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An Application of the Kalman Filter Recursive Algorithm to Estimate the Gaussian Errors by Minimizing the Symmetric Loss Function

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Abstract: Kalman filtering is a linear quadratic estimation (LQE) algorithm that uses a time series of observed data to produce estimations of unknown variables. The Kalman filter (KF) concept is widely used in applied mathematics and signal processing. In this study, we developed a methodology for estimating Gaussian errors by minimizing the symmetric loss function. Relevant applications of the kinetic models are described at the end of the manuscript.

Keywords: unscented Kalman filter (UKF); extended Kalman filter (EKF); recursive algorithm; Gaussian errors; linear quadratic estimation (LQE); contraction property; symmetric loss function

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1. Introduction

Linear quadratic estimation (LQE) is a statistical method with many technological applications. The most common applications of LQE are for navigation, guidance, and the control of vehicles [1]. Other applications [2,3] are based on the lag between the issuing signals and receiving a response, in which the Kalman filter (KF) makes assessments of the condition of the framework and issues directions.

In general, a KF assumes that the noise is normally (Gaussian) distributed and the observation models are linear [4]. Furthermore, the filter gives the posterior conditional probability estimate in the case when the errors follow a Gaussian distribution.

Over the past few decades, generalizations of and extensions to the KF method have been established, such as the unscented Kalman filter (UKF) and the extended Kalman filter (EKF), which both work on nonlinear systems [5,6]. EKF is probably the most widely used estimation algorithm for nonlinear systems [7].

The KF uses a dynamic model, known control inputs, and multiple consecutive measurements to make an estimate of its varying quantities, which leads to a better estimation than if they were obtained by using one measurement alone [8–10].

The Kalman filter is a productive recursive filter that evaluates the inward condition of a direct powerful framework from a progression of noisy estimations. It is utilized in a wide scope of econometric and engineering applications ranging from radar and computer vision to estimations of basic macroeconomic models [11,12], and is an important tool in control system engineering and control framework design. Together with the linear quadratic regulator (LQR), the KF determines the solutions of a direct quadratic Gaussian control issue.

Other related works in the mathematics field have been carried out as well [13–15]. The most representative works were [16–18]. Some mathematical researchers [19,20] proposed the use of a KF for the elimination of noise. An interesting approach [21] uses the extended KF for linearizing the nonlinear systems while improving the applicability of these methods.

Anderson et al. [22] presented the theory and applications of Kalman filtering in a unified and organized fashion, which encompassed major contributions by a substantial number of researchers of this topic. Jazwinski [23] took the Bayesian point of view regarding filtering, while other economists [24,25] set a procedure for the frequency domain design of statically optimal filters.

Starting from what is already known in this area, the novelty of this study consists of using a recursive algorithm of the Kalman filter, the semigroup of the Hamiltonian matrices, and the contraction property to estimate the Gaussian errors. The main difference between our work and classical approaches, such as EKF and LQE, is that it handles the time-varying measurement uncertainty. For this, the covariance matching approach is used to update the observation covariance matrix.

The rest of this paper is organized as follows. The methodology and the formulation of the system equations are described in Section 2. Section 2 also describes the KF model, the general case of the KF, and the Gaussian case. The subsequent section, Section 3, describes the semigroup of the Hamiltonian matrices and the contraction property. The results, including the need for the KF-based techniques under different conditions for the numerical and experimental study, are presented in Section 3. Based on the observed results, some conclusions and areas for future work, as well as possible applications of the proposed methodology, are identified in the last section of the paper.

2. Materials and Methods

2.1. The Kalman Filter Model

The KF is based on the following mathematical theory. Consider $B_m = 1, 2, 3, \dots$ data observed from $p \times 1$ vectors. B_m is a sequence that is presumed to have the following distribution:

$$B_m = D_m + \varphi_m = Z_m A_m + \varphi_m,$$

where

- φ_m —white noise
- D_m —carrier signal.

In this regard, the dynamic representation is as follows:

$$D_m = Z_m A_m, m = 1, 2, 3 \dots$$

whose factors are defined below.

Next, we continue with an observed data model:

$$B_m = Z_m A_m + \varphi_m. \quad (1)$$

Equation (1) is related to the $p \times 1$ vector of B_m data points, with an $r \times 1$ A_m vector. In this case, Z_m is a $p \times r$ matrix, while φ_m is a $p \times 1$ vector with $E[\varphi_m] = 0$ and $\text{Var}[\varphi_m] = M_m$.

The vector A_m is not straightforwardly observable, but we could assume that it follows a Markov chain process of order 1, i.e.,

$$A_m = X_m A_{m-1} + \theta_m, m = 1, 2, 3, \dots \quad (2)$$

The states' behavior is described by Equation (2). In this case, X_m is an $r \times r$ matrix and θ_m is an $r \times 1$ error vector, which is not correlated with θ_m since $E[\theta_m] = 0$ and $\text{Var}[\theta_m] = P_m$.

A_0 , the initial vector, was assumed to have:

$$E[A_0] = x_0 \text{ and } \text{Var}[A_0] = Q_0. \quad (3)$$

Furthermore, we assumed that the error terms were not two-by-two correlated, i.e.,

$$E[\theta_n \varphi'_d] = 0 \text{ for all } n, d = 1, 2, 3, \dots \quad (4)$$

and

$$E[\theta_n A_{0'}] = 0 \text{ and } E[\varphi_n A_{0'}] = 0 \text{ for all } n = 1, 2, 3, \dots \quad (5)$$

We also assumed that the matrices X_m , M_m , P_m , and Z_m , as defined before, are not random matrices. The above two statements (1) and (2) reveal that A_m could be constructed as a linear combination of θ_d and φ_d , for $d \leq m - 1$ and A_0 for any m .

2.2. The General Case of the Kalman Filter

From the least-squares class of estimators, the KF is a recursive estimator.

The matrices X_m , M_m , P_m , and Z_m are presumed to be known for any m .

Consider a sequence Γ_m , which includes all the data observations recorded until moment m , inclusively.

Consider now an optimal estimator $x_{m|m}$ of A_m based on the data observations of Γ_m . The $r \times r$ matrix of the mean of the squared errors (EPM) of $x_{m|m}$ is:

$$Q_{m|m} = E[(A_m - x_{m|m})(A_m - x_{m|m})'], \quad (6)$$

where “ ’ ” stands for the transpose of a matrix.

Considering $x_{m-1|m-1}$ and $Q_{m-1|m-1}$, the estimators $x_{m|m-1}$ and $Q_{m|m-1}$ of A_m and $Q_{m|m}$, which are based on the data points of Γ_{m-1} , can be calculated from the prediction formulas:

$$x_{m|m-1} = X_m x_{m-1|m-1} + \theta_m, \text{ for } m = 1, 2, 3, \dots \quad (7)$$

and

$$\begin{aligned} Q_{m|m-1} &= E[(A_m - x_{m|m-1})(A_m - x_{m|m-1})'] \\ &= E[(X_m A_{m-1} + \theta_m - X_m x_{m-1|m-1})(X_m A_{m-1} + \theta_m - X_m x_{m-1|m-1})'] \\ &= E[(X_m (A_{m-1} - x_{m-1|m-1}) + \theta_m)(X_m (A_{m-1} - x_{m-1|m-1}) + \theta_m)'] \\ &= X_m E[(A_{m-1} - x_{m-1|m-1})(A_{m-1} - x_{m-1|m-1})'] X_m' + E[\theta_m \theta_m'], \end{aligned}$$

i.e.,

$$Q_{m|m-1} = X_m Q_{m-1|m-1} X_m' + P_m, \text{ for } m = 1, 2, 3, \dots \quad (8)$$

By (4) and (5), the average of the connected terms of the multiplication is zero.

Since A_{m-1} is a linear combination of the terms θ_d and φ_d , $d \leq m - 1$, and the A_0 term is also not correlated with φ_m .

The equations (7) and (8) above represent the KF. Given A_0 and Q_0 , the KF gives the optimal estimator of the state when new data points are accessible.

2.3. The Gaussian Case

Now, we consider the following system:

$$A_m = X_m A_{m-1} + \theta_m,$$

$$B_m = Z_m A_m + \varphi_m, m \geq 1, \quad (9)$$

where

- X_m and Z_m are deterministic $r \times r$ and $p \times r$ matrices, respectively, and
- $A_m \in \mathbb{R}^r$, $\theta_m \in \mathbb{R}^q$, φ_m , and $B_m \in \mathbb{R}^p$ are random variables.

Now, we assume that the pair of vectors (θ_m, φ_m) , $m \geq 1$, are independent and have a Gaussian distribution with an average of 0 and a covariance matrix equal to J , where J is the unit matrix and A_0 has a Gaussian distribution with an average of a_0 and a covariance matrix Q_0 . We then assume that the determinant of X_m is not zero, $m \geq 1$.

For any $m \geq 1$, let an N_m σ -algebra be generated by $B_1, \dots, B_m$ [26] and

$$\hat{A}_m = E[A_m | N_m], \quad (10)$$

$$Q_m = E((A_m - \hat{A}_m)(A_m - \hat{A}_m)' | N_m). \quad (11)$$

If N_m is known, $\hat{A}_m$ is the best estimator of A_m and Q_m is the conditional variance–covariance matrix of the errors.

Now, we consider $\mathbf{Q}$, respectively Q_0 , which is the set of symmetric matrices that are non-negatively defined (respectively positively defined) and of $r \times r$ dimensions. For any $Q \in \mathbf{Q}$ and $m \geq 1$, we then define:

$$\xi_m(Q) = (X_m Q X_m' + J)(J + S_m + S_m X_m Q X_m')^{-1}, \quad (12)$$

where $S_m = Z_m' Z_m$ and J is the unit matrix of order r . It follows that ξ_m takes P_0 into itself.

The basic result shown above is related to the fact that the applications ξ_m , $m \geq 1$, are contractions in Q_0 if we enhance Q_0 with the Riemann measurement δ [27], which is invariant to conjugacy. Furthermore, the contractions could be strict or uniform, depending on the controllability and observability properties of the above system (9).

To demonstrate the results in this view, we considered the symplectic group of matrices. The matrices are invariant to P_0 . In this view, they could be considered a generalization of the Perron–Frobenius theorem. In some sense, the study of Bougerol [28] is elementary.

The symplectic matrices utilize some of the already existing ideas [29]. The semigroup Σ was induced by Wojtkowsky [30] in a similar context. Finally, we note that the Riccati transformation [31] has a contractibility property, which is a generalization of the positive matrices under a Hilbert space metric [32].

3. Applications to Kinetic Models

3.1. A Recursive KF Algorithm Application

We considered an object moving along a straight line. Its movements are captured by a video camera. The object is moving along a straight line, with random acceleration a_t , and Δ_t is the time interval. Its location is described by the x -coordinate. We also assumed that the acceleration has a Gaussian distribution with zero mean and constant variance (σ_a^2). The position of the object, denoted as y_t , is also noisy (Gaussian with zero mean and constant variance). We denote the variance as σ_y^2 .

The state vector of this motion is given by $v_t = (x_t, v_t)$, where the speed is the first derivative of the distance, i.e., $v_t = \dot{x}_t$.

We also assumed that there was no process control and get:

$$x_t = \begin{pmatrix} 1 & \Delta_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ v_{t-1} \end{pmatrix} + a_t \begin{pmatrix} \frac{\Delta_t^2}{2} \\ \Delta_t \end{pmatrix} = F_t x_{t-1} + w_t, \quad Q_t = \text{var}(w_t), \quad (13)$$

where

- F_t is a known state transition matrix applied to the $t - 1$ state x_{t-1} ,
- Q_t is the variance matrix, and
- w_t is a process noise vector, which has the joint distribution as a multivariate Gaussian with a variance matrix Q_t and $\mu_{i,t} = E[w_{i,t}] = 0$ for $i = \overline{1, n}$.

Now, let $G_t = \left(\frac{\Delta_t^2}{2}, \Delta_t \right)^T$. Then, we get:

$$Q_t = E[w_t, w_t^T] = G_t E[a_t^2] G_t^T = \sigma_a^2 G_t G_t^T = \sigma_a^2 \begin{bmatrix} \frac{\Delta_t^4}{4} & \frac{\Delta_t^3}{2} \\ \frac{\Delta_t^3}{2} & \Delta_t^2 \end{bmatrix}. \quad (14)$$

This means that F_t , Q_t , and G_t are independent of t . Hence, we will call them just F , Q , and G .

To measure the object's position at time t , we get:

$$y_t = \begin{pmatrix} v_t \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_t = v_t + H_t \cdot x_t, \quad (15)$$

where

- v_t is the observation noise and
- H_t is an observation matrix.

Hence, the variance matrix is:

$$R = E[v_t v_t^T] = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (16)$$

We denote $\hat{x}_{t|t}$ as the estimate of state x_t at time t and $P_{t|t}$ as the variance matrix of the error $x_t - \hat{x}_{t|t}$. Our goal was to minimize $P_{t|t}$.

The initial position of the object is $\hat{x}_{0|0} = (0, 0)^T$. If this position vector is known accurately, then the variance matrix is zero, i.e.,

$$P_{0|0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

otherwise, we will have that:

$$P_{0|0} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where a is real and suitably large, $a > 0$.

When $t = 1$, we first predicted $\hat{x}_{1|0}$ and computed its variance matrix $P_{1|0}$ by using the prediction equations:

$$\hat{x}_{t|t-1} = F_t \hat{x}_{t-1|t-1} + B_t u_t = F \hat{x}_{t-1|t-1}, \quad (17)$$

$$P_{t|t-1} = F_t P_{t-1|t-1} F_t^T + Q_t = F P_{t-1|t-1} F^T + Q_t = F P_{t-1|t-1} + Q, \quad (18)$$

where B_t is the control matrix, which is applied to a control vector u_t .

We then calculated the auxiliary data S_1 and z_1 from the equations:

$$\tilde{z}_t = y_t - H_t \hat{x}_{t|t-1} = y_t - H \hat{x}_{t|t-1}, \quad (19)$$

$$S_t = H_t P_{t|t-1} H_t^T + R_t = H P_{t|t-1} H^T + R, \quad (20)$$

where R_t is the variance matrix of a multivariate Gaussian distribution vector v_t .

Then, we could compute the optimal Kalman gain K_1 and the position vector $\hat{x}_{1|1}$ using the equations:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t \tilde{z}_t, \quad (21)$$

$$K_t = P_{t|t} H_t^T S_t^{-1} = P_{t|t-1} H^T S_t^{-1}. \quad (22)$$

Finally, we could compute $P_{1|1}$ to prepare for time $t = 2$ by using the following equation:

$$P_{t|t} = (I - K_t H_t) P_{t|t-1} = (I - K_t H) P_{t|t-1}, \quad (23)$$

which minimizes the variance matrix $P_{t|t}$.

3.2. EKF Recursive Algorithm Application

We extended our previous example of an object moving along a straight line by adding the orientation of the vehicle θ to the position vector (x, y) . Then, we estimated the state vector p , i.e., $p = (x, y, \theta)$.

The kinematic model is given by the following system:

$$\begin{cases} x_t = x_{t-1} + v_t \cdot \Delta_t \cdot \cos\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right), \\ y_t = y_{t-1} + v_t \cdot \Delta_t \cdot \sin\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right), \\ \theta_t = \theta_{t-1} + w_t \cdot \Delta_t, \end{cases} \quad (24)$$

where Δ_t is the time interval, while v and w are the speed and the yaw rate of the vehicle. We also denote $\hat{v}$ and $\hat{w}$ as the speed and yaw rate measurements. We assumed that their errors followed the Gaussian distribution as $\Delta v_t \sim N(0, \sigma_{v_t}^2)$ and $\Delta w_t \sim N(0, \sigma_{w_t}^2)$.

Now, suppose that the vehicle is equipped with a GPS, and the GPS measurements are denoted by vector z with a measurement model given by:

$$z_t = \mathbf{H}p_t + \gamma_t, \mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (25)$$

where $\gamma \sim N(0, \sigma_\gamma^2)$ is the measurement error that follows a Gaussian distribution with zero mean and variance σ_γ^2 .

This is an EKF model, which could be locally linearized as follows:

$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} \approx \begin{pmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \end{pmatrix} + \mathbf{A}(p_{t-1}, u_t) \begin{pmatrix} \Delta x_{t-1} \\ \Delta y_{t-1} \\ \Delta \theta_{t-1} \end{pmatrix} + \mathbf{B}(p_{t-1}, u_t) \begin{pmatrix} \Delta v_t \\ \Delta w_t \end{pmatrix},$$

$$\begin{cases} \bar{x}_t = x_{t-1} + v_t \cdot \Delta_t \cdot \cos\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right), \\ \bar{y}_t = y_{t-1} + v_t \cdot \Delta_t \cdot \sin\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right), \\ \bar{\theta}_t = \theta_{t-1} + w_t \cdot \Delta_t, \end{cases}$$

$$\mathbf{A}(p_{t-1}, u_t) = \begin{pmatrix} 1 & 0 & -v_t \cdot \Delta_t \cdot \sin\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) \\ 0 & 1 & v_t \cdot \Delta_t \cdot \cos\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B}(p_{t-1}, u_t) = \begin{pmatrix} \Delta t \cdot \cos\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) & -v_t \cdot \Delta t^2 \cdot \sin\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) \\ \Delta t \cdot \sin\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) & v_t \cdot \Delta t^2 \cdot \cos\left(\theta_{t-1} + w_t \cdot \frac{\Delta_t}{2}\right) \\ 0 & \Delta t \end{pmatrix}, \quad (26)$$

where $u_t = (v_t, w_t)$.

As we can see, the matrices $\mathbf{A}(p_{t-1}, u_t)$ and $\mathbf{B}(p_{t-1}, u_t)$ of the state evolution function are Jacobian with respect to p_{t-1} and u_t , respectively.

Hence, we have a locally linearized EKF model. The prediction of the state vector is:

$$\begin{pmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \end{pmatrix} = \begin{pmatrix} \hat{x}_{t-1} + \hat{v}_t \cdot \Delta_t \cdot \cos\left(\hat{\theta}_{t-1} + \hat{w}_t \cdot \frac{\Delta_t}{2}\right) \\ \hat{y}_{t-1} + \hat{v}_t \cdot \Delta_t \cdot \sin\left(\hat{\theta}_{t-1} + \hat{w}_t \cdot \frac{\Delta_t}{2}\right) \\ \hat{\theta}_{t-1} + \hat{w}_t \cdot \Delta_t \end{pmatrix}. \quad (27)$$

3.3. Simulation Using Synthetic Data

We tested the KF performance using synthetic data generated using model (24) and the measurement model (25). Now let:

$$\Delta t = 1, \sigma_w^2 = 0.02^2 \text{ rad}^2/\text{s}^2, \sigma_v^2 = 0.5^2 \text{ m}^2/\text{s}^2, \sigma_\gamma^2 = \text{diag}(10^2, 10^2) \text{ m}^2.$$

Now, we set the ground truth:

$$p_0 = 0, 0 \text{ m}, -\frac{\pi}{2} \text{ rad}, w_t = 0.04 \text{ rad} \cdot \text{s}, v_t = 10 \text{ m/s}.$$

The yaw rate measurements and the speed measurements were synthesized according to $\hat{w}_t \sim N(w_t, \sigma_w^2)$ and $\hat{v}_t \sim N(v_t, \sigma_v^2)$.

The GPS measurements were synthesized according to $\hat{z}_t \sim N(p_t, \sigma_\gamma^2)$.

Then, the KF was applied to the synthetic data and estimates of the state of the vehicle were obtained. The results can be seen in Figure 1, in which:

- The red line represents the estimated vehicle trajectory.
- The black line stands for the ground truth of the vehicle trajectory.
- The blue crosses are the GPS measurements of the position of the vehicle.

Figure 1. Estimates and position measurements for one trial.

As we can see from the above figure, the estimated trajectory of the vehicle was smoother than the jumping measurements. Figure 2 shows the results of 100 Monte Carlo trials. As we can observe, the measurement errors were larger than the estimated errors, just like in the previous example.

We also calculated the orientation estimate errors and the results of the same 100 Monte Carlo simulations are shown in Figure 3. As we can observe, these errors varied around 0.05 radians.

Hence, we concluded that, without using estimations in this application, we could not have any estimate of the vehicle's orientation except a random guess, which could be as large as $\pi/2$ radians.

This last example shows another important utility of the estimation, i.e., a proper estimation of the Gaussian errors could reveal the state part, which is not directly measurable.

Figure 2. Measurement errors and position estimates of 100 Monte Carlo trials.

Figure 3. Orientation estimate errors for 100 Monte Carlo trials.

4. Discussion and Conclusions

The Kalman filter is an adequate recursive filter that evaluates the internal capacity of a linear dynamic system (LDS) using a series of error estimates. Possible applications of the KF are in economics, biology, and the environmental sciences, and it is an important topic in control systems engineering and control theory. Together with the LQR, the KF solves the linear quadratic Gaussian (LQG) control problem. The linear quadratic Gaussian controller, linear quadratic regulator, and Kalman filter are solutions to some of the most important problems in control theory.

In many applications, the few “detectable” measured parameters are smaller than the internal state. In this situation, the Kalman filter could estimate the entire internal state by combining a series of measurements.

The Kalman filter could be adapted to a wide variety of economic applications. It is mainly an ordinal least-square (OLS) procedure that gives the minimum mean square estimators (MMSE) under the Gaussian assumption. Even when the normality assumption

is waived, the Kalman filter minimizes any symmetric loss function. The KF is not only used directly in economic problems but it could also be used as a background to other estimation techniques, such as the estimation of Markov switching models or the quasi-maximum likelihood estimation procedure.

In the environmental sciences, the Kalman filter has wide applications in the areas of the pollution of aquatic systems, measuring atmospheric pollution, environmental policy management, and forecasting hurricanes [33,34]. The KF could also be used to estimate the temporal variations in associations between the radial growth of black oak and physiologically based climate indices [35,36]. In biology, biological samples and bacteriological infections could be detected using a multi-process Kalman filter to provide an online estimate of the inflection point of changes in the growth of an organism [37].

A wide collection of Kalman filters have been developed in the past few decades from the original formulation of the Kalman filter. The information filter, Schmidt's extended filter, or the Kalman–Bucy filter are a wide variety of filters that were developed by Thornton, Bierman, and many others. Probably the most used type of Kalman filter is the phase-locked loop (PLL), which is now omnipresent in television sets, radios, satellite communications receivers, and almost any other communications equipment. In this paper, we presented the KF model, the general case of the KF, and the Gaussian case. The semigroup of the Hamiltonian matrices and contraction property were used in the KF algorithm to estimate the Gaussian errors.

The main result of this study was the use of a recursive algorithm of the Kalman filter to estimate the Gaussian errors.

Several possible variations of our approach exist. Model herring inflow as a state variable and model herring inflow as pure noise with a non-zero mean or model noise as pure white (non-geometric) noise assume perfect observations of the control variables (catch and inflow). These variations were all tested and led to only small changes in the recursive algorithm fit and small variations in the parameter estimates.

The main limitation of this study came from the fact that there were not many applications of the algorithms described in the previous sections. Future work, as well as possible applications of the methodology, could be applied in mathematics, physics, and engineering research areas. Moreover, a future research topic is how to treat more general noise terms, for example, terms including unknown parameters.

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