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Left-Invariant Riemann Solitons of Three-Dimensional Lorentzian Lie Groups

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Abstract: Riemann solitons are generalized fixed points of the Riemann flow. In this note, we study left-invariant Riemann solitons on three-dimensional Lorentzian Lie groups. We completely classify left-invariant Riemann solitons on three-dimensional Lorentzian Lie groups.

Keywords: left-invariant Riemann solitons; three-dimensional Lorentzian Lie groups

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1. Introduction

Riemann solitons are generalized fixed points of the Riemann flow. In the context of contact geometry, Hirica and Udriste proved [1] that if a Sasakian manifold admited a Riemann soliton with potential vector field pointwise collinear with the structure vector field, then it was a Sasakian space form. In [2], Blaga and Latcu studied almost Riemann solitons and almost Ricci solitons in an (α,β)-contact metric manifold satisfying some Ricci symmetry conditions, treating the case when the potential vector field of the soliton was pointwise collinear with the structure vector field. Geometric flows have many physical applications. Here we call attention to certain important applications of the Ricci flow theory in the study of nonlinear sigma models [3–6], research on geometric flow evolution of modified (non) holonomic commutative and noncommutative gravity theories [7–10], and exact solutions for (modified) gravity and geometric flows, Ricci solitons [11–15]. In [16], Calvaruso studied three-dimensional generalized Ricci solitons, both in Riemannian and Lorentzian settings. He determined their homogeneous models, classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups. In [17], Batat and Onda studied algebraic Ricci solitons of three-dimensional Lorentzian Lie groups. They got a complete classification of algebraic Ricci solitons of three-dimensional Lorentzian Lie groups. In [18], Calvaruso completely classify three-dimensional homogeneous manifolds equipped with Einstein-like metrics. In [19], we classify affine Ricci solitons associated to canonical connections and Kobayashi–Nomizu connections and perturbed canonical connections and perturbed Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups with some product structure. In this note, we completely classify the left-invariant Riemann solitons on three-dimensional Lorentzian Lie groups.

2. Left-Invariant Riemann Solitons of Three-Dimensional Lorentzian Lie Groups

Three-dimensional Lorentzian Lie groups have been classified in [20,21] (see Theorems 2.1 and 2.2 in [17]). Throughout this paper, we shall by $\{G_i\}_{i=1,\cdots,7}$, denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,7}$. Let ∇ be the Levi–Civita connection of G_i and R its curvature tensor, taken with the convention:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1}$$

Let $R(X,Y,Z,W) = -g(R(X,Y)Z,W)$. Riemann solitons are defined by a smooth vector field and a real constant λ which satisfy the following equation:

$$R + \frac{1}{2}L_V g \wedge g = \frac{\lambda}{2} g \wedge g, \tag{2}$$

where $L_V g$ denotes the Lie derivative of g and $\wedge$ is the Kulkarni–Nomizu product. Let T_1 and T_2 be two arbitrary $(0,2)$-tensors, then their Kulkarni–Nomizu product is defined by:

$$\begin{aligned} T_1 \wedge T_2(X,Y,Z,W) := {} & T_1(X,W)T_2(Y,Z) + T_1(Y,Z)T_2(X,W) \\ & - T_1(X,Z)T_2(Y,W) - T_1(Y,W)T_2(X,Z), \end{aligned} \tag{3}$$

for any $X,Y,Z,W \in \Gamma(TG_i)$, where $\Gamma(TG_i)$ denotes the set of all vector fields on G_i. By (2) and (3), we can express the Riemann soliton as follows:

$$\begin{aligned} & 2R(X,Y,Z,W) + g(X,W)(L_V g)(Y,Z) + g(Y,Z)(L_V g)(X,W) \\ & - g(X,Z)(L_V g)(Y,W) - g(Y,W)(L_V g)(X,Z) \\ & = 2\lambda[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4}$$

For G_i, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike. Let $V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, where $\lambda_1, \lambda_2, \lambda_3$ are real numbers. Let $R_{ijkl} = R(e_i, e_j, e_k, e_l)$. Then (G_i, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} 2R_{1212} - (L_V g)(e_2,e_2) - (L_V g)(e_1,e_1) = -2\lambda, \\ 2R_{1312} - (L_V g)(e_2,e_3) = 0, \\ 2R_{2312} + (L_V g)(e_1,e_3) = 0, \\ 2R_{1313} - (L_V g)(e_3,e_3) + (L_V g)(e_1,e_1) = 2\lambda, \\ 2R_{2313} + (L_V g)(e_1,e_2) = 0, \\ 2R_{2323} - (L_V g)(e_3,e_3) + (L_V g)(e_2,e_2) = 2\lambda. \end{cases} \tag{5}$$

By Theorem 2.1 in [17], we have for G_1, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_1 satisfies:

$$[e_1,e_2] = \alpha e_1 - \beta e_3, \ \ [e_1,e_3] = -\alpha e_1 - \beta e_2, \ \ [e_2,e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \ \ \alpha \neq 0. \tag{6}$$

By (2.18) in [18], we have for G_1:

$$\begin{aligned} & R_{1212} = -2\alpha^2 - \frac{\beta^2}{4}, \ \ R_{1313} = \frac{\beta^2}{4} - 2\alpha^2, \ \ R_{2323} = \frac{\beta^2}{4}, \\ & R_{1213} = 2\alpha^2, \ \ R_{1223} = -\alpha\beta, \ \ R_{1323} = \alpha\beta. \end{aligned} \tag{7}$$

Let,

$$L_V g = \begin{pmatrix} (L_V g)(e_1,e_1) & (L_V g)(e_1,e_2) & (L_V g)(e_1,e_3) \\ (L_V g)(e_2,e_1) & (L_V g)(e_2,e_2) & (L_V g)(e_2,e_3) \\ (L_V g)(e_3,e_1) & (L_V g)(e_3,e_2) & (L_V g)(e_3,e_3) \end{pmatrix}. \tag{8}$$

By page 7 in [16], we get for G_1,

$$L_V g = \begin{pmatrix} 2\alpha(\lambda_2 - \lambda_3) & -\alpha\lambda_1 & \alpha\lambda_1 \\ -\alpha\lambda_1 & 2\alpha\lambda_3 & -\alpha(\lambda_2 + \lambda_3) \\ \alpha\lambda_1 & -\alpha(\lambda_2 + \lambda_3) & 2\alpha\lambda_2 \end{pmatrix}. \tag{9}$$

By (5), (7), and (9) and $\alpha \neq 0$, we get that (G_1, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} -2\alpha^2 - \frac{\beta^2}{4} - \alpha\lambda_2 = -\lambda, \\ 4\alpha + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 = 2\beta, \\ \frac{\beta^2}{4} - 2\alpha^2 - \alpha\lambda_3 = \lambda, \\ \frac{\beta^2}{2} - 2\alpha\lambda_2 + 2\alpha\lambda_3 = 2\lambda. \end{cases} \tag{10}$$

The first equation plusing the fourth equation in (10), we get $\lambda_2 + \lambda_3 + 4\alpha = 0$. By the fourth equation and the fifth equation in (10), we have $\lambda_2 - 2\lambda_3 - 2\alpha = 0$. Then $\lambda_2 = \lambda_3 = -2\alpha$. By the first equation in (10), we get $\lambda = \frac{\beta^2}{4}$. So we have:

Theorem 1. *(G_1, V, g) is a left-invariant Riemann soliton if and only if $\lambda_1 = 2\beta, \lambda_2 = -2\alpha$, $\lambda_3 = -2\alpha$, $\lambda = \frac{\beta^2}{4}$.*

By Theorem 2.1 in [17], we have for G_2, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_2 satisfies:

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \ \ [e_1, e_3] = -\beta e_2 - \gamma e_3, \ \ [e_2, e_3] = \alpha e_1, \ \ \gamma \neq 0. \tag{11}$$

By page 144 in [17], we have for G_2:

$$\begin{aligned} &R_{1212} = -\gamma^2 - \frac{\alpha^2}{4}, \ \ R_{1313} = \frac{\alpha^2}{4} + \gamma^2, \ \ R_{2323} = -\gamma^2 - \frac{3}{4}\alpha^2 + \alpha\beta, \\ &R_{1213} = \gamma(2\beta - \alpha), \ \ R_{1223} = 0, \ \ R_{1323} = 0. \end{aligned} \tag{12}$$

By page 8 in [16], we get for G_2 (we correct a misprint in [16]),

$$L_V g = \begin{pmatrix} 0 & \gamma\lambda_2 + (\alpha - \beta)\lambda_3 & (-\alpha + \beta)\lambda_2 + \gamma\lambda_3 \\ \gamma\lambda_2 + (\alpha - \beta)\lambda_3 & -2\gamma\lambda_1 & 0 \\ (-\alpha + \beta)\lambda_2 + \gamma\lambda_3 & 0 & -2\gamma\lambda_1 \end{pmatrix}. \tag{13}$$

By (5), (12) and (13), we get that (G_2, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} -\gamma^2 - \frac{\alpha^2}{4} + \gamma\lambda_1 = -\lambda, \\ \gamma(2\beta - \alpha) = 0, \\ (-\alpha + \beta)\lambda_2 + \gamma\lambda_3 = 0, \\ \frac{\alpha^2}{4} + \gamma^2 + \gamma\lambda_1 = \lambda, \\ \gamma\lambda_2 + (\alpha - \beta)\lambda_3 = 0, \\ -\gamma^2 - \frac{3}{4}\alpha^2 + \alpha\beta = \lambda. \end{cases} \tag{14}$$

By the first equation and the fourth equation and $\gamma \neq 0$ in (14), we get $\lambda_1 = 0$ and $\lambda = \frac{\alpha^2}{4} + \gamma^2$. By the second equation and the sixth equation in (14), we get $\lambda = -\frac{\alpha^2}{4} - \gamma^2$. Then $\gamma = 0$ and this is a contradiction. So,

Theorem 2. *(G_2, V, g) is not a left-invariant Riemann soliton.*

By Theorem 2.1 in [17], we have for G_3, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_3 satisfies:

$$[e_1, e_2] = -\gamma e_3, \ \ [e_1, e_3] = -\beta e_2, \ \ [e_2, e_3] = \alpha e_1. \tag{15}$$

By page 146 in [17], we have for G_3:

$$\begin{aligned} &R_{1212} = -(a_1 a_2 + \gamma a_3), \ \ R_{1313} = a_1 a_3 + \beta a_2, \ \ R_{2323} = -(a_2 a_3 + \alpha a_1), \\ &R_{1213} = 0, \ \ R_{1223} = 0, \ \ R_{1323} = 0, \end{aligned} \tag{16}$$

where

$$a_1 = \frac{1}{2}(\alpha - \beta - \gamma), \;\; a_2 = \frac{1}{2}(\alpha - \beta + \gamma), \;\; a_3 = \frac{1}{2}(\alpha + \beta - \gamma). \tag{17}$$

By page 9 in [16], we get for G_3,

$$L_V g = \begin{pmatrix} 0 & (\alpha-\beta)\lambda_3 & (\gamma-\alpha)\lambda_2 \\ (\alpha-\beta)\lambda_3 & 0 & (\beta-\gamma)\lambda_1 \\ (\gamma-\alpha)\lambda_2 & (\beta-\gamma)\lambda_1 & 0 \end{pmatrix}. \tag{18}$$

By (5), (16) and (18), we get that (G_3, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} a_1 a_2 + \gamma a_3 = \lambda, \\ (\beta-\gamma)\lambda_1 = 0, \\ (\alpha-\gamma)\lambda_2 = 0, \\ (\alpha-\beta)\lambda_3 = 0, \\ a_1 a_3 + \beta a_2 = \lambda, \\ a_2 a_3 + \alpha a_1 = -\lambda. \end{cases} \tag{19}$$

Theorem 3. *(G_3, V, g) is a left-invariant Riemann soliton if and only if:*

(i) $\beta = \gamma$, $\alpha \neq \gamma$, $\lambda_2 = \lambda_3 = 0$, $\alpha = 0$, $\lambda = 0$,
(ii) $\alpha = \beta = \gamma$, $\lambda = \frac{1}{4}\alpha^2$,
(iii) $\beta \neq \gamma$, $\alpha = \beta$, $\lambda_1 = \lambda_2 = 0$, $\gamma = 0$, $\lambda = 0$,
(iv) $\beta \neq \gamma$, $\alpha = \gamma$, $\lambda_1 = \lambda_3 = 0$, $\beta = 0$, $\lambda = 0$.

Proof. By the first equation and the fifth equation in (19), we get $a_1(a_2 - a_3) + \gamma a_3 - \beta a_2 = 0$. By (17), then we get $(\alpha - \beta - \gamma)(\beta - \gamma) = 0$. By the fifth equation and the sixth equation in (19), we get $(\alpha + \beta - \gamma)(\alpha - \beta) = 0$ and:

$$\begin{cases} (\beta-\gamma)\lambda_1 = 0, \\ (\alpha-\gamma)\lambda_2 = 0, \\ (\alpha-\beta)\lambda_3 = 0, \\ (\alpha-\beta-\gamma)(\beta-\gamma) = 0, \\ (\alpha+\beta-\gamma)(\alpha-\beta) = 0, \\ \lambda = a_1 a_2 + \gamma a_3. \end{cases} \tag{20}$$

Case (1) $\beta \neq \gamma$, $\alpha \neq \gamma$, $\alpha \neq \beta$. Then by the fourth equation and the fifth equation in (20), we get $\alpha = \gamma$. This is a contradiction and there are no solutions.
Case (2) $\beta = \gamma$, $\alpha \neq \gamma$. Solving (20), we get the case (i).
Case (3) $\alpha = \beta = \gamma$. Solving (20), we get the case (ii).
Case (4) $\beta \neq \gamma$, $\alpha = \beta$. Solving (20), we get the case (iii).
Case (5) $\beta \neq \gamma$, $\alpha = \gamma$. Solving (20), we get the case (iv). □

By Theorem 2.1 in [17], we have for G_4, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_4 satisfies:

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \;\; \eta = 1 \text{ or } -1, \;\; [e_1, e_3] = -\beta e_2 + e_3, \;\; [e_2, e_3] = \alpha e_1. \tag{21}$$

By (2.32) in [18], we have for G_4:

$$\begin{aligned} &R_{1212} = (2\eta - \beta)b_3 - b_1 b_2 - 1, \;\; R_{1313} = b_1 b_3 + \beta b_2 + 1, \;\; R_{2323} = -(b_2 b_3 + \alpha b_1 + 1), \\ &R_{1213} = 2\eta - \beta + b_1 + b_2, \;\; R_{1223} = 0, \;\; R_{1323} = 0, \end{aligned} \tag{22}$$

where

$$b_1 = \frac{\alpha}{2} + \eta - \beta, \;\; b_2 = \frac{\alpha}{2} - \eta, \;\; b_3 = \frac{\alpha}{2} + \eta. \tag{23}$$

By page 11 in [16], we get for G_4,

$$L_V g = \begin{pmatrix} 0 & -\lambda_2 + (\alpha-\beta)\lambda_3 & (\beta-\alpha-2\eta)\lambda_2 - \lambda_3 \\ -\lambda_2 + (\alpha-\beta)\lambda_3 & 2\lambda_1 & 2\eta\lambda_1 \\ (\beta-\alpha-2\eta)\lambda_2 - \lambda_3 & 2\eta\lambda_1 & 2\lambda_1 \end{pmatrix}. \tag{24}$$

By (5), (22) and (24), we get that (G_4, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} (2\eta-\beta)b_3 - b_1b_2 - 1 - \lambda_1 = -\lambda, \\ 2\eta - \beta + b_1 + b_2 - \eta\lambda_1 = 0, \\ (\beta-\alpha-2\eta)\lambda_2 - \lambda_3 = 0, \\ b_1b_3 + \beta b_2 + 1 - \lambda_1 = \lambda, \\ -\lambda_2 + (\alpha-\beta)\lambda_3 = 0, \\ -(b_2b_3 + \alpha b_1 + 1) = \lambda. \end{cases} \tag{25}$$

Theorem 4. (G_4, V, g) *is a left-invariant Riemann soliton if and only if:*
(i) $\beta \neq \eta$, $\alpha = 0$, $\lambda_1 = 2 - 2\eta\beta$, $\lambda_2 = \lambda_3 = 0$, $\lambda = 0$,
(ii) $\alpha - \beta + \eta = 0$, $\lambda_2 = -\eta\lambda_3$, $\lambda_1 = 1 - \eta\beta$, $\lambda = \frac{\alpha^2}{4}$.

Proof. The fourth equation minusing the first equation in (25), we get $b_1b_3 + \beta b_2 + 1 - (2\eta-\beta)b_3 + b_1b_2 + 1 = 2\lambda$. By the sixth equation in (25), we get $\alpha(\alpha - \beta + \eta) = 0$.
Case (1) $\alpha - \beta + \eta \neq 0$. Then $\alpha = 0$, solving (25), we get case (i).
Case (2) $\alpha - \beta + \eta = 0$. Solving (25), we get case (ii). □

By Theorem 2.2 in [17], we have for G_5, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_5 satisfies:

$$[e_1, e_2] = 0, \ [e_1, e_3] = \alpha e_1 + \beta e_2, \ [e_2, e_3] = \gamma e_1 + \delta e_2, \ \alpha + \delta \neq 0, \ \alpha\gamma + \beta\delta = 0. \tag{26}$$

By (2.36) in [18], we have for G_5:

$$\begin{aligned} R_{1212} &= \alpha\delta - \frac{(\beta+\gamma)^2}{4}, \ R_{1313} = -\alpha^2 - \frac{\beta(\beta+\gamma)}{2} - \frac{\beta^2-\gamma^2}{4}, \\ R_{2323} &= -\delta^2 - \frac{\gamma(\beta+\gamma)}{2} + \frac{\beta^2-\gamma^2}{4}, \ R_{1213} = 0, \ R_{1223} = 0, \ R_{1323} = 0. \end{aligned} \tag{27}$$

By page 13 in [16], we get for G_5,

$$L_V g = \begin{pmatrix} 2\alpha\lambda_3 & (\beta+\gamma)\lambda_3 & -\alpha\lambda_1 - \gamma\lambda_2 \\ (\beta+\gamma)\lambda_3 & 2\delta\lambda_3 & -\beta\lambda_1 - \delta\lambda_2 \\ -\alpha\lambda_1 - \gamma\lambda_2 & -\beta\lambda_1 - \delta\lambda_2 & 0 \end{pmatrix}. \tag{28}$$

By (5), (27), and (28), we get that (G_5, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} \alpha\delta - \frac{(\beta+\gamma)^2}{4} - \delta\lambda_3 - \alpha\lambda_3 = -\lambda, \\ \beta\lambda_1 + \delta\lambda_2 = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ -\alpha^2 - \frac{\beta(\beta+\gamma)}{2} - \frac{\beta^2-\gamma^2}{4} + \alpha\lambda_3 = \lambda, \\ (\beta+\gamma)\lambda_3 = 0, \\ -\delta^2 - \frac{\gamma(\beta+\gamma)}{2} + \frac{\beta^2-\gamma^2}{4} + \delta\lambda_3 = \lambda. \end{cases} \tag{29}$$

Theorem 5. (G_5, V, g) *is a left-invariant Riemann soliton if and only if:*
(i) $\beta + \gamma = 0$, $\beta \neq 0$, $\alpha = \delta$, $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda = -\alpha^2$,

(ii) $\beta = \gamma = 0, \alpha = \delta, \alpha \neq 0, \lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda = -\alpha^2.$

Proof. Case (1) $\beta + \gamma \neq 0$. Then $\lambda_3 = 0$. By the fourth equation and the sixth equation in (29), we get $\alpha^2 - \delta^2 + \beta^2 - \gamma^2 = 0$. By the first equation and the fourth equation in (29), we get $\alpha^2 + \beta^2 + \beta\gamma - \alpha\delta = 0$.

Case (1-a) $\beta\gamma - \alpha\delta = 0$. We get $\alpha = \beta = \gamma = \delta = 0$. This is a contradiction.

Case (1-b) $\beta\gamma - \alpha\delta \neq 0$. By the second equation and the third equation in (29), we get $\lambda_1 = \lambda_2 = 0$.

Case (1-b-1) $\alpha = 0$. Then $\delta \neq 0$ and $\beta = 0$, then $\delta = \gamma = 0$ by $\alpha^2 - \delta^2 + \beta^2 - \gamma^2 = 0$. This is a contradiction.

Case (1-b-2) $\alpha \neq 0$. Then $\gamma = -\frac{\beta\delta}{\alpha}$. Then $\beta \neq 0$ and $\alpha \neq \delta$ by $\beta + \gamma \neq 0$. By $\alpha + \delta \neq 0$, then $\alpha^2 \neq \delta^2$. By $\alpha^2 - \delta^2 + \beta^2 - \gamma^2 = 0$, we get $1 + \frac{\beta^2}{\alpha^2} = 0$. This is a contradiction.

Case (2) $\beta + \gamma = 0$. By (29), we have:

$$\begin{cases} \alpha\delta - \delta\lambda_3 - \alpha\lambda_3 = -\lambda, \\ \beta\lambda_1 + \delta\lambda_2 = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ -\alpha^2 + \alpha\lambda_3 = \lambda, \\ -\delta^2 + \delta\lambda_3 = \lambda. \end{cases} \tag{30}$$

By $\alpha\gamma + \beta\delta = 0$, we have $\beta(\alpha - \delta) = 0$.

Case (2-a) $\beta \neq 0$. Then $\alpha = \delta$. Solving (30), we get the case (i).

Case (2-b) $\beta = 0$. Then $\gamma = 0$. So $\delta\lambda_2 = 0$, $\alpha\lambda_1 = 0$.

Case (2-b-1) $\alpha \neq 0$, $\delta \neq 0$. Then $\lambda_1 = \lambda_2 = 0$. Solving (30), we get the case (ii).

Case (2-b-2) $\alpha = 0$, $\delta \neq 0$. Solving (30), we get $\delta = 0$. This is a contradiction.

Case (2-b-3) $\alpha \neq 0$, $\delta = 0$. Solving (30), we get $\alpha = 0$. This is a contradiction. □

By Theorem 2.2 in [17], we have for G_6, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_6 satisfies:

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0. \tag{31}$$

By (2.40) in [18], we have for G_6:

$$\begin{aligned} &R_{1212} = -\alpha^2 + \frac{\beta(\beta - \gamma)}{2} + \frac{\beta^2 - \gamma^2}{4}, \quad R_{1313} = \delta^2 + \frac{\gamma(\beta - \gamma)}{2} + \frac{\beta^2 - \gamma^2}{4}, \\ &R_{2323} = \alpha\delta + \frac{(\beta - \gamma)^2}{4}, \quad R_{1213} = 0, \quad R_{1223} = 0, \quad R_{1323} = 0. \end{aligned} \tag{32}$$

By page 14 in [16], we get for G_6,

$$L_V g = \begin{pmatrix} 0 & \alpha\lambda_2 + \gamma\lambda_3 & -\beta\lambda_2 - \delta\lambda_3 \\ \alpha\lambda_2 + \gamma\lambda_3 & -2\alpha\lambda_1 & (\beta - \gamma)\lambda_1 \\ -\beta\lambda_2 - \delta\lambda_3 & (\beta - \gamma)\lambda_1 & 2\delta\lambda_1 \end{pmatrix}. \tag{33}$$

By (5), (32), and (33), we get that (G_6, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} -\alpha^2 + \frac{\beta(\beta-\gamma)}{2} + \frac{\beta^2-\gamma^2}{4} + \alpha\lambda_1 = -\lambda, \\ (\beta - \gamma)\lambda_1 = 0, \\ \beta\lambda_2 + \delta\lambda_3 = 0, \\ \delta^2 + \frac{\gamma(\beta-\gamma)}{2} + \frac{\beta^2-\gamma^2}{4} - \delta\lambda_1 = \lambda, \\ \alpha\lambda_2 + \gamma\lambda_3 = 0, \\ \alpha\delta + \frac{(\beta-\gamma)^2}{4} - \delta\lambda_1 - \alpha\lambda_1 = \lambda. \end{cases} \tag{34}$$

Theorem 6. *(G_6, V, g) is a left-invariant Riemann soliton if and only if:*

(i) $\beta \neq \gamma$, $\lambda_1 = 0$, $\alpha = \beta = 0$, $\lambda = \frac{\gamma^2}{4}$, $\lambda_3 = 0$, $\delta^2 = \gamma^2$,
(ii) $\beta \neq \gamma$, $\lambda_1 = 0$, $\alpha \neq 0$, $\alpha^2 = \beta^2$, $\delta = \frac{\beta\gamma}{\alpha}$, $\lambda = \frac{(\beta+\gamma)^2}{4}$, $\lambda_2 = -\frac{\gamma}{\alpha}\lambda_3$,
(iii) $\beta = \gamma$, $\beta \neq 0$, $\alpha = \delta$, $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda = \alpha^2$,
(iv) $\lambda_3 \neq 0$, $\lambda_2 = -\frac{\delta}{\beta}\lambda_3$, $\alpha \neq 0$, $\beta \neq 0$, $\beta = \gamma$, $\alpha = \delta$, $\alpha^2 = \beta^2$, $\lambda_1 = 0$, $\lambda = \alpha^2$,
(v) $\beta = \gamma = 0$, $\alpha \neq 0$, $\delta \neq 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\alpha = \delta$, $\lambda = \alpha^2$.

Proof. Case (1) $\beta - \gamma \neq 0$. Then $\lambda_1 = 0$. So by the first, the fourth, and sixth equations in (34), we get:

$$\delta^2 - \alpha^2 + \beta^2 - \gamma^2 = 0, \quad \alpha^2 - \beta^2 + \beta\gamma - \alpha\delta = 0. \tag{35}$$

Case (1-a) $\beta\gamma - \alpha\delta = 0$. So $\alpha^2 = \beta^2$ and $\delta^2 = \gamma^2$ by (35).
Case (1-a-1) $\alpha = 0$. Solving (34), we get the case (i).
Case (1-a-2) $\alpha \neq 0$. Then $\delta = \frac{\beta\gamma}{\alpha}$. Solving (34), we get the case (ii).
Case (1-b) $\beta\gamma - \alpha\delta \neq 0$. So $\lambda_2 = \lambda_3 = 0$.
Case (1-b-1) $\alpha = 0$. So $\delta \neq 0$ and $\beta = 0$. This is a contradiction with $\beta\gamma - \alpha\delta \neq 0$.
Case (1-b-2) $\alpha \neq 0$. We get $\gamma = \frac{\beta\delta}{\alpha}$ and $\alpha^2 = \beta^2$ by (35). Then $\beta\gamma - \alpha\delta = 0$. This is a contradiction.
Case (2) $\beta - \gamma = 0$. Then $\beta(\alpha - \delta) = 0$. By (34), we have:

$$\begin{cases} -\alpha^2 + \alpha\lambda_1 = -\lambda, \\ \beta\lambda_2 + \delta\lambda_3 = 0, \\ \delta^2 - \delta\lambda_1 = \lambda, \\ \alpha\lambda_2 + \gamma\lambda_3 = 0, \\ \alpha\delta - \delta\lambda_1 - \alpha\lambda_1 = \lambda. \end{cases} \tag{36}$$

Case (2-a) $\beta \neq 0$. Then $\alpha = \delta$ and $\lambda_2 = -\frac{\delta}{\beta}\lambda_3 = -\frac{\gamma}{\alpha}\lambda_3$.
Case (2-a-1) $\lambda_3 = 0$. Then we get the case (iii).
Case (2-a-2) $\lambda_3 \neq 0$. Then we get the case (iv).
Case (2-b) $\beta = 0$. Then $\gamma = 0$ and $\delta\lambda_3 = 0$, $\alpha\lambda_2 = 0$.
Case (2-b-1) $\alpha \neq 0$, $\delta \neq 0$. Then $\lambda_2 = \lambda_3 = 0$. Solving (36), we get the case (v).
Case (2-b-2) $\alpha = 0$, $\delta \neq 0$. Solving (36), we get $\delta = 0$. This is a contradiction.
Case (2-b-3) $\alpha \neq 0$, $\delta = 0$. Solving (36), we get $\alpha = 0$. This is a contradiction. □

By Theorem 4.2 in [17], we have for G_7, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_7 satisfies:

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \ [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \ [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,, \ \alpha + \delta \neq 0, \ \alpha\gamma = 0. \tag{37}$$

By (2.44) in [18], we have for G_7:

$$R_{1212} = \alpha\delta - \alpha^2 - \beta\gamma - \frac{\gamma^2}{4}, \ R_{1313} = \alpha\delta - \alpha^2 - \beta\gamma + \frac{\gamma^2}{4},$$
$$R_{2323} = -\frac{3}{4}\gamma^2, \ R_{1213} = \alpha^2 - \alpha\delta + \beta\gamma, \ R_{1223} = 0, \ R_{1323} = 0. \tag{38}$$

By page 16 in [16], we get for G_7,

$$L_V g = \begin{pmatrix} -2\alpha(\lambda_2 - \lambda_3) & \alpha\lambda_1 - \beta\lambda_2 + (\beta+\gamma)\lambda_3 & -\alpha\lambda_1 + (\beta-\gamma)\lambda_2 - \beta\lambda_3 \\ \alpha\lambda_1 - \beta\lambda_2 + (\beta+\gamma)\lambda_3 & 2\beta\lambda_1 + 2\delta\lambda_3 & -2\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3 \\ -\alpha\lambda_1 + (\beta-\gamma)\lambda_2 - \beta\lambda_3 & -2\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3 & 2\beta\lambda_1 + 2\delta\lambda_2 \end{pmatrix}. \tag{39}$$

By (5), (38), and (39), we get that (G_7, V, g) is a left-invariant Riemann soliton if and only if:

$$\begin{cases} \alpha\delta - \alpha^2 - \beta\gamma - \frac{\gamma^2}{4} - (\beta\lambda_1 + \delta\lambda_3) + \alpha(\lambda_2 - \lambda_3) = -\lambda, \\ 2(\alpha^2 - \alpha\delta + \beta\gamma) + 2\beta\lambda_1 + \delta\lambda_2 + \delta\lambda_3 = 0, \\ -\alpha\lambda_1 + (\beta - \gamma)\lambda_2 - \beta\lambda_3 = 0, \\ \alpha\delta - \alpha^2 - \beta\gamma + \frac{\gamma^2}{4} - \beta\lambda_1 - \delta\lambda_2 - \alpha(\lambda_2 - \lambda_3) = \lambda, \\ \alpha\lambda_1 - \beta\lambda_2 + (\beta + \gamma)\lambda_3 = 0, \\ -\frac{3}{4}\gamma^2 - \delta\lambda_2 + \delta\lambda_3 = \lambda. \end{cases} \tag{40}$$

Theorem 7. *(G_7, V, g) is a left-invariant Riemann soliton if and only if:*

(i) $\alpha = 0$, $\delta \neq 0$, $\beta = \gamma = 0$, $\lambda_2 = \lambda_3 = \lambda = 0$,

(ii) $\alpha = 0$, $\delta \neq 0$, $\gamma = 0$, $\beta \neq 0$, $\lambda_2 = \lambda_3$, $\lambda = 0$, $\lambda_1 = -\frac{\delta}{\beta}\lambda_2$,

(iii) $\alpha \neq 0$, $\gamma = 0$, $\alpha = \delta$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda = 0$.

Proof. Case (1) $\alpha = 0$. Then $\delta \neq 0$. By (40), we have:

$$\begin{cases} -\beta\gamma - \frac{\gamma^2}{4} - (\beta\lambda_1 + \delta\lambda_3) = -\lambda, \\ 2\beta\gamma + 2\beta\lambda_1 + \delta\lambda_2 + \delta\lambda_3 = 0, \\ (\beta - \gamma)\lambda_2 - \beta\lambda_3 = 0, \\ -\beta\gamma + \frac{\gamma^2}{4} - \beta\lambda_1 - \delta\lambda_2 = \lambda, \\ -\beta\lambda_2 + (\beta + \gamma)\lambda_3 = 0, \\ -\frac{3}{4}\gamma^2 - \delta\lambda_2 + \delta\lambda_3 = \lambda. \end{cases} \tag{41}$$

Case (1-a) $\gamma \neq 0$. Then $\lambda_2 = \lambda_3 = 0$ by the third equation and the fifth equation in (41). By (41), we have:

$$\begin{cases} -\beta\gamma - \frac{\gamma^2}{4} - \beta\lambda_1 = -\lambda, \\ \beta\gamma + \beta\lambda_1 = 0, \\ -\beta\gamma + \frac{\gamma^2}{4} - \beta\lambda_1 = \lambda, \\ -\frac{3}{4}\gamma^2 = \lambda. \end{cases} \tag{42}$$

Case (1-a-1) $\beta = 0$. By (42), we get $\gamma = 0$. This is a contradiction.
Case (1-a-2) $\beta \neq 0$. By (42), we get $\lambda_1 = -\gamma$ and $\gamma = 0$. This is a contradiction.
Case (1-b) $\gamma = 0$. By (41), we have:

$$\begin{cases} \beta\lambda_1 + \delta\lambda_3 = \lambda, \\ 2\beta\lambda_1 + \delta\lambda_2 + \delta\lambda_3 = 0, \\ \beta(\lambda_2 - \lambda_3) = 0, \\ -\beta\lambda_1 - \delta\lambda_2 = \lambda, \\ -\delta\lambda_2 + \delta\lambda_3 = \lambda. \end{cases} \tag{43}$$

Case (1-b-1) $\beta = 0$. Solving (43), we get the case (i).
Case (1-b-2) $\beta \neq 0$. Solving (43), we get the case (ii).
Case (2) $\alpha \neq 0$. Then $\gamma = 0$. By (40), we get:

$$\begin{cases} \alpha\delta - \alpha^2 - (\beta\lambda_1 + \delta\lambda_3) + \alpha(\lambda_2 - \lambda_3) = -\lambda, \\ 2(\alpha^2 - \alpha\delta) + 2\beta\lambda_1 + \delta\lambda_2 + \delta\lambda_3 = 0, \\ -\alpha\lambda_1 + \beta\lambda_2 - \beta\lambda_3 = 0, \\ \alpha\delta - \alpha^2 - \beta\lambda_1 - \delta\lambda_2 - \alpha(\lambda_2 - \lambda_3) = \lambda, \\ -\delta\lambda_2 + \delta\lambda_3 = \lambda. \end{cases} \tag{44}$$

By the third equation in (44), we have $\lambda_1 = \frac{\beta}{\alpha}(\lambda_2 - \lambda_3)$. By the first, the second, and the fourth equations in (44), we get $\alpha = \delta$ and $2\beta\lambda_1 + \delta\lambda_2 + \delta\lambda_3 = 0$. By the fourth and the fifth equations in (44), we get $\beta\lambda_1 + \delta\lambda_2 = 0$ and $\lambda_2 = \lambda_3$. Then by the fifth equation in (44), we get $\lambda = 0$. So $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This is the case (iii). □

3. Conclusions

During the last years, geometric evolution equations have been used to study geometric questions like isoperimetric inequalities, the Poincare conjecture, and Thurston's geometrization conjecture. In particular, the geometric flow enjoys rapid growth. The Riemann flow is an important geometric flow. Riemann solitons are generalized fix points of the Riemann flow. Thus it is interesting to study Riemann solitons. In this note, a classification of Riemann solitons on three dimensional Lorentzian Lie group was given. In particular, (G_2, V, g) was not a left-invariant Riemann soliton, while (G_i, V, g) for $i = 1, 3, 4, 5, 6$, and 7, were left invariant Riemann solitons if and only if the parameters satisfied particular conditions.

Our classified theorems are proven by some algebraic calculations. In fact, by (5), we needed to compute the geometric objects R_{ijkl} and $L_V(e_i, e_j)$. Moreover our classified theorems will have some geometric applications.

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