

A Note on Effects of Generalized and Extended Uncertainty Principles on Jüttner Gas

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Abstract: In recent years, the implications of the generalized (GUP) and extended (EUP) uncertainty principles on Maxwell–Boltzmann distribution have been widely investigated. However, at high energy regimes, the validity of Maxwell–Boltzmann statistics is under debate and instead, the Jüttner distribution is proposed as the distribution function in relativistic limit. Motivated by these considerations, in the present work, our aim is to study the effects of GUP and EUP on a system that obeys the Jüttner distribution. To achieve this goal, we address a method to get the distribution function by starting from the partition function and its relation with thermal energy which finally helps us in finding the corresponding energy density states.

Keywords: generalized uncertainty principle; extended uncertainty principle; Jüttner distribution



Citation: Moradpour, H.; Aghababaei, S.; Ziaie, A.H. A Note on Effects of Generalized and Extended Uncertainty Principles on Jüttner Gas. *Symmetry* **2021**, *13*, 213. <https://doi.org/10.3390/sym13020213>

Academic Editor: Chris Fields

Received: 25 December 2020

Accepted: 26 January 2021

Published: 28 January 2021

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1. Introduction

A general prediction of any quantum gravity theory is the possibility of the existence of a minimal length in nature, known as the Planck length, below which no other length can be observed. It is commonly believed that in the vicinity of the Planck length, the smooth structure of spacetime is replaced by a foamy structure due to quantum gravity effects [1–3]. Therefore, the Planck scale can be regarded as a separation line between classical and quantum gravity regimes. There is a general consensus that in the scale of this minimal size, the characteristics of different physical systems would be altered. For instance, the introduction of a minimal length scale results in a generalization of the Heisenberg uncertainty principle (HUP) in such a way that it incorporates gravitationally induced uncertainty, postulated as the generalized uncertainty principle (GUP) [4]. In fact, the HUP breaks down for energies near the Planck scale, i.e., when the Schwarzschild radius is comparable to the Compton wavelength and both are close to the Planck length. This deficiency is removed by revising the characteristic scale through the modification of HUP to GUP.

In recent decades, numerous studies on the effects of GUP in various classical and quantum mechanical systems have been performed [5–30]. Uncertainty in momentum is also bounded from below and it is proposed that its minimum is non-zero, a proposal which modifies HUP to the extended uncertainty principle (EUP) [31–35]. In the presence of EUP and GUP, the general form of modified HUP is proposed as

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha (\Delta x)^2 + \eta (\Delta p)^2 + \gamma \right), \quad (1)$$

in which α , η , and γ are positive deformation parameters [35,36]. It should be noted that there is another formulation of GUP and EUP [37], and also that extended forms of HUP like GUP may break the fundamental symmetries such as Lorentz invariance and CPT [38].

On the other hand, it is known that heavy ions can be accelerated to very high kinetic energies constituting an ensemble of ideal gas with relativistic velocities in large

particle accelerators [39]. In such high energy regimes, minimal length effects may appear and could have their own influences on the statistics of ideal gases. Therefore, particle accelerators could provide a setting to examine the phenomena related to short-distance physics [40,41]. Based on minimum observable length, the quantum gravity implications on the statistical properties of ideal gases have been investigated in many studies, see, e.g., [42–45] and references therein. In the framework of GUP: (i) deformed density of states and an improved definition of the statistical entropy have been introduced in [46,47], (ii) Maxwell–Boltzmann statistics have been investigated in [48], and (iii) employing Maxwell–Boltzmann statistics, the thermodynamics of relativistic ideal gas has also been analyzed in [49]. In the same manner, there have been some studies on the deformation of statistical concepts in the EUP framework [50,51].

Jüttner distribution is a generalization of Maxwell–Boltzmann statistics to the relativistic regimes, which appears in high energy physics. Since quantum gravity is a high energy physics scenario, its statistical effects may be more meaningful in the framework of Jüttner distribution function compared to the Maxwell–Boltzmann distribution [52]. Here, our main aim is to study the effects of GUP and EUP on Maxwell–Boltzmann and Jüttner distributions and density of states in energy space. To achieve this goal, we begin by providing an introductory note on Maxwell–Boltzmann and Jüttner distributions. We then address a way to find these functions by starting from the partition function of the system. The effects of GUP and EUP on these statistics are also studied in the subsequent sections, respectively. The last section is devoted to a summary of the work.

2. The Maxwell–Boltzmann and Jüttner Distribution Functions

We begin by considering an ideal gas composed of non-interacting particles and set the units so that $\omega_0 = 2\pi\hbar = 1$, where ω_0 denotes the fundamental volume of each cell in the two-dimensional phase-space. This value of ω_0 originates from the well-known commutation relation between canonical coordinates x and p , and indeed, it is the direct result of HUP [22]. Therefore, any changes in HUP can affect ω_0 .

2.1. Non-Relativistic Gas

Let us consider a 3-dimensional classical gas consisting of N identical non-interacting particles of mass m with $E = mv^2/2$, where E and v denote the energy and velocity of each particle, respectively. At temperature T , the Maxwell–Boltzmann (MB) distribution function is given by

$$f_{MB}(v, \beta) = Z_{MB} \exp\left(-\frac{\beta mv^2}{2}\right), \quad (2)$$

where Z_{MB} is a normalization constant, and $\beta \equiv 1/K_B T$ with K_B being the Boltzmann constant. In terms of E , we have

$$f_{MB} = 4\pi v^2(E) f_{MB}(v(E), \beta) \frac{dv}{dE} = Z_{MBE} E^{\frac{1}{2}} \exp(-\beta E), \quad (3)$$

in which Z_{MBE} is a new normalization constant and $E^{\frac{1}{2}}$ denotes the density of states with energy E . The normalization constants can be calculated using the normalization constraint

$$\int_0^\infty f_{MB}(v, \beta) d^3v = \int_0^\infty f_{MB}(E, \beta) dE = 1. \quad (4)$$

The extremum of $f_{MB}(E, \beta)$ is located at $E = 1/2\beta \equiv E_{MB}^{ext}$ or equally at velocity $v = 1/\sqrt{\beta m} \equiv v_{MB}^{ext}$. One can also evaluate the partition function of the mentioned gas (with Hamiltonian $H = p^2/2m$) as

$$Q_N = \frac{1}{N!} \int \dots \int_{3N} \exp(-\beta H) d^{3N}x d^{3N}p = \frac{(Q_1^{NR})^N}{N!}, \quad (5)$$

where

$$Q_1^{NR} = \int \exp(-\beta H) d^3x d^3p = V \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}}, \quad (6)$$

denotes the single partition function of a nonrelativistic gas and V refers to the total volume of the system. In this manner, the corresponding thermal energy per particle (U) takes the form

$$U^{NR} = \int_0^\infty E f_{MB}(E, \beta) dE = -\frac{\partial \ln Q_1^{NR}}{\partial \beta} = \frac{3}{2\beta} = 3E_{MB}^{ext}. \quad (7)$$

Although the use of Equation (7) dates back to before the discovery of special relativity theory by Einstein, the ultra-relativistic expression of E produces interesting results in this framework [53].

2.2. Relativistic Gas

In the relativistic situations, where $E = \sqrt{p^2 c^2 + m^2 c^4}$ in which c denotes the light velocity and m is the rest mass, one can employ Equation (5) to get

$$Q_1^R = Q_1^{NR} \Psi(\sigma), \quad \Psi(\sigma) = \frac{i^{\frac{3}{2}} m^3 c^6 \sqrt{\frac{\pi}{2}} H_2^{(1)}(i\sigma)}{(i\sigma)^{\frac{5}{2}}}, \quad (8)$$

as the partition function of a single particle [52,54,55]. Finally, we obtain the thermal energy per particle as

$$U^R = \frac{1}{\beta} \left[1 - i\sigma \frac{H_2^{(1)}(i\sigma)}{H_2^{(1)}(i\sigma)} \right] = -\frac{\partial}{\partial \beta} \ln Q_1^R. \quad (9)$$

In the above equations, $\sigma = \beta m c^2$, $H_n^{(j)}(i\sigma)$ is the n -th order Hankel function of the j -th kind, and prime denotes a derivative with respect to the argument of the function. The above results were first reported in 1911 by Jüttner [52], who attempted to calculate the energy of a relativistic ideal gas using the conventional theory of relativistic statistical mechanics. According to Jüttner's results, a comprehensive study of a 3-dimensional relativistic system requires the Jüttner distribution (f_J)

$$f_J(\gamma, \beta) = Z_J (\gamma^2 - 1)^{\frac{1}{2}} \gamma \exp(-\beta m \gamma), \quad (10)$$

instead of MB distribution (f_{MB}) [56–61]. Jüttner distribution is indeed the relativistic extension of generalized isotropic MB distribution when $E(p) = m\gamma(p)c^2$. Here, Z_J is the normalization constant and $\gamma = \frac{1}{\sqrt{1-v^2}}$ refers to the Lorentz factor, where the units have been set so that $c = 1$. In terms of E , simple calculations give us Jüttner distribution as

$$f_J(E, \beta) = Z_J a_J(E) \exp(-\beta E), \quad (11)$$

where $a_J(E) = E\sqrt{E^2 - m^2}$ denotes the density of states in energy representation, and in terms of v one finds

$$f_J = (v, \beta) = Z_{JV} \left(\frac{1}{\sqrt{1 - v^2}} \right)^5 \exp \left(-\frac{m\beta}{\sqrt{1 - v^2}} \right), \quad (12)$$

in which Z_{JE} and Z_{JV} are new normalization constants [57]. These constants can be evaluated using the normalization condition

$$\begin{aligned} \int_1^\infty f_J(\gamma, \beta) d\gamma &= \int_m^\infty f_J(E, \beta) dE = 1 \\ &= \int_0^1 f_J(\gamma(v), \beta) \frac{d\gamma}{dv} dv = \int_0^1 f_J(v, \beta) d^3v, \end{aligned} \quad (13)$$

which clearly states that

$$f_J(v, \beta) = \frac{1}{4\pi v^2} f_J(\gamma(v), \beta) \frac{d\gamma}{dv}. \quad (14)$$

It is finally useful to note that the extremum of $f_J(v, \beta)$ is located at $v = \sqrt{1 - \left(\frac{\beta m}{5}\right)^2} \equiv v_J^{ext}$ leading to $E_J^{ext} = \frac{5}{\beta}$, a solution which is valid only when $\beta m < 5$. There are also other proposals for Jüttner function ($f_J(\gamma, \beta)$) [56–61], but the standard form Equation (11) considered in this paper is confirmed by some previous studies [58–60]. The corresponding thermal energy per particle (i.e., $\langle \gamma \rangle$) (or equally, the ratio U/N in Equation (12)) can also be obtained by using $f_J(\gamma, \beta)$, as

$$U^R \equiv m \langle \gamma \rangle = \int_m^\infty E f_J(E, \beta) dE = -\frac{\partial}{\partial \beta} \ln Q_1^R. \quad (15)$$

Although Equations (7) and (15) are simple examples, they confirm that the mean value of energy (or equally, thermal energy) can be calculated by using either the partition function or the distribution function. Moreover, employing these equations, one can find the distribution functions whenever the partition function is known. Indeed, if the phase-space geometry is deformed, then the partition function will also be modified. Therefore, one can find the corresponding modified MB and Jüttner distributions by directly using Equations (7) and (15) for the non-relativistic and relativistic cases, repetitively.

3. Generalized Uncertainty Principle, Partition and Distribution Functions

In the units of $\hbar = c = 1$, the relation $[x_k, p_l] = i\delta_{kl}$ is the standard commutation relation between the canonical coordinates x and p . This relation leads to HUP in the framework of quantum mechanics and is the backbone of calculating ω_0 [48,53]. Thus, the volume element $d^3x d^3p$ changes whenever different coordinates (commutation relations) are used [42–44,48]. For GUP, we have [14,31]

$$\Delta X \Delta P \geq \frac{1}{2} [1 + \eta(\Delta P)^2 + \dots], \quad (16)$$

where η denotes the GUP parameter and it is based on modified commutation relations

$$\begin{aligned} [X_k, P_l] &= i(\delta_{kl}(1 + \eta P^2) + \eta' P_k P_l), \\ [P_k, P_l] &= 0, \\ [X_k, X_l] &= i \frac{2\eta - \eta' + (2\eta + \eta')\eta P^2}{1 + \eta P^2} (P_k X_l - P_l X_k), \end{aligned} \quad (17)$$

where $k, l = 1, 2, 3$ for a 3-dimensional space [43,62]. P and X are generalized coordinates which are not necessarily equal to the canonical coordinates p and x . In this manner, assuming $\eta' = 0$ and η is independent of $\hbar$, one finds

$$d^3x d^3p \rightarrow \frac{d^3X d^3P}{(1 + \eta P^2)^3}, \quad (18)$$

which must be considered as the volume element in X - P space instead of $d^3x d^3p$ [42–44,48]. This means that the density of states in the X - P phase space is affected by the factor of $(1 + \eta P^2)$ [42]. In this situation, the single particle partition function can also be found as

$$Q_1^{GUP} = \int \exp(-\beta H(P, X)) \frac{d^3X d^3P}{(1 + \eta P^2)^3}, \quad (19)$$

where $H(P, X)$ denotes Hamiltonian in generalized coordinates [42,43,50]. The corresponding thermal energy (U^{GUP}) can be calculated using the relation

$$U^{GUP} = -\frac{\partial}{\partial \beta} \ln Q_1^{GUP}, \quad (20)$$

along with Equation (19), which finally gives

$$U^{GUP} = \frac{\int H(P) \exp(-\beta H(P)) \frac{d^3P}{(1 + \eta P^2)^3}}{\int \exp(-\beta H(P)) \frac{d^3P}{(1 + \eta P^2)^3}}. \quad (21)$$

In obtaining this equation, the fact that $H(\equiv E)$ is independent of X has been used which cancels integration over d^3X . Indeed, the density of states in phase-space is changed under the shadow of GUP [42,43,48], a result which affects the distribution function.

For a single free particle with $H = \frac{P^2}{2m}$, the ideal gas law is still valid, and therefore

$$Q_1^{NR,GUP} = Q_1^{NR} I\left(\frac{2m\eta}{\beta}, 3\right), \quad (22)$$

while the explicit form of the function $I\left(\frac{2m\eta}{\beta}, 3\right)$ can be followed in [44] and Q_1^{NR} is introduced in Equation (6). The effects of GUP are stored in $I\left(\frac{2m\eta}{\beta}, 3\right)$, and in the limit of $\eta \rightarrow 0$, one gets the ordinary single partition function of a free particle. Correspondingly, the partition function of a single free relativistic particle can also be evaluated using $H^2 = P^2 + m^2$ in Equation (19). By doing so one finds

$$Q_1^{R,GUP} = \int \exp(-\beta \sqrt{P^2 + m^2}) \frac{d^3X d^3P}{(1 + \eta P^2)^3}, \quad (23)$$

for which the solution reads

$$Q_1^{R,GUP} = Q_1^R \left(1 - \eta \frac{15}{2} \frac{1}{\beta m}\right), \quad (24)$$

when $m \gg \frac{1}{\beta}$ [50].

3.1. Maxwell–Boltzmann Statistics

Bearing in mind the recipe which led to the expression for $f_{MB}(E, \beta)$, one can get the modified MB distribution in the X - P space as

$$\begin{aligned} f_{MB}^{GUP}(E, \beta) &= 4\pi P^2(E) \frac{\exp(-\beta E)}{(1 + \eta P^2(E))^3} \frac{dP}{dE} \\ &= Z_{MBE}^{GUP} \frac{E^{\frac{1}{2}} \exp(-\beta E)}{(1 + 2\eta m E)^3}, \end{aligned} \quad (25)$$

where Z_{MBE}^{GUP} denotes the normalization constant in the presence of GUP. The thermal energy then reads

$$U^{GUP} = \int_0^\infty E f_{MB}^{GUP}(E, \beta) dE. \quad (26)$$

One can also find the normalization constant Z_{MBE}^{GUP} as

$$Z_{MBE}^{GUP} = \left[\int_0^\infty \frac{E^{\frac{1}{2}} \exp(-\beta E)}{(1 + 2\eta m E)^3} dE \right]^{-1}, \quad (27)$$

which is equal to Z_{MBE} in the limit where $\eta \rightarrow 0$. Obviously, the MB distribution $f_{MB}(E, \beta)$, is recovered through Equation (25) at the appropriate limit of $\eta = 0$. For the density of states in HUP framework we have $a_{MB}(E) = \sqrt{E}$. This relation is modified in the presence of GUP effects and thus, the density of states will take the following form

$$a_{MB}^{GUP}(E) = \frac{\sqrt{E}}{(1 + 2\eta m E)^3}, \quad (28)$$

which is in agreement with the results of [18]. The extremum of $f_{MB}^{GUP}(E, \beta)$ is also located at

$$\varepsilon_{MB}^{ext} = \frac{1 + 10m\eta E_{MB}^{ext}}{4m\eta} \left(\sqrt{1 + \frac{8m\eta E_{MB}^{ext}}{(1 + 10m\eta E_{MB}^{ext})^2}} - 1 \right), \quad (29)$$

which clearly indicates $\varepsilon_{MB}^{ext} \rightarrow E_{MB}^{ext}$ whenever $\eta \rightarrow 0$. In Figure 1, the effects of GUP on the distribution function in MB statistics are shown where the temperature is considered to be constant ($\beta = 1$).

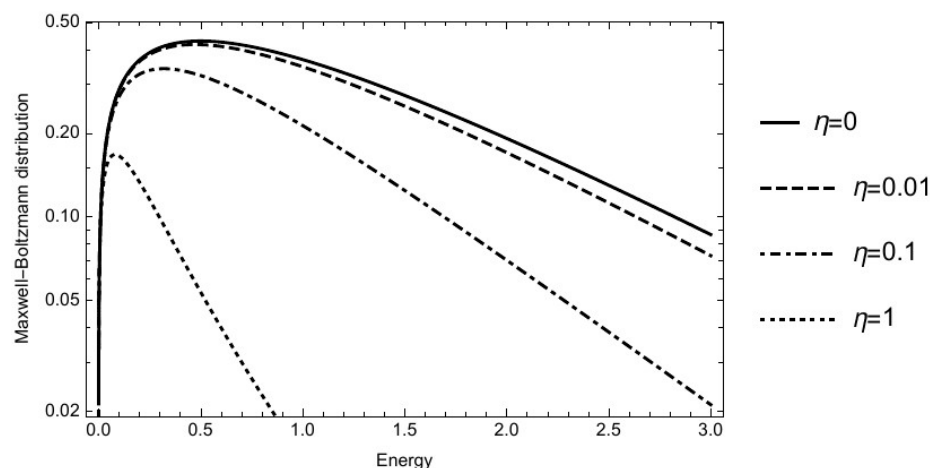


Figure 1. Maxwell–Boltzmann (MB) distribution versus energy for $\eta = 0.5, 1, 1.5$. The ordinary MB distribution is denoted by the solid curve. Here, we have set the units so that $\hbar = c = K_B = 1$.

3.2. Jüttner Statistics

In the relativistic situation, where $H = \sqrt{P^2 + m^2} (\equiv E)$, following the above recipe, we get the modified Jüttner distribution as

$$f_J^{GUP}(E, \beta) = Z_{JE}^{GUP} \frac{E \sqrt{E^2 - m^2} \exp(-\beta E)}{(1 + \eta[E^2 - m^2])^3}, \quad (30)$$

which recovers $f_J(E, \beta)$ in the limit where $\eta \rightarrow 0$. Here, Z_{JE}^{GUP} is also a normalization constant which can be calculated by utilizing the normalization constraint $\int_m^\infty f_J^{GUP}(E, \beta) dE = 1$. We also find

$$a_J^{GUP}(E) = \frac{E \sqrt{E^2 - m^2}}{(1 + \eta[E^2 - m^2])^3}, \quad (31)$$

as the density of states in Jüttner statistics in the presence of GUP. Figure 2 shows the behavior of $f_J^{GUP}(E, \beta)$ for some positive values of η parameter.

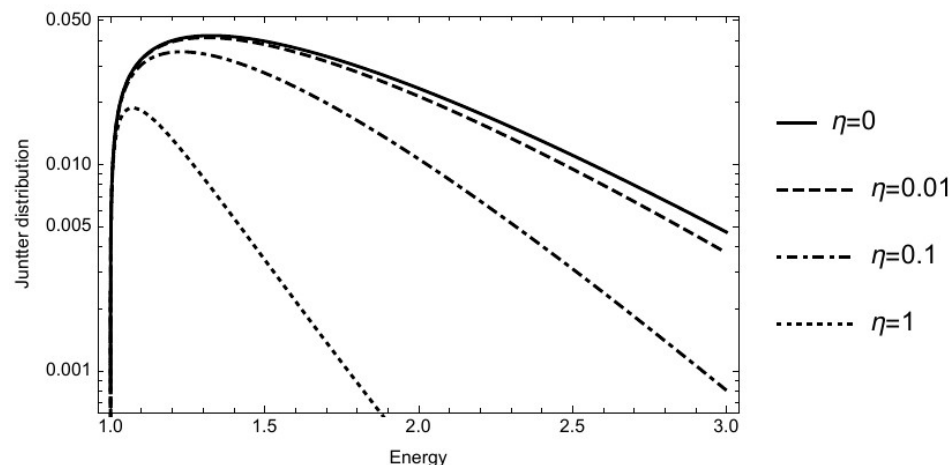


Figure 2. Behavior of Jüttner distribution against energy for $\eta = 0.5, 1, 1.5$. The solid curve belongs to the ordinary Jüttner distribution and we have set the units so that $\hbar = c = K_B = 1$.

4. Extended Uncertainty Principle, Partition and Distribution Functions

The modified Heisenberg algebra in the EUP framework can be recast into the following form

$$[X_i, P_j] = i(\delta_{ij} + \alpha X_i X_j), \quad (32)$$

where α is a small positive parameter known as the EUP parameter. In the limit of $\alpha \rightarrow 0$, the canonical commutation relation of the standard quantum mechanics is recovered. Based on the commutation relation Equation (32), the HUP is modified by

$$(\Delta X_i)(\Delta P_i) \geq \frac{1}{2} [1 + \alpha(\Delta X_i)^2 + \dots], \quad (33)$$

which leads to a non-zero minimum uncertainty in momentum as $(\Delta P_i)_{\min} = \sqrt{\alpha}$. Here, we apply the coordinate representation of the operators X_i and P_i expressed as

$$\begin{aligned} X_i &= x_i, \\ P_i &= (\delta_{ij} + \alpha x_i x_j) p_j, \end{aligned} \quad (34)$$

where x_i and p_j satisfy the standard commutation relation of ordinary quantum mechanics. This representation yields the following commutation relation for the momentum operator

$$[P_i, P_j] = i\alpha(x_i p_j - p_i x_j). \quad (35)$$

In the X - P space, the modified volume element

$$\frac{d^3 X d^3 P}{(1 + \alpha X^2)^3}, \quad (36)$$

should be considered instead of $d^3 x d^3 p$ [20]. We then proceed to consider the consequences of such a modification in calculating the partition and distribution functions. For a single particle, the partition function in X - P space can be found as

$$Q_1^{EUP} = \int \exp(-\beta H(P, X)) \frac{d^3 X d^3 P}{(1 + \alpha X^2)^3}, \quad (37)$$

whence we get the corresponding thermal energy as

$$U^{EUP} = -\frac{\partial}{\partial \beta} \ln Q_1^{EUP}. \quad (38)$$

The above expression can also be combined with Equation (37) to give Equation (21). For the free non-relativistic and relativistic particles, one finds

$$Q_1^{NR,EUP} = V_{eff}(\alpha, r) \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} = \frac{V_{eff}(\alpha, r)}{V} Q_1^{NR}, \quad (39)$$

and

$$Q_1^{R,EUP} = \frac{V_{eff}(\alpha, r)}{V} Q_1^R, \quad (40)$$

respectively, where we have defined $V_{eff}(\alpha, r) = \int_0^r \frac{d^3X}{(1+\alpha X^2)^3}$ as the effective volume, and in the limit of $\alpha \rightarrow 0$, the usual volume V is recovered. Since $V_{eff}(\alpha, r)$ is independent of β , the thermal energy related to EUP is the same as what we obtained in Equations (7) and (9), respectively. Consequently, EUP does not affect the Maxwell–Boltzmann and Jüttner distribution functions, because the corresponding effective volume has no dependence on β .

5. Conclusions

The Jüttner function is the relativistic version of MB distribution and is proper for studying relativistic (high energy) systems. On the other hand, the minimal length comes into play in the realms of high energy physics. Hence, compared with MB distribution, the study of its effects on Jüttner distribution would be more meaningful. Thus, our attempt in the present work was to address an algorithm with the help of which, one can get the distribution function, starting from the partition function. Motivated then by the abovementioned arguments, we studied the effects of GUP and EUP (two aspects of quantum gravity) on Jüttner distribution and the corresponding density of states in energy space. We also addressed the consequence of applying our approach to the MB distribution in order to find the density of states Equation (28) which is in agreement with previous reports [42,48], a result which confirms our approach. The results of our study are summarized in Tables 1 and 2 for the non-relativistic and relativistic regimes, respectively.

Table 1. Non-relativistic ideal gas ($2\pi\hbar = 1$).

	HUP	GUP	EUP
The volume of phase space element	1	$(1 + \eta P^2)^3$	$(1 + \alpha X^2)^3$
Density of States	$\sqrt{E}$	$\frac{\sqrt{E}}{(1+2m\eta E)^3}$	$\sqrt{E}$
Single Partition Function	$V \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}}$	$V \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} I \left(\frac{2m\eta}{\beta}, 3 \right)$	$V_{eff}(\alpha, r) \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}}$

Table 2. Relativistic ideal gas ($2\pi\hbar = 1$).

	HUP	GUP	EUP
The volume of phase space element	1	$(1 + \eta P^2)^3$	$(1 + \alpha X^2)^3$
Density of States	$E\sqrt{E^2 - m^2}$	$\frac{E\sqrt{E^2 - m^2}}{(1+\eta(E^2 - m^2))^3}$	$E\sqrt{E^2 - m^2}$
Single Partition Function	$V \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \Psi(\sigma)$	$V \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \Psi(\sigma) \left(1 - \eta \frac{15}{2} \frac{1}{\beta m} \right)$ when $m \gg \frac{1}{\beta}$	$V_{eff}(\alpha, r) \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \Psi(\sigma)$

It is obvious from Figures 1 and 2, that the effects of the existence of a non-zero minimal length ($\eta \neq 0$) on distribution functions become more sensible as energy increases. This means that the probability of achieving high energy states when $\eta \neq 0$ is smaller than

the $\eta = 0$ case. It is also worth mentioning that though there exist some proposals to test observable effects of the minimal length [63], the Planck scale is currently far beyond our reach. Since by comparing the Planck energy ($\approx 10^{16}$ TeV) [64] to the energy achieved in the Large Hadron Collider (≈ 10 TeV) [65], or the Planck length ($\approx 10^{-35}$ m) to the uncertainty within the position of the LIGO mirrors ($\approx 10^{-18}$ m) [66] or the Planck time ($\approx 10^{-44}$ s) to the shortest light pulse produced in laboratory ($\approx 10^{-17}$ s) [67], we observe that we are at best 15 orders of magnitude away from achieving the Planck scale. In this regard, future developments within these experimental setups are expected in order to search for the footprints of GUP effects in nature.

Finally, regarding the results reported in [68] and [69] the usefulness of Tsallis distribution function in high energy physics is expected. In line with these results, some researchers study the possibility of describing the distribution of transverse momentum in the Large Hadron Collider and Relativistic Heavy Ion Collider, employing the Tsallis distribution, expressed as [70–73]

$$f_T(q, \beta) = Z_T [1 - (1 - q)\beta E]^{\frac{1}{1-q}}. \quad (41)$$

Here Z_T and q denote the normalization constant and non-extensivity parameter, respectively. Although utilizing our approach to investigate the effects of GUP and EUP on Equation (41) is straightforward, such a study needs more careful analysis owing to the issues raised by [38] which states a criterion on the domains of validity of Maxwell–Boltzmann, Jüttner, and Tsallis distributions as a special high energy phenomenon. Therefore, it can be considered as an attractive topic for future studies.

Author Contributions: All authors have the same contribution. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The study did not report any data.

Conflicts of Interest: The authors declare no conflict of interest.

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