   (2) If $\mathbb{H}_{AF}(\mathfrak{C}_{CQ-1}) = \mathbb{H}_{AF}(\mathfrak{C}_{CQ-2})$, then $\mathfrak{C}_{CQ-1} = \mathfrak{C}_{CQ-2}$.

3. Proposed Complex q-Rung Orthopair Uncertain Linguistic Sets

The purpose of this section is to present the novel approach of complex q-rung orthopair uncertain linguistic sets (CQROULSs) and their fundamental laws. The notion of CQROULS composes two kind of information, uncertain linguistic variable (ULV) and CQROFS, with a condition that the sum of q-power of the real parts (also for imaginary parts) of the supporting and supporting against grades cannot be exceeded from a unit interval.

Definition 6. *A CQROULS is initiated by*

$$\mathfrak{C}_{CQUL} = \left\{ \left(x, \left([\mathcal{L}_\theta, \mathcal{L}_\varphi], \left(\mathfrak{M}_{\mathfrak{C}_{CQUL}}, \mathfrak{N}_{\mathfrak{C}_{CQUL}}\right)\right)\right) : x \in X_{UNI} \right\} \tag{5}$$

where $\mathfrak{M}_{\mathfrak{C}_{CQ}} = \mathfrak{M}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP}})}$ and $\mathfrak{N}_{\mathfrak{C}_{CQ}} = \mathfrak{N}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP}})}$ expresses the grade of complex supporting and complex supporting against with conditions: $0 \leq \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP}} \leq 1$ and $0 \leq \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP}} \leq 1$ for ULVs $[\mathcal{L}_\theta, \mathcal{L}_\varphi]$. Additionally, the grade of refusal is stated by $\Re_{\mathfrak{C}_{CQ}} = \Re_{\mathfrak{C}_{RP}} e^{i2\pi(\Re_{\mathfrak{C}_{IP}})} = \left(1 - \left(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP}}\right)\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP}}))^{\frac{1}{q_{SC}}}}$. The complex q-rung orthopair uncertain fuzzy number (CQROUFN) is expressed by $\mathfrak{C}_{CQUL} = \left([\mathcal{L}_\theta, \mathcal{L}_\varphi], \left(\mathfrak{M}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP}})}, \mathfrak{N}_{\mathfrak{C}_{RP}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP}})}\right)\right)$.

Definition 7. *For any two CQROULSs* $\mathfrak{C}_{CQUL-1} = \left(\left[\mathcal{L}_{\theta_1}, \mathcal{L}_{\varphi_1}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}})}, \mathfrak{N}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}})}\right)\right)$ *and* $\mathfrak{C}_{CQUL-2} = \left(\left[\mathcal{L}_{\theta_2}, \mathcal{L}_{\varphi_2}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-2}})}\right)\right)$*, we have*

1. $\mathfrak{C}_{CQUL-1} \oplus_{CQUL} \mathfrak{C}_{CQUL-2} =$
$$\left(\begin{array}{c}\left[\mathcal{L}_{\theta_1+\theta_2-\frac{\theta_1\theta_2}{k}}, \mathcal{L}_{\varphi_1+\varphi_2-\frac{\varphi_1\varphi_2}{k}}\right], \\ \left(\begin{array}{c}\left(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-2}} - \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-2}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-2}} - \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-2}})^{\frac{1}{q_{SC}}}}, \\ \mathfrak{N}_{\mathfrak{C}_{RP-1}} \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}} \mathfrak{N}_{\mathfrak{C}_{IP-2}})}\end{array}\right)\end{array}\right);$$
2. $\mathfrak{C}_{CQUL-1} \otimes_{CQUL} \mathfrak{C}_{CQUL-2} =$
$$\left(\begin{array}{c}\left[\mathcal{L}_{\frac{\theta_1\theta_2}{k}}, \mathcal{L}_{\frac{\varphi_1\varphi_2}{k}}\right]', \\ \left(\begin{array}{c}\mathfrak{M}_{\mathfrak{C}_{RP-1}} \mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}} \mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \\ \left(\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-2}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-2}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-2}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}} \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-2}})^{\frac{1}{q_{SC}}}}\end{array}\right)\end{array}\right);$$
3. $\delta_{SC}\mathfrak{C}_{CQUL-1} =$
$$\left(\begin{array}{c}\left[\mathcal{L}_{k-k(1-\frac{\theta_1}{k})^{\delta_{SC}}}, \mathcal{L}_{k-k(1-\frac{\varphi_1}{k})^{\delta_{SC}}}\right], \\ \left(\left(1-\left(1-\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}})^{\delta_{SC}})^{\frac{1}{q_{SC}}}}, \mathfrak{N}^{\delta_{SC}}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}^{\delta_{SC}}_{\mathfrak{C}_{IP-1}})}\right)\end{array}\right), \delta_{SC} > 0;$$
4. $\mathfrak{C}^{\delta_{SC}}_{CQUL-1} =$
$$\left(\begin{array}{c}\left[\mathcal{L}_{k(\frac{\theta_1}{k})^{\delta_{SC}}}, \mathcal{L}_{k(\frac{\varphi_1}{k})^{\delta_{SC}}}\right], \\ \left(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{IP-1}})}, \left(1-\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}})^{\delta_{SC}})^{\frac{1}{q_{SC}}}}\right)\end{array}\right), \delta_{SC} > 0.$$

Definition 8. *For any two CQROULSs* $\mathfrak{C}_{CQUL-1} = \left(\left[\mathcal{L}_{\theta_1}, \mathcal{L}_{\varphi_1}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-1}})}, \mathfrak{N}_{\mathfrak{C}_{RP-1}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-1}})}\right)\right)$ *and* $\mathfrak{C}_{CQUL-2} = \left(\left[\mathcal{L}_{\theta_2}, \mathcal{L}_{\varphi_2}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-2}})}, \mathfrak{N}_{\mathfrak{C}_{RP-2}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-2}})}\right)\right)$*, the score and accuracy function are initiated by*

$$\mathbb{S}_{SF}(\mathfrak{C}_{CQUL-1}) = \mathcal{L}_{\frac{((\theta_1+\varphi_1)(1+\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} - \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}}))}{4}} \tag{6}$$

$$\mathbb{H}_{AF}(\mathfrak{C}_{CQUL-1}) = \mathcal{L}_{\frac{((\theta_1+\varphi_1)(1+\mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{M}^{q_{SC}}_{\mathfrak{C}_{IP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-1}} + \mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-1}}))}{4}} \tag{7}$$

To examine the interrelationship between any two CQROULSs, we use the following rules:

1. If $\mathbb{S}_{SF}(\mathfrak{C}_{CQUL-1}) > \mathbb{S}_{SF}(\mathfrak{C}_{CQUL-2})$, then $\mathfrak{C}_{CQUL-1} > \mathfrak{C}_{CQUL-2}$;
2. If $\mathbb{S}_{SF}(\mathfrak{C}_{CQUL-1}) = \mathbb{S}_{SF}(\mathfrak{C}_{CQUL-2})$, then
   (1) If $\mathbb{H}_{AF}(\mathfrak{C}_{CQUL-1}) > \mathbb{H}_{AF}(\mathfrak{C}_{CQUL-2})$, then $\mathfrak{C}_{CQUL-1} > \mathfrak{C}_{CQUL-2}$;
   (2) If $\mathbb{H}_{AF}(\mathfrak{C}_{CQUL-1}) = \mathbb{H}_{AF}(\mathfrak{C}_{CQUL-2})$, then $\mathfrak{C}_{CQUL-1} = \mathfrak{C}_{CQUL-2}$.

4. Partitioned Bonferroni Mean Operators Based on CQROULSs

In this section, we propose partitioned Bonferroni mean (PBM) operators based on CQROULSs, called complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean (CQROULPBM) operators. We also investigate their special cases of CQROULPBM operators. Furthermore, we choose the weight vectors whose representation is of the form $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-n})^T$ with the condition $\sum_{i=1}^{n} \omega_{W-i} = 1, \omega_{W-i} \in [0,1]$,

and then present the complex q-rung orthopair uncertain linguistic weighted partitioned Bonferroni mean (CQROULWPBM) operator.

Definition 9. *For a family of CQROULSs* $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i}, \mathcal{L}_{\varphi_i}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, \mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})}\right)\right)$, $i = 1, 2, \ldots, n$, *the CQROULPBM operator is initiated by*

$$\begin{aligned} &CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n}\right) \\ &= \frac{1}{m}\left(\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|}\sum_{i\in\mathcal{P}_{\mathcal{H}}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}} \otimes \left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|-1}\sum_{\substack{j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right) \\ &= \frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}} \otimes \mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}} \end{aligned} \tag{8}$$

where $|\mathcal{P}_{\mathcal{H}}|$ expresses the order of $\mathcal{P}_{\mathcal{H}}$, and $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ is a partition set with m different categories. The symbols $\mathcal{S}_{SC}, \mathcal{T}_{SC} \geq 0$, with a condition of $\mathcal{S}_{SC} + \mathcal{T}_{SC} > 0$.

Theorem 1. *For a family of CQROULSs* $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i}, \mathcal{L}_{\varphi_i}\right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, \mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})}\right)\right)$ $i = 1, 2, \ldots, n$, *by using the Equation (8) we get*

$$\begin{aligned} &CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n}\right) \\ &= \left(\begin{array}{l} \left[\begin{array}{l} \mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\frac{\theta_i}{k}\right)^{\mathcal{S}_{SC}}\left(\frac{\theta_j}{k}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}}, \\ \mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\frac{\varphi_i}{k}\right)^{\mathcal{S}_{SC}}\left(\frac{\varphi_j}{k}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}} \end{array}\right], \\ \left(\begin{array}{l} \left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\mathfrak{M}_{\mathfrak{C}_{RP-i}}^{\mathcal{S}_{SC}}\mathfrak{M}_{\mathfrak{C}_{RP-j}}^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}}\times \\ e^{i2\pi\left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\mathfrak{M}_{\mathfrak{C}_{IP-i}}^{\mathcal{S}_{SC}}\mathfrak{M}_{\mathfrak{C}_{IP-j}}^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}}\right)\right)} \end{array}, \right. \\ \left.\begin{array}{l} \left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}}\times \\ e^{i2\pi\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}}\right)} \end{array}\right) \end{array}\right) \end{aligned} \tag{9}$$

where $\delta_{SC} = \frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}$.

Proof. By using the Definition (7), we get

$$\mathfrak{C}^{\delta_{SC}}_{CQUL-i} = \begin{pmatrix} \left[\mathcal{L}_{k(\frac{\theta_i}{k})^{\mathcal{S}_{SC}}}, \mathcal{L}_{k(\frac{\varphi_i}{k})^{\mathcal{S}_{SC}}}\right], \\ \left(\mathfrak{M}^{\mathcal{S}_{SC}}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}^{\mathcal{S}_{SC}}_{\mathfrak{C}_{IP-i}})}, \left(1-\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-i}}\right)^{\mathcal{S}_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-i}})^{\mathcal{S}_{SC}})^{\frac{1}{q_{SC}}}}\right) \end{pmatrix}$$

$$\mathfrak{C}^{\mathcal{T}_{SC}}_{CQUL-j} = \begin{pmatrix} \left[\mathcal{L}_{k(\frac{\theta_j}{k})^{\mathcal{T}_{SC}}}, \mathcal{L}_{k(\frac{\varphi_j}{k})^{\mathcal{T}_{SC}}}\right], \\ \left(\mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{RP-j}} e^{i2\pi(\mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{IP-j}})}, \left(1-\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-j}}\right)^{\mathcal{T}_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-j}})^{\mathcal{T}_{SC}})^{\frac{1}{q_{SC}}}}\right) \end{pmatrix}$$

$$\mathfrak{C}^{\delta_{SC}}_{CQUL-i} \otimes_{CQUL} \mathfrak{C}^{\mathcal{T}_{SC}}_{CQUL-j} = \begin{pmatrix} \left[\mathcal{L}_{k(\frac{\theta_i}{k})^{\mathcal{S}_{SC}}(\frac{\theta_j}{k})^{\mathcal{T}_{SC}}}, \mathcal{L}_{k(\frac{\varphi_i}{k})^{\mathcal{S}_{SC}}(\frac{\varphi_j}{k})^{\mathcal{T}_{SC}}}\right]' \\ \left(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{RP-i}} \mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{RP-j}} e^{i2\pi(\mathfrak{M}^{\delta_{SC}}_{\mathfrak{C}_{IP-i}} \mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{IP-j}})}, \right. \\ \left. \left(1-\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-i}}\right)^{\delta_{SC}}\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-j}}\right)^{\mathcal{T}_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi(1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-i}})^{\delta_{SC}}(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-j}})^{\mathcal{T}_{SC}})^{\frac{1}{q_{SC}}}} \right) \end{pmatrix}$$

$$\delta_{SC} \sum_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} \mathfrak{C}^{\mathcal{S}_{SC}}_{CQUL-i} \otimes \mathfrak{C}^{\mathcal{T}_{SC}}_{CQUL-j} = \begin{pmatrix} \left[\begin{matrix} \mathcal{L}_{k-k(\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} (1-(\frac{\theta_i}{k})^{\mathcal{S}_{SC}}(\frac{\theta_j}{k})^{\mathcal{T}_{SC}}))^{\delta_{SC}}}, \\ \mathcal{L}_{k-k(\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} (1-(\frac{\varphi_i}{k})^{\mathcal{S}_{SC}}(\frac{\varphi_j}{k})^{\mathcal{T}_{SC}}))^{\delta_{SC}}} \end{matrix}\right], \\ \left(\left(1-\left(\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} \left(1-\left(\mathfrak{M}^{\mathcal{S}_{SC}}_{\mathfrak{C}_{RP-i}} \mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{RP-j}}\right)^{q_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{q_{SC}}} \times \right. \\ \left. e^{i2\pi(1-(\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} (1-(\mathfrak{M}^{\mathcal{S}_{SC}}_{\mathfrak{C}_{IP-i}} \mathfrak{M}^{\mathcal{T}_{SC}}_{\mathfrak{C}_{IP-j}})^{q_{SC}}))^{\delta_{SC}})^{\frac{1}{q_{SC}}}}, \right. \\ \left. \left(\left(\left(\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} \left(1-\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-i}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{RP-j}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)\right)^{\frac{1}{q_{SC}}} \times \right. \right. \\ \left. \left. e^{i2\pi((\prod_{\substack{i,j \in \mathcal{P}_{\mathcal{H}} \\ i \neq j}} (1-(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-i}})^{\mathcal{S}_{SC}}(1-\mathfrak{N}^{q_{SC}}_{\mathfrak{C}_{IP-j}})^{\mathcal{T}_{SC}})^{\delta_{SC}}))^{\frac{1}{q_{SC}}}} \right) \end{pmatrix}$$

$$\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}$$

$$=\left(\begin{array}{l}\left[\begin{array}{l}\mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\frac{\theta_i}{k}\right)^{\mathcal{S}_{SC}}\left(\frac{\theta_j}{k}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)}^{\frac{1}{m}},\\ \mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\frac{\varphi_i}{k}\right)^{\mathcal{S}_{SC}}\left(\frac{\varphi_j}{k}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)}^{\frac{1}{m}}\end{array}\right],\\ \left(\begin{array}{l}\left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\mathfrak{M}_{\mathfrak{C}_{RP-i}}^{\mathcal{S}_{SC}}\mathfrak{M}_{\mathfrak{C}_{RP-j}}^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}}\times\\ e^{i2\pi\left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\mathfrak{M}_{\mathfrak{C}_{IP-i}}^{\mathcal{S}_{SC}}\mathfrak{M}_{\mathfrak{C}_{IP-j}}^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}}\right)}\end{array}\right),\\ \left(\begin{array}{l}\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}}\times\\ e^{i2\pi\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}}}\end{array}\right)\end{array}\right)$$

The proof is completed. □

Theorem 2. *For a family of CQROULSs* $\mathfrak{C}_{CQUL-i}=\left(\left[\mathcal{L}_{\theta_i},\mathcal{L}_{\varphi_i}\right],\left(\mathfrak{M}_{\mathfrak{C}_{RP-i}}e^{i2\pi\left(\mathfrak{M}_{\mathfrak{C}_{IP-i}}\right)},\mathfrak{N}_{\mathfrak{C}_{RP-i}}e^{i2\pi\left(\mathfrak{N}_{\mathfrak{C}_{IP-i}}\right)}\right)\right)$ $i=1,2,\ldots,n$, *the following properties of Idempotency, Boundedness and Commutativity are hold.*

1. **Idempotency:** If $\mathfrak{C}_{CQUL-i}=\mathfrak{C}_{CQUL}$, then

$$
\begin{aligned}
&CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\
&\quad=CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL},\mathfrak{C}_{CQUL},\mathfrak{C}_{CQUL},\ldots,\mathfrak{C}_{CQUL}\right)\\
&\quad=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\
&\quad=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\
&\quad=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL}^{\mathcal{S}_{SC}+\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\mathfrak{C}_{CQUL}^{\mathcal{S}_{SC}+\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\
&\quad=\mathfrak{C}_{CQUL}
\end{aligned}
$$

2. **Boundedness:** If $\mathfrak{C}_{CQUL}^{-}=\left(\left[\mathcal{L}_{min\theta_i},\mathcal{L}_{min\varphi_i}\right],\left(min\mathfrak{M}_{\mathfrak{C}_{RP-i}}e^{i2\pi(min\mathfrak{M}_{\mathfrak{C}_{IP-i}})},\right.\right.$ $\left.\left.max\mathfrak{N}_{\mathfrak{C}_{RP-i}}e^{i2\pi(max\mathfrak{N}_{\mathfrak{C}_{IP-i}})}\right)\right)$ and $\mathfrak{C}_{CQUL}^{+}=\left(\left[\mathcal{L}_{max\theta_i},\mathcal{L}_{max\varphi_i}\right],\right.$ $\left.\left(max\mathfrak{M}_{\mathfrak{C}_{RP-i}}e^{i2\pi(max\mathfrak{M}_{\mathfrak{C}_{IP-i}})},min\mathfrak{N}_{\mathfrak{C}_{RP-i}}e^{i2\pi(min\mathfrak{N}_{\mathfrak{C}_{IP-i}})}\right)\right)$, then

$$\mathfrak{C}_{CQUL}^{-}\leq CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\leq\mathfrak{C}_{CQUL}^{+}$$

3. **Commutativity:** If $\mathfrak{C}_{CQUL-i}^{'}$ is a permutation of the $\mathfrak{C}_{CQUL-i}$, then

$$
\begin{aligned}
&CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1}^{'},\mathfrak{C}_{CQUL-2}^{'},\mathfrak{C}_{CQUL-3}^{'},\ldots,\mathfrak{C}_{CQUL-n}^{'}\right)\\
&=CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)
\end{aligned}
$$

If $\mathfrak{C}_{CQUL-i}^{'}$ is a permutation of the $\mathfrak{C}_{CQUL-i}$, then

$$
\begin{aligned}
&CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\
&\quad=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\
&\quad=\frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}'^{\mathcal{S}_{SC}}_{CQUL-i}\otimes\mathfrak{C}'^{\mathcal{T}_{SC}}_{CQUL-j}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\
&\quad=CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1}^{'},\mathfrak{C}_{CQUL-2}^{'},\mathfrak{C}_{CQUL-3}^{'},\ldots,\mathfrak{C}_{CQUL-n}^{'}\right)
\end{aligned}
$$

Proof. The proof of this part is straightforward. □

Additionally, we discuss the special cases of the CQROULPBM operator. The special cases are discussed below.

Case 1. *By using the values of parameter $\mathcal{T}_{SC} = 1$ and $|\mathcal{P}_1| = n$, the explored CQROULPBM operator is reduced into a complex q-rung orthopair uncertain linguistic Bonferroni mean operator, such that:*

$$\begin{aligned} &CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\ &= \frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_1|(|\mathcal{P}_1|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\ &= \left(\frac{\sum_{i,j=1}^{n}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}}{n(n-1)}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\\ &= CQROULBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\end{aligned}$$

Case 2. *By using the values of parameter $\mathcal{S}_{SC} = 1$ and $\mathcal{T}_{SC} = 0$, the explored CQROULPBM operator is reduced into a complex q-rung orthopair uncertain linguistic mean operator, such that:*

$$\begin{aligned} &CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\ &= \frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_1|(|\mathcal{P}_1|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{C}_{CQUL-i}\right)\end{aligned}$$

Case 3. *By using the values of parameter $\mathcal{S}_{SC} = 2$ and $\mathcal{T}_{SC} = 0$, the explored CQROULPBM operator is reduced into a complex q-rung orthopair uncertain linguistic geometric mean operator, such that:*

$$\begin{aligned} &CQROULPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\ &= \frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_1|(|\mathcal{P}_1|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{C}_{CQUL-i}^{2}\right)^{\frac{1}{2}}\end{aligned}$$

Definition 10. *For a family of CQROULSs $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i},\mathcal{L}_{\varphi_i}\right],\left(\mathfrak{M}_{\mathfrak{C}_{RP-i}}e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})},\right.\right.$ $\left.\left.\mathfrak{N}_{\mathfrak{C}_{RP-i}}e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})}\right)\right)$ $i = 1, 2, \ldots, n$, the CQROULWPBM operator is initiated by:*

$$\begin{aligned} &CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\ldots,\mathfrak{C}_{CQUL-n}\right)\\ &= \frac{1}{m}\left(\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|}\sum_{i\in\mathcal{P}_{\mathcal{H}}}\omega_{W-i}^{\mathcal{S}_{SC}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|-1}\sum_{\substack{j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\omega_{W-j}^{\mathcal{T}_{SC}}\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\\ &= \frac{1}{m}\sum_{\mathcal{H}=1}^{m}\left(\frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}\sum_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\omega_{W-i}^{\mathcal{S}_{SC}}\mathfrak{C}_{CQUL-i}^{\mathcal{S}_{SC}}\otimes\omega_{W-j}^{\mathcal{T}_{SC}}\mathfrak{C}_{CQUL-j}^{\mathcal{T}_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\end{aligned} \tag{10}$$

where $|\mathcal{P}_{\mathcal{H}}|$ expresses the order of $\mathcal{P}_{\mathcal{H}}$, and $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ is a partition set with m different categories. The symbols $\mathcal{S}_{SC}, \mathcal{T}_{SC} \geq 0$ are with a condition that is $\mathcal{S}_{SC} + \mathcal{T}_{SC} > 0$. Additionally, the term $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-n})^T, \omega_{W-i} \in [0,1]$ with a condition that is $\sum_{i=1}^{n} \omega_{W-i} = 1$.

Theorem 3. *For a family of CQROULSs* $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i}, \mathcal{L}_{\varphi_i} \right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, \mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})} \right) \right)$ $i = 1, 2, \ldots, n$, *by using Equation (10), we get:*

$$
\begin{aligned}
&CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n}) \\
&= \left(\left[\begin{array}{l} \mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\left(1-\frac{\theta_i}{k}\right)^{\omega_{W-i}}\right)^{\mathcal{S}_{SC}}\left(1-\left(1-\frac{\theta_j}{k}\right)^{\omega_{W-j}}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)}^{\frac{1}{m}}, \\ \mathcal{L}_{k-k\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\left(1-\frac{\varphi_i}{k}\right)^{\omega_{W-i}}\right)^{\mathcal{S}_{SC}}\left(1-\left(1-\frac{\varphi_j}{k}\right)^{\omega_{W-i}}\right)^{\mathcal{T}_{SC}}\right)\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)}^{\frac{1}{m}} \end{array} \right], \right. \\
&\left. \left(\left(1-\left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\left(1-\mathfrak{M}_{\mathfrak{C}_{RP-i}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{M}_{\mathfrak{C}_{RP-j}}\right)^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}} \times \right. \right. \\
&\left. \left. e^{i2\pi\left(1-\left(1-\left(1-\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(\left(1-\mathfrak{M}_{\mathfrak{C}_{IP-i}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{M}_{\mathfrak{C}_{IP-j}}\right)^{\mathcal{T}_{SC}}\right)^{q_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{q_{SC}}}\right)\right)} \right. \right. \\
&\left. \left. \left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{RP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}} \times \right. \right. \right. \\
&\left. \left. \left. e^{i2\pi\left(\prod_{\mathcal{H}=1}^{m}\left(1-\left(1-\left(\prod_{\substack{i,j\in\mathcal{P}_{\mathcal{H}}\\ i\neq j}}\left(1-\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-i}}^{q_{SC}}\right)^{\mathcal{S}_{SC}}\left(1-\mathfrak{N}_{\mathfrak{C}_{IP-j}}^{q_{SC}}\right)^{\mathcal{T}_{SC}}\right)^{\delta_{SC}}\right)^{\frac{1}{\mathcal{S}_{SC}+\mathcal{T}_{SC}}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{m}}\right)} \right) \right)
\end{aligned}
\tag{11}
$$

where $\delta_{SC} = \frac{1}{|\mathcal{P}_{\mathcal{H}}|(|\mathcal{P}_{\mathcal{H}}|-1)}$.

Proof. Straightforward. □

Theorem 4. *For a family of CQROULSs* $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i}, \mathcal{L}_{\varphi_i} \right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, \mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})} \right) \right)$ $i = 1, 2, \ldots, n$, *the following properties hold:*

1. **Idempotency:** If $\mathfrak{C}_{CQUL-i} = \mathfrak{C}_{CQUL}$, then

$$
\begin{aligned}
&CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n}\right) \\
&= CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL}, \mathfrak{C}_{CQUL}, \mathfrak{C}_{CQUL}, \ldots, \mathfrak{C}_{CQUL}\right) = \mathfrak{C}_{CQUL}
\end{aligned}
$$

2. **Boundedness:** If $\mathfrak{C}^{-}_{CQUL} = \left(\left[\mathcal{L}_{min\theta_i}, \mathcal{L}_{min\varphi_i} \right], \left(min\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(min\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, max\mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(max\mathfrak{N}_{\mathfrak{C}_{IP-i}})} \right) \right)$ and $\mathfrak{C}^{+}_{CQUL} = \left(\left[\mathcal{L}_{max\theta_i}, \mathcal{L}_{max\varphi_i} \right], \left(max\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(max\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, min\mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(min\mathfrak{N}_{\mathfrak{C}_{IP-i}})} \right) \right)$, then

$$\mathfrak{C}^{-}_{CQUL} \leq CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}} \left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n} \right) \leq \mathfrak{C}^{+}_{CQUL}$$

3. **Commutativity:** If $\mathfrak{C}^{'}_{CQUL-i}$ is a permutation of the $\mathfrak{C}_{CQUL-i}$, then

$$CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}} \left(\mathfrak{C}^{'}_{CQUL-1}, \mathfrak{C}^{'}_{CQUL-2}, \mathfrak{C}^{'}_{CQUL-3}, \ldots, \mathfrak{C}^{'}_{CQUL-n} \right) = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}} \left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n} \right)$$

If $\mathfrak{C}^{'}_{CQUL-i}$ is a permutation of the $\mathfrak{C}_{CQUL-i}$, then

$$CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}} \left(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \ldots, \mathfrak{C}_{CQUL-n} \right) = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}} \left(\mathfrak{C}^{'}_{CQUL-1}, \mathfrak{C}^{'}_{CQUL-2}, \mathfrak{C}^{'}_{CQUL-3}, \ldots, \mathfrak{C}^{'}_{CQUL-n} \right)$$

5. MADM Method Based on CQROULSs with Application in Antivirus Mask Selection

The purpose of this section is to explore the MADM issue based on CQROULSs with its CQROULWPBM operator to examine the proficiency and reliability of the presented approach. To proficiently examine the approaches, we choose the family of alternatives and their attributes based on weight vectors, whose expressions are summarized as follows: $\mathbb{A}_{AL} = \{\mathbb{A}_{AL-1}, \mathbb{A}_{AL-2}, \ldots, \mathbb{A}_{AL-n}\}$ and $\mathbb{C}_{AT} = \{\mathbb{C}_{AT-1}, \mathbb{C}_{AT-2}, \ldots, \mathbb{C}_{AT-m}\}$ by using the weight vectors $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-n})^T$ with a condition that is $\sum_{i=1}^{n} \omega_{W-i} = 1, \omega_{W-i} \in [0,1]$. To evaluate the above information, we construct the decision matrix whose information in the form of CQROULNs with $\mathfrak{C}_{CQUL-i} = \left(\left[\mathcal{L}_{\theta_i}, \mathcal{L}_{\varphi_i} \right], \left(\mathfrak{M}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{M}_{\mathfrak{C}_{IP-i}})}, \mathfrak{N}_{\mathfrak{C}_{RP-i}} e^{i2\pi(\mathfrak{N}_{\mathfrak{C}_{IP-i}})} \right) \right)$. To resolve the above issue, the steps of the algorithm is summarized as follows:

Step 1: Construct the decision matrix, whose entries are in the form of CQROULNs.
Step 2: By using Equation (11), we aggregate the information which is given in step 1.
Step 3: By using Equation (6), we examine the score values of the aggregated values of step 2.
Step 4: Rank to all alternatives, and examine the best one.
Step 5: The end.

5.1. On Antivirus Mask Selection for the COVID-19 Pandemic

In the current extreme instance of COVID-19 transmission, there are six sorts of veils that are generally accessible in the market, including clinical careful covers, particulate respirators (N95/KN95 or more), clinical defensive covers, dispensable clinical covers, conventional non-clinical covers, and gas covers. An individual needs to purchase an antivirus veil from the six up-and-comer antivirus covers. In addition, the individual in question assesses the antivirus veils by thinking about four measures, in particular, spillage rate (C1), that is the adhesiveness of the veil structure configuration to cover the human face; reusability (C2); nature of crude materials (C3); and filtration productivity (C4), which implies that the filtration productivity of non-slick 0.3 μm particles is more noteworthy than 95%, and it should likewise have clinical assurance prerequisites, for example, surface dampness opposition and blood obstruction. The mask and their representations are presented in Figure 1.



Figure 1. Representation of mask and their names.

The individual gives the assessment estimation of standard C_i $(i = 1, 2, 3, 4)$ concerning every A_i $(i = 1, 2, 3, 4, 5, 6)$ in CQROULSs. Since the proposed MADM based on CQROULSs with its CQROULPBM operator has different q values, it generates some usefully special cases. For example, when $q = 1$, it becomes complex intuitionistic uncertain linguistic numbers (CIULNs), and when $q = 2$, it becomes complex Pythagorean uncertain linguistic numbers (CPULNs). To examine the usefulness and proficiency of the proposed approach, we next present three cases of $q = 1$, $q = 2$, and $q = 7$.

We first consider the case of $q = 1$, that is, CIULNs. These decision matrix items in the form of CIULNs are shown in Table 1.

Table 1. Decision matrix, whose items are in the form of CIULNs.

Data	$\mathbb{C}_{AT-1}$	$\mathbb{C}_{AT-2}$	$\mathbb{C}_{AT-3}$	$\mathbb{C}_{AT-4}$
$\mathbb{A}_{AL-1}$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_2], \\ 0.3e^{i2\pi(0.4)}, \\ 0.2e^{i2\pi(0.1)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_3], \\ 0.31e^{i2\pi(0.41)}, \\ 0.21e^{i2\pi(0.11)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_4], \\ 0.32e^{i2\pi(0.42)}, \\ 0.22e^{i2\pi(0.12)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_5], \\ 0.33e^{i2\pi(0.43)}, \\ 0.23e^{i2\pi(0.13)} \end{array} \right)$
$\mathbb{A}_{AL-2}$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_3], \\ 0.4e^{i2\pi(0.5)}, \\ 0.3e^{i2\pi(0.2)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_4], \\ 0.41e^{i2\pi(0.51)}, \\ 0.31e^{i2\pi(0.21)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_5], \\ 0.42e^{i2\pi(0.52)}, \\ 0.32e^{i2\pi(0.22)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_3], \\ 0.43e^{i2\pi(0.53)}, \\ 0.33e^{i2\pi(0.23)} \end{array} \right)$
$\mathbb{A}_{AL-3}$	$\left(\begin{array}{c} [\mathcal{L}_3, \mathcal{L}_4], \\ 0.5e^{i2\pi(0.6)}, \\ 0.4e^{i2\pi(0.3)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_3, \mathcal{L}_5], \\ 0.51e^{i2\pi(0.61)}, \\ 0.41e^{i2\pi(0.31)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_4], \\ 0.52e^{i2\pi(0.62)}, \\ 0.42e^{i2\pi(0.32)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_5], \\ 0.53e^{i2\pi(0.63)}, \\ 0.43e^{i2\pi(0.33)} \end{array} \right)$
$\mathbb{A}_{AL-4}$	$\left(\begin{array}{c} [\mathcal{L}_0, \mathcal{L}_2], \\ 0.1e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.2)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0, \mathcal{L}_1], \\ 0.11e^{i2\pi(0.31)}, \\ 0.41e^{i2\pi(0.21)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0, \mathcal{L}_3], \\ 0.12e^{i2\pi(0.32)}, \\ 0.42e^{i2\pi(0.22)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0, \mathcal{L}_4], \\ 0.13e^{i2\pi(0.33)}, \\ 0.43e^{i2\pi(0.23)} \end{array} \right)$
$\mathbb{A}_{AL-5}$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_4], \\ 0.7e^{i2\pi(0.5)}, \\ 0.1e^{i2\pi(0.2)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_5], \\ 0.71e^{i2\pi(0.51)}, \\ 0.11e^{i2\pi(0.21)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_3], \\ 0.72e^{i2\pi(0.52)}, \\ 0.12e^{i2\pi(0.22)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_2], \\ 0.73e^{i2\pi(0.53)}, \\ 0.13e^{i2\pi(0.23)} \end{array} \right)$
$\mathbb{A}_{AL-6}$	$\left(\begin{array}{c} [\mathcal{L}_3, \mathcal{L}_5], \\ 0.8e^{i2\pi(0.7)}, \\ 0.1e^{i2\pi(0.1)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_3, \mathcal{L}_4], \\ 0.81e^{i2\pi(0.71)}, \\ 0.11e^{i2\pi(0.11)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2, \mathcal{L}_3], \\ 0.82e^{i2\pi(0.72)}, \\ 0.12e^{i2\pi(0.12)} \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_5], \\ 0.83e^{i2\pi(0.73)}, \\ 0.13e^{i2\pi(0.13)} \end{array} \right)$

To resolve the above issue, the steps of the algorithm are summarized as follows:

Step 1: We construct the decision matrix, whose entries in the form of CQROULNs are discussed in Table 1.
Step 2: By using Equation (11), we aggregate the information which is given in *Step 1*, for $\mathcal{S}_{SC} = \mathcal{T}_{SC} = 1, q = 1$, as shown in Table 1. We obtain the following:

$$\mathbb{A}_{\boldsymbol{AL-1}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_{3.4354}], \\ \left(\begin{array}{c} 0.32243e^{i2\pi(0.42635)}, \\ 0.223535e^{i2\pi(0.126353)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-2}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_{1.9846}, \mathcal{L}_{3.4655}], \\ \left(\begin{array}{c} 0.424645e^{i2\pi(0.527327)}, \\ 0.3232323e^{i2\pi(0.228483)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-3}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_{2.345}, \mathcal{L}_{4.3545}], \\ \left(\begin{array}{c} 0.52723672e^{i2\pi(0.627836)}, \\ 0.4286428e^{i2\pi(0.328682)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-4}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_0, \mathcal{L}_{2.134}], \\ \left(\begin{array}{c} 0.12824648e^{i2\pi(0.32288)}, \\ 0.42383627e^{i2\pi(0.227846278)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-5}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_{3.12342}], \\ \left(\begin{array}{c} 0.72726274e^{i2\pi(0.52846387)}, \\ 0.1238483e^{i2\pi(0.228483)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-6}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_{1.34562}, \mathcal{L}_{3.3636}], \\ \left(\begin{array}{c} 0.828483e^{i2\pi(0.7283286)}, \\ 0.123278236e^{i2\pi(0.1282382)} \end{array} \right) \end{array} \right)$$

Step 3: By using Equation (6), we examine the score values of the aggregated values of *Step 2* with

$$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{1.5512}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{0.5453}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{0.6663}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{-0.1070}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{0.9312}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{1.5366}$$

Step 4: We rank all alternatives with $\mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$. Therefore, the alternative $\mathbb{A}_{\boldsymbol{AL-1}}$ is the best one according to the rank of *Step 4*.
Step 5: The end.

Additionally, we consider the case of $q = 2$, that is, CPULNs. These decision matrix items in the form of CPULNs are shown in Table 2.

By using Equation (11), we aggregate the information given in step 1, for $\mathcal{S}_{SC} = \mathcal{T}_{SC} = 1, q = 2$. We obtain the following:

$$\mathbb{A}_{\boldsymbol{AL-1}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_1, \mathcal{L}_{3.4354}], \\ \left(\begin{array}{c} 0.82243e^{i2\pi(0.92635)}, \\ 0.223535e^{i2\pi(0.126353)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-2}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_{1.9846}, \mathcal{L}_{3.4655}], \\ \left(\begin{array}{c} 0.724645e^{i2\pi(0.827327)}, \\ 0.3232323e^{i2\pi(0.228483)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-3}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}(\mathfrak{C}_{CQUL-1}, \mathfrak{C}_{CQUL-2}, \mathfrak{C}_{CQUL-3}, \mathfrak{C}_{CQUL-4}) = \left(\begin{array}{c} [\mathcal{L}_{2.345}, \mathcal{L}_{4.3545}], \\ \left(\begin{array}{c} 0.62723672e^{i2\pi(0.727836)}, \\ 0.4286428e^{i2\pi(0.328682)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-4}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c}[\mathcal{L}_0,\mathcal{L}_{2.134}],\\ \left(\begin{array}{c}0.62824648e^{i2\pi(0.82288)},\\ 0.42383627e^{i2\pi(0.227846278)}\end{array}\right)\end{array}\right)$$

$$\mathbb{A}_{\boldsymbol{AL-5}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_{3.12342}],\\ \left(\begin{array}{c}0.62726274e^{i2\pi(0.72846387)},\\ 0.1238483e^{i2\pi(0.228483)}\end{array}\right)\end{array}\right)$$

$$\mathbb{A}_{\boldsymbol{AL-5}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c}[\mathcal{L}_{1.34562},\mathcal{L}_{3.3636}],\\ \left(\begin{array}{c}0.928483e^{i2\pi(0.8283286)},\\ 0.123278236e^{i2\pi(0.1282382)}\end{array}\right)\end{array}\right)$$

Table 2. Decision matrix whose items are the form of CPULNs.

Data	$\mathbb{C}_{\boldsymbol{AT-1}}$	$\mathbb{C}_{\boldsymbol{AT-2}}$	$\mathbb{C}_{\boldsymbol{AT-3}}$	$\mathbb{C}_{\boldsymbol{AT-4}}$
$\mathbb{A}_{\boldsymbol{AL-1}}$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_2],\\ \left(\begin{array}{c}0.8e^{i2\pi(0.9)},\\ 0.2e^{i2\pi(0.1)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_3],\\ \left(\begin{array}{c}0.81e^{i2\pi(0.91)},\\ 0.21e^{i2\pi(0.11)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_4],\\ \left(\begin{array}{c}0.82e^{i2\pi(0.92)},\\ 0.22e^{i2\pi(0.12)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_5],\\ \left(\begin{array}{c}0.83e^{i2\pi(0.93)},\\ 0.23e^{i2\pi(0.13)}\end{array}\right)\end{array}\right)$
$\mathbb{A}_{\boldsymbol{AL-2}}$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_3],\\ \left(\begin{array}{c}0.7e^{i2\pi(0.8)},\\ 0.3e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_4],\\ \left(\begin{array}{c}0.71e^{i2\pi(0.81)},\\ 0.31e^{i2\pi(0.21)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_5],\\ \left(\begin{array}{c}0.72e^{i2\pi(0.82)},\\ 0.32e^{i2\pi(0.22)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_3],\\ \left(\begin{array}{c}0.73e^{i2\pi(0.83)},\\ 0.33e^{i2\pi(0.23)}\end{array}\right)\end{array}\right)$
$\mathbb{A}_{\boldsymbol{AL-3}}$	$\left(\begin{array}{c}[\mathcal{L}_3,\mathcal{L}_4],\\ \left(\begin{array}{c}0.6e^{i2\pi(0.7)},\\ 0.4e^{i2\pi(0.3)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_3,\mathcal{L}_5],\\ \left(\begin{array}{c}0.61e^{i2\pi(0.71)},\\ 0.41e^{i2\pi(0.31)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_4],\\ \left(\begin{array}{c}0.62e^{i2\pi(0.72)},\\ 0.42e^{i2\pi(0.32)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_5],\\ \left(\begin{array}{c}0.63e^{i2\pi(0.73)},\\ 0.43e^{i2\pi(0.33)}\end{array}\right)\end{array}\right)$
$\mathbb{A}_{\boldsymbol{AL-4}}$	$\left(\begin{array}{c}[\mathcal{L}_0,\mathcal{L}_2],\\ \left(\begin{array}{c}0.6e^{i2\pi(0.8)},\\ 0.4e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_0,\mathcal{L}_1],\\ \left(\begin{array}{c}0.61e^{i2\pi(0.81)},\\ 0.41e^{i2\pi(0.21)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_0,\mathcal{L}_3],\\ \left(\begin{array}{c}0.62e^{i2\pi(0.82)},\\ 0.42e^{i2\pi(0.22)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_0,\mathcal{L}_4],\\ \left(\begin{array}{c}0.63e^{i2\pi(0.83)},\\ 0.43e^{i2\pi(0.23)}\end{array}\right)\end{array}\right)$
$\mathbb{A}_{\boldsymbol{AL-5}}$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_4],\\ \left(\begin{array}{c}0.7e^{i2\pi(0.5)},\\ 0.1e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_5],\\ \left(\begin{array}{c}0.71e^{i2\pi(0.51)},\\ 0.11e^{i2\pi(0.21)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_3],\\ \left(\begin{array}{c}0.72e^{i2\pi(0.52)},\\ 0.12e^{i2\pi(0.22)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_2],\\ \left(\begin{array}{c}0.73e^{i2\pi(0.53)},\\ 0.13e^{i2\pi(0.23)}\end{array}\right)\end{array}\right)$
$\mathbb{A}_{\boldsymbol{AL-6}}$	$\left(\begin{array}{c}[\mathcal{L}_3,\mathcal{L}_5],\\ \left(\begin{array}{c}0.8e^{i2\pi(0.7)},\\ 0.1e^{i2\pi(0.1)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_3,\mathcal{L}_4],\\ \left(\begin{array}{c}0.81e^{i2\pi(0.71)},\\ 0.11e^{i2\pi(0.11)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_2,\mathcal{L}_3],\\ \left(\begin{array}{c}0.82e^{i2\pi(0.72)},\\ 0.12e^{i2\pi(0.12)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_5],\\ \left(\begin{array}{c}0.83e^{i2\pi(0.73)},\\ 0.13e^{i2\pi(0.13)}\end{array}\right)\end{array}\right)$

By using Equation (6), we examine the score values of the aggregated values and have:

$$\begin{aligned}\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{2.7371}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{2.7975}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{2.7324}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{0.9817}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}})\\ = \mathcal{L}_{1.9137}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{2.9625}\end{aligned} \tag{12}$$

We rank all alternatives with $\mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$; the alternative $\mathbb{A}_{\boldsymbol{AL-6}}$ is found to be the best.

Furthermore, we consider a general CQROULN with $q = 7$; its decision matrix items are shown in Table 3.

By using Equation (11), we aggregate the information given in step 1, for $\mathcal{S}_{SC} = \mathcal{T}_{SC} = 1, q = 7$ with:

$$\mathbb{A}_{\boldsymbol{AL-1}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c}[\mathcal{L}_1,\mathcal{L}_{3.4354}],\\ \left(\begin{array}{c}0.82243e^{i2\pi(0.92635)},\\ 0.723535e^{i2\pi(0.126353)}\end{array}\right)\end{array}\right)$$

$$\mathbb{A}_{\boldsymbol{AL-2}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c} [\mathcal{L}_{1.9846},\mathcal{L}_{3.4655}], \\ \left(\begin{array}{c} 0.724645e^{i2\pi(0.827327)}, \\ 0.8232323e^{i2\pi(0.928483)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-3}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c} [\mathcal{L}_{2.345},\mathcal{L}_{4.3545}], \\ \left(\begin{array}{c} 0.62723672e^{i2\pi(0.727836)}, \\ 0.7286428e^{i2\pi(0.628682)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-4}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c} [\mathcal{L}_{0},\mathcal{L}_{2.134}], \\ \left(\begin{array}{c} 0.62824648e^{i2\pi(0.82288)}, \\ 0.42383627e^{i2\pi(0.227846278)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-5}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c} [\mathcal{L}_{1},\mathcal{L}_{3.12342}], \\ \left(\begin{array}{c} 0.72726274e^{i2\pi(0.52846387)}, \\ 0.8238483e^{i2\pi(0.928483)} \end{array} \right) \end{array} \right)$$

$$\mathbb{A}_{\boldsymbol{AL-5}} = CQROULWPBM^{\mathcal{S}_{SC},\mathcal{T}_{SC}}\left(\mathfrak{C}_{CQUL-1},\mathfrak{C}_{CQUL-2},\mathfrak{C}_{CQUL-3},\mathfrak{C}_{CQUL-4}\right) = \left(\begin{array}{c} [\mathcal{L}_{1.34562},\mathcal{L}_{3.3636}], \\ \left(\begin{array}{c} 0.828483e^{i2\pi(0.7283286)}, \\ 0.123278236e^{i2\pi(0.1282382)} \end{array} \right) \end{array} \right)$$

Table 3. Decision matrix whose items are in the form of CQROULNs.

Data	$\mathbb{C}_{\boldsymbol{AT-1}}$	$\mathbb{C}_{\boldsymbol{AT-2}}$	$\mathbb{C}_{\boldsymbol{AT-3}}$	$\mathbb{C}_{\boldsymbol{AT-4}}$
$\mathbb{A}_{\boldsymbol{AL-1}}$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_2], \\ \left(\begin{array}{c} 0.8e^{i2\pi(0.9)}, \\ 0.7e^{i2\pi(0.8)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.81e^{i2\pi(0.91)}, \\ 0.71e^{i2\pi(0.81)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.82e^{i2\pi(0.92)}, \\ 0.72e^{i2\pi(0.82)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.83e^{i2\pi(0.93)}, \\ 0.73e^{i2\pi(0.83)} \end{array} \right) \end{array} \right)$
$\mathbb{A}_{\boldsymbol{AL-2}}$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.7e^{i2\pi(0.8)}, \\ 0.8e^{i2\pi(0.9)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.71e^{i2\pi(0.81)}, \\ 0.81e^{i2\pi(0.91)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.72e^{i2\pi(0.82)}, \\ 0.82e^{i2\pi(0.92)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.73e^{i2\pi(0.83)}, \\ 0.83e^{i2\pi(0.93)} \end{array} \right) \end{array} \right)$
$\mathbb{A}_{\boldsymbol{AL-3}}$	$\left(\begin{array}{c} [\mathcal{L}_3,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.6e^{i2\pi(0.7)}, \\ 0.7e^{i2\pi(0.6)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_3,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.61e^{i2\pi(0.71)}, \\ 0.71e^{i2\pi(0.61)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.62e^{i2\pi(0.72)}, \\ 0.72e^{i2\pi(0.62)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.63e^{i2\pi(0.73)}, \\ 0.73e^{i2\pi(0.63)} \end{array} \right) \end{array} \right)$
$\mathbb{A}_{\boldsymbol{AL-4}}$	$\left(\begin{array}{c} [\mathcal{L}_0,\mathcal{L}_2], \\ \left(\begin{array}{c} 0.6e^{i2\pi(0.8)}, \\ 0.4e^{i2\pi(0.2)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0,\mathcal{L}_1], \\ \left(\begin{array}{c} 0.61e^{i2\pi(0.81)}, \\ 0.41e^{i2\pi(0.21)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.62e^{i2\pi(0.82)}, \\ 0.42e^{i2\pi(0.22)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_0,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.63e^{i2\pi(0.83)}, \\ 0.43e^{i2\pi(0.23)} \end{array} \right) \end{array} \right)$
$\mathbb{A}_{\boldsymbol{AL-5}}$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.7e^{i2\pi(0.5)}, \\ 0.8e^{i2\pi(0.9)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.71e^{i2\pi(0.51)}, \\ 0.81e^{i2\pi(0.91)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.72e^{i2\pi(0.52)}, \\ 0.82e^{i2\pi(0.92)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_2], \\ \left(\begin{array}{c} 0.73e^{i2\pi(0.53)}, \\ 0.83e^{i2\pi(0.93)} \end{array} \right) \end{array} \right)$
$\mathbb{A}_{\boldsymbol{AL-6}}$	$\left(\begin{array}{c} [\mathcal{L}_3,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.8e^{i2\pi(0.7)}, \\ 0.1e^{i2\pi(0.1)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_3,\mathcal{L}_4], \\ \left(\begin{array}{c} 0.81e^{i2\pi(0.71)}, \\ 0.11e^{i2\pi(0.11)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_2,\mathcal{L}_3], \\ \left(\begin{array}{c} 0.82e^{i2\pi(0.72)}, \\ 0.12e^{i2\pi(0.12)} \end{array} \right) \end{array} \right)$	$\left(\begin{array}{c} [\mathcal{L}_1,\mathcal{L}_5], \\ \left(\begin{array}{c} 0.83e^{i2\pi(0.73)}, \\ 0.13e^{i2\pi(0.13)} \end{array} \right) \end{array} \right)$

By using Equation (6), we examine the score values of the aggregated values with

$$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{1.9247}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{0.7078}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{1.6725}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{0.6890}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{1.4162}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{2.1921}$$

We rank all alternatives with $\mathbb{A}_{\mathbf{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$; the alternative $\mathbb{A}_{\boldsymbol{AL-6}}$ is found to be the best one.

From the above analysis (Tables 1–3), it is clear that the proposed approach is proficient and applicable, even for the special cases of CIULNs and CPULNs.

5.2. Comparative Analysis and Graphical Interpretations of the Proposed Approach

Additionally, a comparative analysis of the explored operator with some existing operators was done to examine the reliability and effectiveness of the proposed approach. The comparative analysis is based on information in Tables 1–3. The existing operators are as follows: Liu and Jin [21] explored the notion of aggregation operator based on intuitionistic uncertain linguistic variables. Lu and Wei [22] presented the theory of Pythagorean uncertain linguistic aggregation operators, and Liu et al. [23] explored the theory of q-rung orthopair uncertain linguistic aggregation operators.

The comparative results of Table 1 are shown in Table 4. From the above analysis, it is clear that the explored operator provides the same ranking result with the best alternative, $\mathbb{A}_{\boldsymbol{AL-1}}$. However, these existing operators of Liu and Jin [21], Lu and Wei [22], and Liu et al. [23] fail to make a decision.

Table 4. Analysis of the explored operator with existing operators in Table 1.

Methods	Score Values	Ranking Results
Liu and Jin [21]	*Cannot be Calculated*	*Failed*
Lu and Wei [22]	*Cannot be Calculated*	*Failed*
Liu et al. [23]	*Cannot be Calculated*	*Failed*
Proposed work for $q = 1$	$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{1.5512}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{0.5453}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{0.6663}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{-0.1070}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{0.9312}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{1.536}$	$\mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$
Proposed work for $q = 2$	$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{1.422}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{0.378}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{0.389}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{-0.246}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{0.411}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{1.418}$	$\mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$
Proposed work for $q = 3$	$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{1.412}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{0.349}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{0.354}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{-0.456}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{0.367}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{1.377}$	$\mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$

The comparative results of Table 2 are shown in Table 5. From the analysis, it is clear that the explored operator provides the same ranking result with the best alternative, $\mathbb{A}_{\boldsymbol{AL-6}}$. However, these existing operators of Liu and Jin [21], Lu and Wei [22], and Liu et al. [23] fail to make a decision.

Table 5. Comparative analysis of the explored operator with existing operators in Table 2.

Methods	Score Values	Ranking Results
Liu and Jin [21]	*Cannot be Calculated*	*Failed*
Lu and Wei [22]	*Cannot be Calculated*	*Failed*
Liu et al. [23]	*Cannot be Calculated*	*Failed*
Proposed work for $q = 1$	*Cannot be Calculated*	*Failed*
Proposed work for $q = 2$	$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{2.7371}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{2.7975}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{2.7324}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{0.9817}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{1.9137}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{2.9625}$	$\mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$
Proposed work for $q = 3$	$\mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-1}}) = \mathcal{L}_{2.867}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-2}}) = \mathcal{L}_{2.912}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-3}}) = \mathcal{L}_{2.845}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-4}}) = \mathcal{L}_{0.810}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-5}}) = \mathcal{L}_{1.811}, \mathbb{S}_{SF}(\mathbb{A}_{\boldsymbol{AL-6}}) = \mathcal{L}_{2.978}$	$\mathbb{A}_{\boldsymbol{AL-6}} \geq \mathbb{A}_{\boldsymbol{AL-2}} \geq \mathbb{A}_{\boldsymbol{AL-1}} \geq \mathbb{A}_{\boldsymbol{AL-3}} \geq \mathbb{A}_{\boldsymbol{AL-5}} \geq \mathbb{A}_{\boldsymbol{AL-4}}$

Furthermore, the comparative results of Table 3 are shown in Table 6. From the analysis, it is clear that the explored operator provides the same ranking result with the best alternative, $\mathbb{A}_{AL-6}$. However, these existing operators of Liu and Jin [21], Lu and Wei [22], and Liu et al. [23] fail to make a decision.

Table 6. Comparative analysis of the explored operator with existing operators in Table 3.

Methods	Score Values	Ranking Results
Liu and Jin [21]	*Cannot be Calculated*	*Failed*
Lu and Wei [22]	*Cannot be Calculated*	*Failed*
Liu et al. [23]	*Cannot be Calculated*	*Failed*
Proposed work for $q = 1$	*Cannot be Calculated*	*Failed*
Proposed work for $q = 2$	*Cannot be Calculated*	*Failed*
Proposed work for $q = 7$	$\mathbb{S}_{SF}(\mathbb{A}_{AL-1}) = \mathcal{L}_{1.9247}, \mathbb{S}_{SF}(\mathbb{A}_{AL-2}) = \mathcal{L}_{0.7078}, \mathbb{S}_{SF}(\mathbb{A}_{AL-3}) = \mathcal{L}_{1.6725}, \mathbb{S}_{SF}(\mathbb{A}_{AL-4}) = \mathcal{L}_{0.6890}, \mathbb{S}_{SF}(\mathbb{A}_{AL-5}) = \mathcal{L}_{1.4162}, \mathbb{S}_{SF}(\mathbb{A}_{AL-6}) = \mathcal{L}_{2.1921}$	$\mathbb{A}_{AL-6} \geq \mathbb{A}_{AL-1} \geq \mathbb{A}_{AL-3} \geq \mathbb{A}_{AL-5} \geq \mathbb{A}_{AL-2} \geq \mathbb{A}_{AL-4}$

We then use the graphical representations for Tables 4–6. The graphical representations for Tables 4–6 are shown in Figures 2–4, respectively. From these figures, it is clear that they contains six series, which express the family of alternatives with different colors. The best alternative in Figure 2 is $\mathbb{A}_{AL-1}$ and the best alternative in Figures 3 and 4 is $\mathbb{A}_{AL-6}$. It is seen that the proposed approach is superior and more effective than these existing methods [21–23].

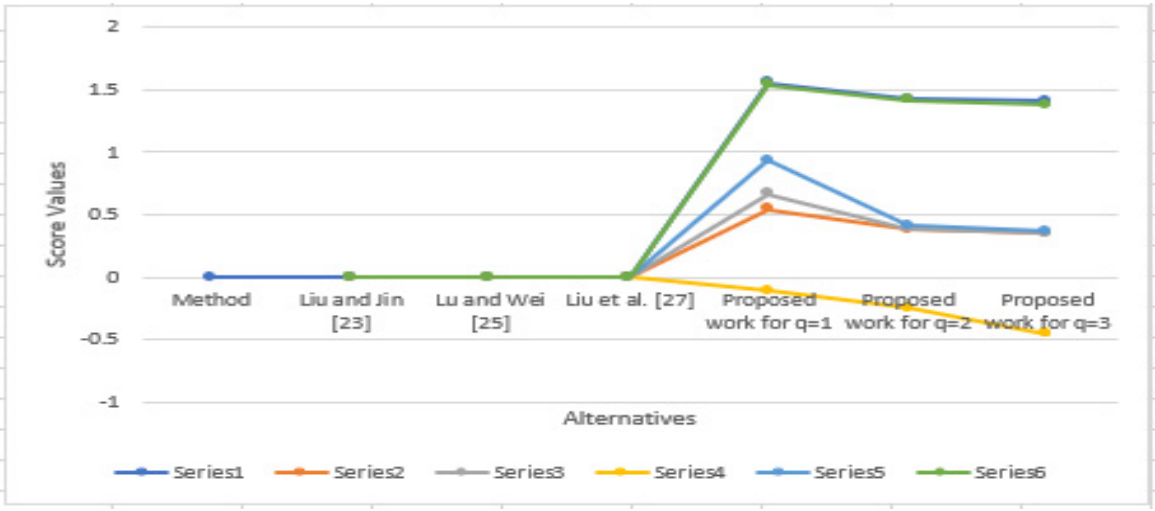


Figure 2. Graphical representation of the information in Table 4.

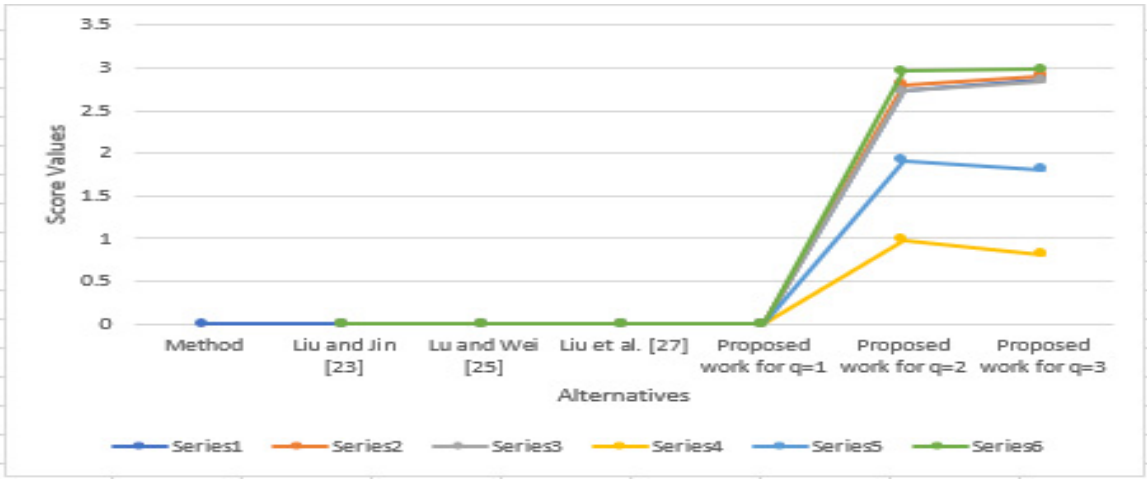


Figure 3. Graphical representation of the information in Table 5.

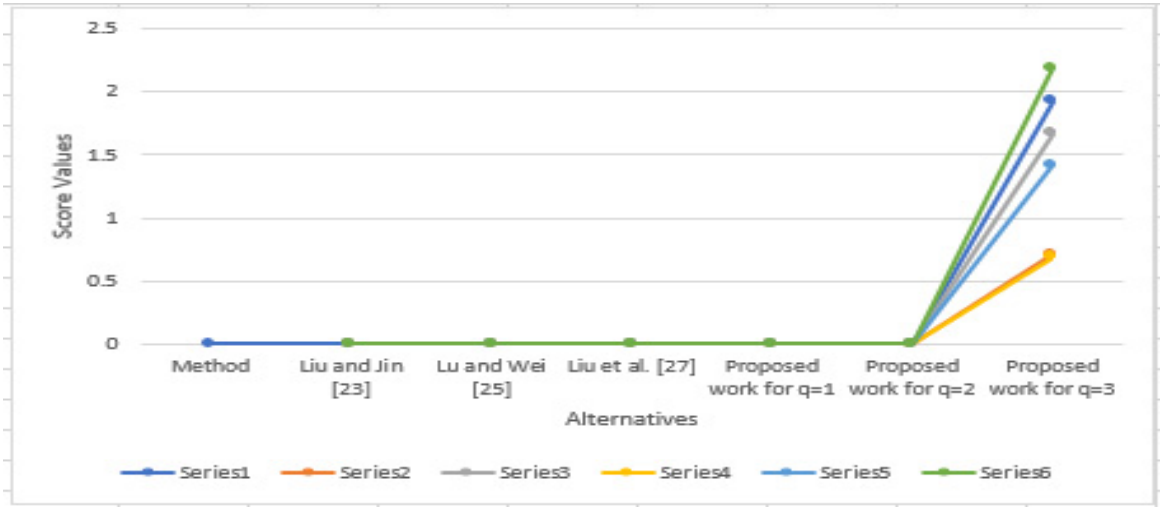


Figure 4. Graphical representation of the information in Table 6.

6. Conclusions

Since Yager [9] proposed q-rung orthopair fuzzy sets (QROFSs), QROFSs have been widely studied and applied in various areas, especially in MADM. To broaden the scope of QROFSs from real part in a real plane to real and imaginary parts in a complex plane, complex QROFSs are considered. On the other hand, uncertain linguistic sets are satisfactory in presenting the DMs' assessment data for handling MADM issues. Thus, complex q-rung orthopair uncertain linguistic sets (CQROULSs) are first proposed in the paper. Regarding MADM issues in which the assessed estimation of each trait is as CQROULS, the method of MADM based on CQROULSs is then created. Based on CQROULSs, another aggregation operator, called the partitioned Bonferroni mean (PBM) operator, is then used to manage the circumstances where the properties are parceled into various parts with interrelationships among different properties in the same part while the qualities in various parts are not related. To wipe out preposterous assessment estimations of characteristics, we stretched out the PBM operator to CQROULSs and then proposed the complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean (CQROULPBM) operator. Furthermore, we propose the complex q-rung orthopair uncertain linguistic weighted partitioned Bonferroni mean (CQROULWPBM) operator based on weighting. The properties of idempotency, boundedness, and commutativity for the CQROULWPBM operator are shown to hold. Finally, we present an MADM method based on CQROULSs with the CQROULWPBM op-

erator and apply it to an antivirus mask selection for the COVID-19 pandemic. We further analyze its advantages, and also make a comparative analysis with graphical interpretation. These comparative analysis and application results demonstrate the good aspects of the proposed CQROULSs and CQROULWPBM operator with effectiveness and usefulness in practice. In our future works, we will extend the proposed method to complex neutrosophic hesitant fuzzy sets [26], T-spherical fuzzy sets [27,28], and plithogenic hypersoft sets [29]. Furthermore, we will use the fuzzy symmetry concept [24,25] to the proposed CQROULSs and CQROULWPBM operator and then give more uncertain linguistic set applications in symmetry.

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