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Geometry of k -Yamabe Solitons on Euclidean Spaces and Its Applications to Concurrent Vector Fields

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Abstract: In this paper, we give some classifications of the k -Yamabe solitons on the hypersurfaces of the Euclidean spaces from the vector field point of view. In several results on k -Yamabe solitons with a concurrent vector field on submanifolds in Riemannian manifolds, is proved that a k -Yamabe soliton (M^n, g, v^T, λ) on a hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is contained either in a hypersphere or a hyperplane. We provide an example to support this study and all of the results in this paper can be implemented to Yamabe solitons for k -curvature with $k = 1$.

Keywords: concurrent vector fields; k -Yamabe solitons; hypersurface; Euclidean spaces

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1. Introduction and Motivations

The geometric flows is an interesting tool in geometric analysis from the singularities point of view. In this respect, we can study an extension of the scalar curvature R which is a k -curvature. If $k = 1$, then k -curvature is a usual scalar curvature R . This concept is defined in [1] between the Schouten tensor $A_g = \frac{1}{n-2} \left(Ric - \frac{R}{2(n-1)} g \right)$ and the Weyl tensor W , exists the following relation: $Rm = A_g \odot g + W$. Here the symbol $\odot$ denotes the Kulkarni-Nomizu product. Therefore, the metric g associated with k -curvature is connected as

$$\sigma_k(g) = \sigma_k \left(\mu(g^{-1}A_g) \right) = \sum_{i_1 < \dots < i_k} \mu_{i_1} \cdots \mu_{i_k}, \quad \text{for } 1 \leq k \leq n,$$

where $\mu(g^{-1}A_g) = (\mu_1 \cdots \mu_n)$ is a collection eigenvalue of $g^{-1}A_g$ and $\sigma_k(\mu) = \sum_{i_1 < \dots < i_k} \mu_{i_1} \cdots \mu_{i_k}$. In this case, for a locally conformally flat manifold (M^n, g) , we take into account the geometric flow

$$\frac{d}{dt}g(t) = - \left(\log \sigma_k(g(t)) - \log r_k(g(t)) \right) g(t) \quad \text{and} \quad g(0) = g_0 \quad (1)$$

such that $r_k(g(t)) = \exp \left(\frac{1}{\text{vol}(g(t))} \int \log \sigma_k(g(t)) d\text{vol}_{g(t)} \right)$ and $\sigma_k(g(t))$ is positive. If the flow (1) has a self-similar solution, then it is called k -Yamabe soliton and it is defined as follows:

$$\mathcal{L}_X g = 2 \left(\log \sigma_k(g) - \lambda \right) g \quad (2)$$

where λ is a constant and X is a vector field. The k -Yamabe solitons can be classified as expanding, steady or shrinking according to the following values of λ : $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. If X is a gradient vector field $X = \nabla f$, then (2) is equivalent to the following

$$\nabla^2 f = (\log \sigma_k(g) - \lambda)g. \quad (3)$$

In this case, (M^n, g) is recognized as a gradient of the k -Yamabe soliton. For more details, please see [2,3]. It is exciting to know that the concurrent vector field plays a crucial role in classified Riemannian geometry. For example, Chen-Deshmukh in [4] gave a classification of the Ricci soliton associated with the concurrent vector field and also, they proved many new and important results. They proved that every Ricci soliton (M^n, g, v, λ) associated with a concurrent potential vector field v is a gradient and also that any Ricci soliton (M^n, g, v^T, λ) on a totally umbilical submanifold is a trivial Ricci soliton. From the viewpoint of the vector fields on a hypersurface, Chen-Deshmukh [5] have classified the Ricci solitons on some Euclidean hypersurfaces. In the same pattern, Yamabe soliton is considered a very influential soliton type. Therefore, k -Yamabe soliton has also become useful in the geometric analysis because it is a generalization of Yamabe soliton for $k = 1$. For instance, in [2], the authors acknowledged the full nonlinear Yamabe flow. They proved that on a closed locally conformally flat manifold, if the k -curvature is positive for the k -Yamabe soliton, it must have constant σ_k -curvature. It was shown that if the first nonzero eigenvalue μ_1 satisfied $\mu_1 \leq \frac{R}{n-1}$ for a compact k -Yamabe soliton having constant scalar curvature, then k -Yamabe soliton is trivial [2]. More triviality results can be found in [3], that is, every compact gradient k -Yamabe soliton must have constant k -curvature and certain conditions over the gradient. On the other hand, Yamabe solitons and quasi-Yamabe solitons with concurrent vector fields are discussed in [6] and also in a great number of good results in [7–18]. Motivated by some previous results regarding the classification of the theory of solitons geometry; we shall study some geometric classifications notes for k -Yamabe solitons on Euclidean hypersurfaces, if it is a potential field, originated from their position vector fields.

2. Background and Notations

Assuming M^n is isometrically immersed into a Riemannian manifold $\tilde{M}^m$ among induced connections $\nabla^\perp$ and ∇ on the normal bundle $T^\perp M$ and the tangent bundle TM of M^n , in the same order, then the Weingarten and Gauss formulae are

$$\tilde{\nabla}_{W_1} \xi = -A_\xi W_1 + \nabla_{W_1}^\perp \xi \quad (4)$$

$$\tilde{\nabla}_{W_1} W_2 = \nabla_{W_1} W_2 + h(W_1, W_2) \quad (5)$$

for each $W_1, W_2 \in \mathfrak{X}(TM)$ and $\xi \in \mathfrak{X}(T^\perp M)$. Furthermore, A_ξ and h are the shape operator and respectively the second fundamental form for an embedding of M^n into $\tilde{M}^m$. The relation between both of them can be given as follows:

$$g(h(W_1, W_2), \xi) = g(A_\xi W_1, W_2).$$

The Gauss equation for a submanifold M^n is given by:

$$\tilde{R}(W_1, W_2, Y_1, Y_2) = R(W_1, W_2, Y_1, Y_2) + g(h(W_1, Y_1), h(W_2, Y_2)) - g(h(W_1, Y_2), h(W_2, Y_1)) \quad (6)$$

for any $W_1, W_2, Y_1, Y_2 \in \mathfrak{X}(TM)$, where the curvature tensors on $\tilde{M}^m$ and M^n are denoted by $\tilde{R}$ and R , respectively. Now we give some classifications of submanifold M^n as follows:

- (i) The mean curvature $\mathbf{H}$ of M^n in $\tilde{M}^m$ is expressed as $\mathbf{H} = \frac{1}{n} \text{trace}(h)$. If $\mathbf{H} = 0$, then M^n is minimal in $\tilde{M}^m$ [19].
- (ii) If for h , holds the following relation $h(X_1, X_2) = g(X_1, X_2)\mathbf{H}$, then M^n is totally umbilical. It is referred to be totally geodesic when $h = 0$.

- (iii) Let the shape operator be endowed with the eigenvalue of multiplicity δ . If this condition holds, a hypersurface of $(n + 1)$ -dimensional $\mathbb{R}^{n+1}$ is said to be a quasisumbilical hypersurface. On the subset μ of M^n such that $\text{mult}(\delta) = n - 1$, a characterized direction of an quasisumbilical hypersurface has an eigenvector with the eigenvalue of multiplicity one.
- (iv) Let M be a smooth n -dimensional manifold. A smooth map $\psi : M \rightarrow \mathbb{R}^{n+1}$ is a hypersurface (an immersion) if its differential is injective. It is an embedding if it is also a homeomorphism onto its image $\psi(M)$. In this case, it is called orientation hypersurface of M [20].

The gradient and the Hessian positive function ψ defined on M^n can be written as follows:

$$\nabla\psi = \sum_{i=1}^n e_i(\psi)e_i, \quad \text{and} \quad \|\nabla\psi\|^2 = \sum_{i=1}^n ((\psi)e_i)^2. \quad (7)$$

$$H^\psi(e_i, e_j) = e_i(e_j\psi) - (\nabla_{e_i}e_j)\psi. \quad (8)$$

If $\Psi : (M, g) \rightarrow \mathbb{R}^m$ is an isometric embedding from the Riemannian submanifold M^n with $\dim M = n$ into an Euclidean space $\mathbb{R}^m$ with $\dim \mathbb{R}^m = m$, then the components of a position vector v of M^n in $\mathbb{R}^m$ are decomposed as follows:

$$v = v^T + v^\xi \quad (9)$$

where v^T and v^ξ are tangential and respectively, the normal components of v . Another interesting theorem that called the Hodge-de Rham decomposition theorem [21]. It is stated

Theorem 1 ([21]). *Any vector field X a compact oriented Riemannian manifold M can be decomposed as the sum of a divergence free vector field Y and the gradient of a function ω such that*

$$X = \nabla\omega + Y \quad (10)$$

where $\text{div}Y = 0$. and ω is the Hodge-de Rham potential function.

The relation between Lie derivatives of X and Y is defined as

$$\frac{1}{2}\mathcal{L}_Xg = \nabla^2\omega + \frac{1}{2}\mathcal{L}_Yg. \quad (11)$$

3. Main Results

Now we are going to give our new results.

Theorem 2. *A submanifold (M^n, g) of an Euclidean space $\mathbb{R}^m$ is a k -Yamabe soliton endowed with v^T as its soliton vector, if and only if h satisfies:*

$$\tilde{g}(h(W_1, W_2), v^\xi) = (\log \sigma_k(g) - \lambda - 1)g(W_1, W_2) \quad (12)$$

for any $W_1, W_2 \in \mathfrak{X}(M)$.

Proof. The position vector v of the manifold M^n from $\mathbb{R}^m$ is a concurrent vector field, therefore it satisfies

$$\tilde{\nabla}_{W_1}v = W_1. \quad (13)$$

Using (9) in the above equation, we have

$$\tilde{\nabla}_{W_1}v^T + \tilde{\nabla}_{W_1}v^\xi = W_1.$$

Now making use of the Weingarten (4) and the Gauss (5) formulas in the above equation, one obtains:

$$\nabla_{W_1} v^T + h(W_1, v^T) - A_{v^\xi} W_1 + \nabla_{W_1}^\perp v^\xi = W_1. \quad (14)$$

Equating the normal and tangential components in the previous equations, we find that:

$$h(W_1, v^T) = -\nabla_{W_1}^\perp v^\xi \quad \text{and} \quad \nabla_{W_1} v^T = A_{v^\xi} W_1 + W_1. \quad (15)$$

On the other hand, applying Lie derivative definition, one obtains:

$$(\mathcal{L}_{v^T} g)(W_1, W_2) = g(\nabla_{W_1} v^T, W_2) + g(\nabla_{W_2} v^T, W_1). \quad (16)$$

From (2) and (14), it implies that

$$\begin{aligned} 2(\log \sigma_k(g) - \lambda)g(W_1, W_2) &= g(W_1, W_2) + g(A_{v^\xi} W_1, W_2) \\ &\quad + g(A_{v^\xi} W_2, W_1) + g(W_1, W_2). \end{aligned}$$

The above equation is equivalent with:

$$2(\log \sigma_k(g) - \lambda)g(W_1, W_2) = 2\tilde{g}(W_1, W_2) + 2\tilde{g}(h(W_1, W_2), v^\xi) \quad (17)$$

for any W_1, W_2 which are tangent to M^n . From Equations (2) and (17), we find that M^n is a k -Yamabe soliton with soliton vector v^T if and only if condition (12) is satisfied. $\square$

The following result is a consequences of Theorem 2.

Corollary 1. *An isometric embedding $\Psi : M^n \rightarrow \mathbb{S}^{m-1} \subset \mathbb{R}^m$ of M^n into the hypersurface $\mathbb{S}^{m-1}$ with center o and radius r , is a k -Yamabe soliton. Here, v^T is a soliton vector field if and only if σ_k -curvature $\log \sigma_k(g)$ of (M^n, g) is constant.*

Proof. For an isometric embedding $\Psi : M^n \rightarrow \mathbb{S}^{m-1} \subset \mathbb{R}^m$ of M^n into the $\mathbb{S}^{m-1}$, we have $v^\xi = v$. Here, the second fundamental form h of M^n in $\mathbb{S}^{m-1}$ holds

$$h(W_1, W_2) = h'(W_1, W_2) - \frac{g(W_1, W_2)}{r^2}v \quad (18)$$

which follows from the Lemma 3.5 in [22]. Taking the inner product with v^ξ in (18), one obtains

$$\tilde{g}(h(W_1, W_2), v^\xi) = \tilde{g}(h'(W_1, W_2), v^\xi) - \frac{g(W_1, W_2)}{r^2}g(v, v^\xi)$$

which implies that because $v = v^\xi$, one obtains:

$$\tilde{g}(h(W_1, W_2), v^\xi) = -g(W_1, W_2). \quad (19)$$

Therefore, the Equation (12) is satisfied if and only if $\log \sigma_k(g) - \lambda = 0$ holds. This conclude that σ_k -curvature $\log \sigma_k(g)$ is constant as λ is constant. $\square$

Theorem 3. *Any k -Yamabe soliton (M, g, v^T, λ) on hypersurfaces in $\mathbb{R}^{n+1}$ is contained either in a hypersphere or a hyperplane.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on M^n . Assuming that φ is any support function on M^n and denoting by α the mean curvature, then $\varphi = \tilde{g}(\xi, v)$ for any unit normal vector field ξ and a position vector v , and $\mathbf{H} = \alpha\xi$. From Theorem 2, we have

$$\begin{aligned} (\log \sigma_k(g) - \lambda - 1)g_{ij} &= \tilde{g}(h(e_i, e_j), v^\xi) = \tilde{g}(g(A_\xi(e_i), e_j)\xi, v) \\ &= \tilde{g}(\kappa_i g_{ij}\xi, v) \\ &= \kappa_i g_{ij}\varphi, \end{aligned}$$

such that $h(e_i, e_j) = g(A_\xi(e_i), e_j)\xi = \kappa_i g_{ij}\xi$, where $A_\xi(e_i) = \kappa_i e_i$ ($i = 1, \dots, n$), and κ_i is a principle curvature. Therefore, from the above equation, we get:

$$n(\log \sigma_k(g) - \lambda - 1) = \kappa_i \varphi. \quad (20)$$

Tracing the above equation, we finally get

$$\log \sigma_k(g) - \lambda - 1 = \varphi \alpha. \quad (21)$$

Combining (20) and (21), we get:

$$\kappa_i = \frac{1}{n} \alpha. \quad (22)$$

This implies that M^n is totally umbilical submanifold and hence h satisfies $h(W_1, W_2) = \frac{1}{n} \alpha g(W_1, W_2)\xi$. Now, let $0 = \tilde{\nabla}_{W_1} \tilde{g}(\xi, \xi) = 2\tilde{g}(\nabla_{W_1} \xi, \xi) = 2\tilde{g}(\nabla_{W_1}^\perp \xi, \xi)$. This shows that the normal connection $\nabla_{W_1}^\perp \xi$ is flat, i.e., $\nabla_{W_1}^\perp \xi = 0$ and hence ξ is constant. Then, the covariant derivative $\tilde{\nabla}_{W_1} h$ is given by

$$\begin{aligned} (\tilde{\nabla}_{W_1} h)(W_2, W_3) &= \nabla_{W_1}^\perp h(W_2, W_3) - h(\nabla_{W_1} W_2, W_3) - h(\nabla_{W_1} W_3, W_2) \\ &= \frac{1}{n} W_1(\alpha) g(W_2, W_3) \xi \end{aligned} \quad (23)$$

for any vector fields W_1, W_2, W_3 are tangent to M^n . From the Codazzi equation and the fact that $R^\perp$ is flat, one obtains:

$$(\tilde{\nabla}_{W_1} h)(W_2, W_3) = (\tilde{\nabla}_{W_2} h)(W_1, W_3). \quad (24)$$

From (23) and (24), we found that $W_1(\alpha)W_2 = W_2(\alpha)W_1$. This means that W_1 and W_2 are linearly independents and hence α is a constant. First, we consider $\alpha = 0$, then using the fact that $\tilde{\nabla}_{W_1} \xi = -A_\xi(W_1) = -\frac{\alpha}{n} W_1 = 0$, ξ is restricted to M^n , and α is constant in $\mathbb{R}^n$, thus we define

$$\begin{aligned} \tilde{\nabla}_{W_1} \tilde{g}(v, \xi) &= \tilde{g}(\tilde{\nabla}_{W_1} v, \xi) + \tilde{g}(v, \tilde{\nabla}_{W_1} \xi) \\ &= \tilde{g}(W_1, \xi) \\ &= 0. \end{aligned}$$

We conclude that $\tilde{g}(v, \xi)$ is constant in $\mathbb{R}^{n+1}$ such that v and ξ are defined on M^n . Hence, M^n is contained in hyper-plane perpendicular to ξ . On the other hand, if $\alpha \neq 0$, then we define

$$\begin{aligned} \tilde{\nabla}_{W_1}(v + n\alpha^{-1}\xi) &= \tilde{\nabla}_{W_1} v + n\alpha^{-1}\tilde{\nabla}_{W_1} \xi \\ &= W_1 + n\alpha^{-1}(-A_{W_1}(\xi)) \\ &= W_1 - W_1 \\ &= 0. \end{aligned}$$

Therefore, the vector field $v + n\alpha^{-1}\xi$ is restricted to M^n and degenerates to be a constant in $\mathbb{R}^{n+1}$. This shows that M^n is contained in the hypersphere. The further part follows from [18,23]. This completes the proof of the theorem. $\square$

Another interesting result is the following one:

Theorem 4. *If a k -Yamabe soliton (M^n, g, v^T, λ) on a submanifold M^n of a Riemannian manifold $\tilde{M}$ is minimal, then $\log \sigma_k(g)$ is constant.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on M^n .

$$(\log \sigma_k(g) - \lambda - 1)g_{ij} = g(A_{v^\xi}(e_i), e_j).$$

From the Equation (20), we have $n(\log \sigma_k(g) - \lambda - 1) = \alpha\varphi$ and since we have that $\alpha\varphi = \tilde{g}(\mathbf{H}, v^\xi)$, one obtains:

$$n(\log \sigma_k(g) - \lambda - 1) = \tilde{g}(\mathbf{H}, v^\xi). \quad (25)$$

As we assumed that (M^n, g, v^T, λ) is minimal, then $\log \sigma_k(g) = \lambda + 1$. It means that $\log \sigma_k(g)$ is constant as λ is constant. $\square$

In [24], authors showed that the canonical vector field v^T of a submanifold M of the Euclidean m -space $\mathbb{R}^m$ is a conformal vector field if and only if M is umbilical with respect to the normal component v^ξ of the position vector field. Using this concept and Theorem 2, we give the following result.

Corollary 2. *If an Euclidean submanifold of (M, g) of $\mathbb{R}^m$ is a k -Yamabe solitons with canonical vector field v^T as its soliton vector field, then v^T is a conformal vector field.*

Proof. Assuming that the canonical vector field v^T is a soliton vector field, then from (12), we have

$$\tilde{g}(h(W_1, W_2), v^\xi) = (\log \sigma_k(g) - \lambda - 1)g(W_1, W_2) \quad (26)$$

for the vectors W_1, W_2 which are tangent to M . On the other hand, we have

$$A_{v^\xi}(W_1) = (\log \sigma_k(g) - \lambda - 1)W_1$$

and λ is a constant, then M is umbilical with respect to v^ξ . Now, applying Theorem 3.1 to [24], we conclude that v^T is a conformal vector field. $\square$

We obtain following corollary

Corollary 3. *The scalar curvature of a compact k -Yamabe solitons on a minimal submanifold in $\tilde{M}$ is vanished*

Proof. For a minimal submanifold and using Theorem 4, we have $\log \sigma_k(g) = \lambda + 1$. Since from (3), we get $\Delta f = n$. This we get f is a constant function by implementing the maximum principle. From [2], we have the following

$$(n-1)\Delta \log \sigma_k(g) + \frac{1}{2}g(\nabla R, \nabla f) + R(\log \sigma_k(g) - \lambda) = 0. \quad (27)$$

From the above equation we get $R = 0$ as f is a constant. $\square$

4. Gradient k -Yamabe Soliton

In this section, we will study about gradient k -Yamabe Soliton with the Hodge-de Rham decomposition

Theorem 5. Let (M^n, g, v^T, λ) be a k -Yamabe soliton on submanifold M^n of Riemannian manifold $\tilde{M}$. Then (M^n, g, v^T, λ) is a gradient k -Yamabe soliton.

Proof. Let us consider that $2f = \tilde{g}(v, v)$. Thus

$$g(\nabla f, W_1) = W_1(f) = \frac{1}{2}W_1g(v, v) = g(v, \nabla_{W_1}v).$$

As v is concurrent vector field then we arrive at

$$g(\nabla f, W_1) = \tilde{g}(v, W_1).$$

This means that if $\nabla f = v^T$, then the soliton vector field v^T is a gradient vector field. Hence, (M^n, g, v^T, λ) is a gradient k -Yamabe soliton. This completes the proof of the theorem. $\square$

Corollary 4. If the scalar curvature R of a compact gradient k -Yamabe soliton is constant then k -curvature $\log \sigma_k(g)$ is also constant.

Proof. Taking integration in (27) and using Stokes Theorem, we get

$$\int g(\nabla R, \nabla f) dV = 2 \int R(\lambda - \log \sigma_k(g)) dV.$$

As we assumed that the scalar curvature R is constant from above equation, we get

$$R \int (\lambda - \log \sigma_k(g)) dV = 0.$$

This implies that $\lambda = \log \sigma_k(g)$ and hence $\log \sigma_k(g)$ is constant. $\square$

Theorem 6. Let $(M, g, \nabla f, X, \lambda)$ be a gradient k -Yamabe soliton of dimension n which is compact, with potential function f . Then upto a constant, f agrees with the Hodge-de Rham potential ω .

Proof. It follows that from (2) for a k -Yamabe soliton $(M, g, \nabla f, X, \lambda)$

$$(\log \sigma_k(g) - \lambda)n = \operatorname{div} X. \quad (28)$$

Then from Hodge-de-Rham decomposition (10) along with (11), we get

$$\operatorname{div} X = \Delta \omega.$$

Therefore, we obtain the following from (28)

$$(\log \sigma_k(g) - \lambda)n = \Delta \omega. \quad (29)$$

Tracing Equation (3), we derive

$$(\log \sigma_k(g) - \lambda)n = \Delta f. \quad (30)$$

Combining (29) and (30), we get the following

$$\Delta(f - \omega) = 0.$$

From the above it implies that $f = \omega + C$ for any constant C . The proof is completed. $\square$

Leyang et al. [2] defined the following example for a k -Yamabe soliton.

Example 1. [2] Assuming the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ having the metric

$$g = dr^2 + \frac{1}{2} \left(B_{n-1}^k - B_{n-1}^{k-1} \right)^{\frac{1}{k}} e^{-\frac{\lambda(t)}{k}} g_{\mathbb{S}^{n-1}}$$

such that λ is a function which depend on t . Now consider $k < \frac{n}{2}$ for positive k -curvature and boundary curvature B^k [25]. For indices i, j, k ($2 \leq i, j, k, \leq n$) and r denote the direction, we assume that $C = \frac{1}{2} \left(B_{n-1}^k - B_{n-1}^{k-1} \right)^{\frac{1}{k}} e^{-\frac{\lambda(t)}{k}} g_{\mathbb{S}^{n-1}}$, then scalar curvature, Ricci curvature and the Schouten tensor of the metric g are defined as follows:

$$\begin{aligned} R &= C^{-1}(n-1)(n-1) \\ Ric_{ij} &= (n-2)(g_{\mathbb{S}^{n-1}})_{ij}, \quad Ric_{ir} = Ric_{rr} = 0 \\ A_{ij} &= \frac{1}{2}(g_{\mathbb{S}^{n-1}})_{ij} \quad A_{ir} = 0, \quad A_{rr} = -\frac{1}{2C}. \end{aligned}$$

Taking into account the above relations, the k -curvature is obtained as

$$\sigma_k(g^{-1}A) = B_{n-1}^k \left(\frac{1}{2C} \right)^k - B_{n-1}^{k-1} \left(\frac{1}{2C} \right)^k = e^{\lambda(t)}.$$

It is constant at any fixed time t . Therefore, we have

$$\nabla^2 f = 0 = \left(\log \sigma_k(g) - \lambda(t) \right) g$$

for the potential function $f = cr$ where c is any constant. This shows that for any k -Yamabe soliton on a complete noncompact manifold, the condition of constant k -curvature does not imply that the potential function f is constant.

5. Conclusions

In the present paper, we studied k -Yamabe soliton which is a natural extension of the Yamabe flow where the evolving metric satisfies the partial differential Equation (1). In the differential geometry, for the k -Yamabe soliton, an important question is to find conditions under which this soliton become a trivial k -Yamabe solitons and also a gradient k -Yamabe soliton. On the other hand, the geometric flows represents a topic of active research interest in both mathematics and physics. One of the well-known geometric flows in mathematics is the heat flow [26]. Stable solutions of Yamabe flow are said to be Yamabe solitons. The Yamabe flow is an intrinsic geometric flow, a process which deforms the metric of a Riemannian manifold and it was introduced by Richard S. Hamilton. A Yamabe flow is defined for noncompact manifolds and is the negative L^2 -gradient flow of the (normalized) total scalar curvature, restricted to a given conformal class. If this flow converges then, this can be regarded as a deformation of the Riemannian metric to a conformal metric of constant scalar curvature. The main importance of the Hamilton conjecture is that for every initial metric, the flow converges to a conformal metric of constant scalar curvature and later it becomes a central tool in applications to various areas of sciences and economics. Our results are important in this respect.

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