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# Axial $U_A(1)$ Anomaly: A New Mechanism to Generate Massless Bosons

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**Abstract:** Prior to the establishment of $QCD$ as the correct theory describing hadronic physics, it was realized that the essential ingredients of the hadronic world at low energies are chiral symmetry and its spontaneous breaking. Spontaneous symmetry breaking is a non-perturbative phenomenon, and, thanks to massive $QCD$ simulations on the lattice, we have at present a good understanding of the vacuum realization of the non-abelian chiral symmetry as a function of the physical temperature. As far as the $U_A(1)$ anomaly is concerned, and especially in the high temperature phase, the current situation is however far from satisfactory. The first part of this article is devoted to reviewing the present status of lattice calculations, in the high temperature phase of $QCD$, of quantities directly related to the $U_A(1)$ axial anomaly. In the second part, some recently suggested interesting physical implications of the $U_A(1)$ anomaly in systems where the non-abelian axial symmetry is fulfilled in the vacuum are analyzed. More precisely it is argued that, if the $U_A(1)$ symmetry remains effectively broken, the topological properties of the theory can be the basis of a mechanism, other than Goldstone's theorem, to generate a rich spectrum of massless bosons at the chiral limit.

**Keywords:** chiral transition; lattice $QCD$; $U(1)$ anomaly; topology; massless bosons

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## 1. Introduction

Nowadays, we know that symmetries play an important role in determining the Lagrangian of a quantum field theory. There are essentially two types of symmetry, local ones, or gauge symmetries, and global ones. The gauge symmetries are characterized by transformations which depend on the space-time coordinates, while, in global symmetries, the transformations are space-time independent. In addition, gauge symmetries serve to fix the couplings of the Lagrangian and global symmetries allow us to to assign quantum numbers to the particles and to predict the existence of massless bosons when a continuous global symmetry is spontaneously broken.

In what concerns $QCD$, the theory of the strong interaction, and prior to the establishment of this theory as the correct theory describing hadronic physics, it was realized that the essential ingredients of the hadronic world at low energies are chiral symmetry and its spontaneous breaking. Indeed, these two properties of the strong interaction have important phenomenological implications and allow us to understand some puzzling phenomena such as why pions have much smaller masses than the proton mass and why we do not see degenerate masses for chiral partners in the boson sector and parity partners in the baryon sector.

Chiral symmetry breaking by the vacuum state of QCD is a non-perturbative phenomenon, which results from the interaction of many microscopic degrees of freedom and can be investigated mainly through lattice QCD simulations. As a matter of fact, lattice $QCD$ is the most powerful technique for investigating non-perturbative effects from first principles. However, putting chiral symmetry onto the lattice turned out to be a difficult task. The underlying reason is that a naive lattice regularization suffers from the doubling problem. The addition of the Wilson term to the naive action solves the doubling problem but breaks chiral symmetry explicitly, even for massless quarks. This is usually not considered to be a fundamental problem because we expect that the symmetry is restored in the continuum limit. However, at finite lattice spacing, chiral symmetry may still be rather strongly violated by lattice effects.

On the other hand, staggered fermions cope to the doubling problem reducing the number of species from sixteen to four, and to reduce the number of fermion species from four to one, a rooting procedure has been used. Even if controversial, the rooting procedure has allowed obtaining very accurate results in lattice $QCD$ simulations with two and three flavors.

The doubling problem cannot be simply overcome because there is a fundamental theorem by Nielsen and Ninomiya which states that, on the lattice, one cannot implement chiral symmetry as in the continuum formulation, and at the same time have a theory free of doublers. However, despite this difficulty, the problem of chiral symmetry on the lattice was solved at the end of the past century with a generalization of chiral symmetry, through the so-called Ginsparg–Wilson equation for the lattice Dirac operator, which replaces the standard anticommutation relation of the continuum formulation $D\gamma_5 + \gamma_5 D = 0$ by $D\gamma_5 + \gamma_5 D = aD\gamma_5 D$. With this new concept, a clean implementation of chiral symmetry on the lattice has been achieved. The axial transformations reduce to the continuum transformations in the naive continuum limit, but at finite lattice spacing, $a$, an axial transformation involves also the gauge fields, and this is how the Ginsparg–Wilson formulation evades the Nielsen–Ninomiya theorem.

All these features are well established in the lattice community, and the interested reader can find in [1], for instance, a very good guide.

Returning to the topic of $QCD$ phenomenology, there is also another puzzling phenomenon which is known as the $U(1)$ problem. The $QCD$ Lagrangian for massless quarks is invariant under the chiral group $U_V(N_f) \times U_A(N_f) = SU_V(N_f) \times SU_A(N_f) \times U_V(1) \times U_A(1)$, with $V$ and $A$ denoting vector and axial vector transformations respectively. Below $1GeV$, the flavor index $f$ runs from 1 to 3 (up, down, and strange quarks), and the chiral symmetry group is $U_V(3) \times U_A(3)$. The lightweight pseudoscalars found in Nature suggest, as stated above, that the $U_A(3)$ axial symmetry is spontaneously broken in the chiral limit, but in such a case we would have nine Goldstone bosons. The pions, $K$-meson, and $\eta$-meson are eight of them but the candidate for the ninth Goldstone boson, the $\eta'$-meson, has too great a mass to be a quasi-Goldstone boson. This is the axial $U(1)$ problem that 't Hooft solved by realizing that the $U_A(1)$ axial symmetry is anomalous at the quantum level. 't Hooft's resolution of the U(1) problem suggests in a natural way the introduction of a $CP$ violating term in the $QCD$ Lagrangian, the $\theta$-term, thus generating another long standing problem, the strong $CP$ problem.

Thanks to massive QCD simulations on the lattice, we have at present a good qualitative and quantitative understanding on the vacuum realization of the non-abelian $SU_A(N_f)$ chiral symmetry, as a function of the physical temperature, but as far as $U_A(1)$ anomaly and its associated $\theta$ parameter are concerned, and especially in the high temperature phase, the current situation is far from satisfactory, and this makes understanding the role of the $\theta$ parameter in QCD, as well as its connection with the strong CP problem, one of the biggest challenges for high energy theorists [2].

The aim to elucidate the existence of new low-mass weakly interacting particles from a theoretical, phenomenological, and experimental point of view is intimately related to this issue. The light particle that has gathered the most attention has been the axion, predicted by Weinberg [3] and Wilczek [4], in the Peccei and Quinn mechanism [5], to explain the absence of parity and temporal invariance violations induced by the QCD vacuum. The axion is one of the more interesting candidates to make the dark matter of the universe, and the axion potential, which determines the dynamics of the axion field, plays a fundamental role in this context.

The calculation of the topological susceptibility in QCD is already a challenge, but calculating the complete potential requires a strategy to deal with the so called sign problem, that is, the presence of a highly oscillating term in the path integral. Indeed, Euclidean lattice gauge theory has not been able to help us much because of the imaginary contribution to the action, coming from the $\theta$-term, which prevents the applicability of the importance sampling method [6].

The $QCD$ axion model relates the topological susceptibility $\chi_T$ at $\theta = 0$ with the axion mass $m_a$ and decay constant $f_a$ through the relation $\chi_T = m_a^2 f_a^2$. The axion mass is, on the other hand, an essential ingredient in the calculation of the axion abundance in the Universe. Therefore, a precise computation of the temperature dependence of the topological susceptibility in $QCD$ becomes of primordial interest in this context.

This article focuses on the current status of the lattice calculations, in the high temperature chirally symmetric phase of $QCD$, of quantities directly related to the $U_A(1)$ axial anomaly, as the topological and axial $U_A(1)$ susceptibilities, and screening masses, as well as discusses on some interesting physical implications of the $U_A(1)$ axial anomaly in systems where the non-abelian axial symmetry is fulfilled in the vacuum. In Section 2, some theoretical prejudices about the effects of the axial anomaly in the high temperature phase of $QCD$ are briefly reviewed, and what the results of the numerical simulations on the lattice suggest on the effectiveness of the axial anomaly in this phase is analyzed. In Section 3, it is argued that the topological properties of a quantum field theory, with $U_A(1)$ anomaly and exact non-abelian axial symmetry, as for instance $QCD$ in the high temperature phase, can be the basis of a mechanism, other than Goldstone's theorem, to generate a rich spectrum of massless bosons at the chiral limit. The two-flavor Schwinger model, which was analyzed by Coleman [7] many years ago, is an excellent test bed for verifying the predictions of Section 3, and Section 4 contains the results of this test. The last section contains a discussion of the results reported in this article.

## 2. Theoretical Biases Versus Numerical Results

The large mass of the $\eta'$ meson should come from the effects of the $U_A(1)$ axial anomaly and its related gauge field topology, both present in $QCD$. Despite the difficulty of computing the contribution of disconnected diagrams to the $\eta'$ correlator in lattice simulations, these obstacles have been overcome and lattice calculations [8–10] give a mass for the $\eta'$ meson compatible with its experimental value, and this can be seen as an indirect confirmation that the effects of the anomaly are present in the low temperature phase of QCD.

Conversely, the current situation regarding the fate of the axial anomaly in the high temperature phase of $QCD$, where the non-abelian axial symmetry is not spontaneously broken, is unclear, and this is quite unsatisfactory. The nature of the chiral phase transition in two-flavor $QCD$, for instance, is affected by the way in which the effects of the $U_A(1)$ axial anomaly manifest themselves around the critical temperature [11]. Indeed, if the $U_A(1)$ axial symmetry remains effectively broken, we expect a continuous chiral transition belonging to the three-dimensional $O(4)$ vector universality class, which shows a critical exponent $\delta = 4.789(6)$ [12], while, if $U_A(1)$ is effectively restored, the chiral transition is first order or second order with critical exponents belonging to the $U_V(2) \times U_A(2) \to U_V(2)$ universality class ($\delta = 4.3(1)$) [13].

The first investigations on the fate of the $U_A(1)$ axial anomaly in the chiral symmetry restored phase of $QCD$ started a long time ago. The idea that the chiral symmetry restored phase of two-flavor QCD could be symmetric under $U_V(2) \times U_A(2)$ rather than $SU_V(2) \times SU_A(2)$ was raised by Shuryak in 1994 [14], based on an instanton liquid-model study. In 1996, Cohen [15] showed, using the continuum formulation of two-flavor QCD, and assuming the absence of the zero mode's contribution, that all the disconnected contributions to the two-point correlation functions in the $SU_A(2)$ symmetric phase at high temperature vanish in the chiral limit. The main conclusion of this work is that the eight scalar and pseudoscalar mesons should have the same mass in the chiral limit, the typical effects of the $U_A(1)$ axial anomaly being absent in this phase. In addition, Cohen argued in [16] that the analyticity of the free energy density in the quark mass $m$, around $m = 0$, in the high temperature phase, imposes constraints on the spectral density of the Dirac operator around the origin which are enough to guarantee the previous results.

Later on, Aoki et al. [17] obtained constraints on the Dirac spectrum of overlap fermions, strong enough for all of the $U(1)_A$ breaking effects among correlation functions of scalar and pseudoscalar operators to vanish, and they concluded that there is no remnant of the $U(1)_A$ anomaly above the critical temperature in two-flavor $QCD$, at least in these correlation functions. Their results were obtained under the assumptions that $m$-independent observables are analytic functions of the square quark-mass $m^2$, at $m = 0$, and that the Dirac spectral density can be expanded in Taylor series near the origin, with a non-vanishing radius of convergence.

The range of applicability of the assumptions made by [17] is however unclear. As stated by the authors, their result strongly relies on their assumption that the vacuum expectation values of quark-mass independent observables, as the topological susceptibility, are analytic functions of the square quark-mass, $m^2$, if the non-abelian chiral symmetry is restored. The two-flavor Schwinger model has a non-spontaneously broken $SU_A(2)$ chiral symmetry and $U_A(1)$ axial anomaly, and Coleman's result for the topological susceptibility in this model [7]

$$\chi_T \propto m^{\frac{4}{3}} e^{\frac{2}{3}}$$

shows explicitly a non-analytic quark-mass dependence, and thus casts doubt on the general validity of the assumptions made in [17].

In [18] a Ginsparg–Wilson fermion lattice regularization is used, and it is argued that, if the vacuum energy density is an analytical function of the quark mass in the high temperature phase of two-flavor $QCD$, all effects of the axial anomaly should disappear. The main conclusion of [18] was that either the typical effects of the axial $U_A(1)$ anomaly disappear in the symmetric high temperature phase or the vacuum energy density shows a singular behavior in the quark mass at the chiral limit.

On the other hand, an analysis of chiral and $U_A(1)$ symmetry restoration based on Ward identities and $U(3)$ chiral perturbation theory is carried out in [19,20]. The authors showed in their work that, in the limit of exact $O(4)$ restoration, understood in terms of $\delta - \eta$ partner degeneration, the Ward identities analyzed yield also $O(4) \times U_A(1)$ restoration in terms of $\pi - \eta$ degeneration, and the pseudo-critical temperatures for restoration of $O(4)$ and $O(4) \times U_A(1)$ tend to coincide in the chiral limit.

The first lattice simulations to investigate the fate of the $U_A(1)$ axial anomaly [21,22] also started in the 1990s. In Ref. [21] the authors report results of a numerical simulation of the two-flavor model with staggered quarks. They computed two order parameters, $\chi_\pi - \chi_\sigma$ for the $SU_A(2)$ chiral symmetry and $\chi_\pi - \chi_\delta$ for the $U_A(1)$ axial symmetry, where $\chi_\pi$, $\chi_\sigma$, and $\chi_\delta$ are the pion, $\sigma$, and $\delta$-meson susceptibilities, respectively, and they showed evidence for a restoration of the $SU_V(2) \times SU_A(2)$ chiral symmetry, just above the crossover, but not of the axial $U_A(1)$ symmetry. Ref. [22] contains the results of a similar calculation in two-flavor $QCD$ using also a staggered fermion lattice regularization. As stated by the authors, the relatively coarse lattice spacing in their simulations, $a \sim \frac{1}{3}$ Fermi, does not allow for conclusive results on the effectiveness of the $U(1)_A$ anomaly.

After these pioneering works, this issue has been extensively investigated using numerical simulations on the lattice, and the works in [23–43] are representative of that. We focus below on the most recently obtained results.

In [29], $(2+1)$-flavor $QCD$ is simulated, using chiral domain wall fermions, for temperatures between 139 and 196 MeV. The light-quark mass is chosen so that the pion mass is held fixed at a heavier-than-physical 200 MeV value, while the strange quark mass is set to its physical value. The authors reported results for the chiral condensates, connected and disconnected susceptibilities, and the Dirac eigenvalue spectrum and find a pseudocritical temperature $T_c \sim 165$ MeV and clear evidence for $U_A(1)$ symmetry breaking above $T_c$.

Ref. [31] also provided a study of $QCD$ with $(2+1)$-flavors of highly improved staggered quarks. The authors investigated the temperature dependence of the anomalous $U_A(1)$

symmetry breaking in the high temperature phase, and to this end they employed the overlap Dirac operator, exploiting its property of preserving the index theorem even at non-vanishing lattice spacing. The pion mass is fixed to 160 MeV, and, by quantifying the contribution of the near-zero eigenmodes to $\chi_\pi - \chi_\delta$, the authors concluded that the anomalous breaking of the axial symmetry in $QCD$ is still visible in the range $T_c \leqslant T \leqslant 1.5T_c$.

The thermal transition of $QCD$ with two degenerate light flavors is analyzed in [34] by lattice simulations, using $O(a)$-improved Wilson quarks and the unimproved Wilson plaquette action. In this work, the authors investigated the strength of the anomalous breaking of the $U_A(1)$ symmetry in the chiral limit by computing the symmetry restoration pattern of screening masses in various isovector channels, and, to quantify the strength of the $U_A(1)$-anomaly, they used the difference between scalar and pseudoscalar screening masses. They concluded that their results suggest that the $U_A(1)$-breaking is strongly reduced at the transition temperature, and that this disfavors a chiral transition in the O(4) universality class.

Results for mesonic screening masses in the temperature range 140 MeV $\leqslant T \leqslant$ 2500 MeV in $(2+1)$-flavor $QCD$, using the highly improved staggered quark action, are also reported by the $HotQCD$ Collaboration in [41], with a physical value for the strange quark mass, and two values of the light quark mass corresponding to pion masses of 160 and 140 MeV. Comparing screening masses for chiral partners, related through the chiral $SU_L(2) \times SU_R(2)$ and the axial $U_A(1)$ transformations, respectively, the authors found, in the case of light–light mesons, evidence for the degeneracy of screening masses related through the chiral $SU_L(2) \times SU_R(2)$ at or very close to the pseudocritical temperature, $T_{pc}$, while screening masses related through an axial $U_A(1)$ transformation start becoming degenerate only at about $1.3T_{pc}$.

A recent calculation in $(2+1)$-flavor $QCD$ [42], using also the highly improved staggered quark action, shows, after continuum and chiral extrapolations, that the axial anomaly remains manifested in two-point correlation functions of scalar and pseudoscalar mesons in the chiral limit, at a temperature of about 1.6 times the chiral phase transition temperature. The analysis is based on novel relations between the nth-order light quark mass derivatives of the Dirac eigenvalue spectrum, $\rho(\lambda, m_l)$, and the $(n+1)$-point correlations among the eigenvalues of the massless Dirac operator, and the calculations were carried out at the physical value of the strange quark mass, three lattice spacings, and light quark masses corresponding to pion masses in the range 55–160 MeV.

Ref. [43] provided the latest results of the JLQCD collaboration. In this work, the authors investigated the fate of the $U_A(1)$ axial anomaly in two-flavor $QCD$ at temperatures 190–330 MeV using domain wall fermions, reweighted to overlap fermions, at a lattice spacing of 0.07 fm. They measured the axial $U_A(1)$ susceptibility, $\chi_\pi - \chi_\delta$, and examined the degeneracy of $U_A(1)$ partners in meson and baryon correlators. Their conclusion is that all the data above the critical temperature indicate that the axial $U_A(1)$ violation is consistent with zero within statistical errors.

All the results discussed thus far mainly refer to the temperature dependence of the axial susceptibility $U_A(1)$, screening masses, and related quantities. The topological susceptibility, $\chi_T$, is another observable that can be useful in investigating the fate of the axial anomaly in the high-temperature phase of $QCD$, and its dependence on temperature has also been extensively investigated [35–37,39,43].

The authors of Ref. [35] explored $N_f = 2+1$ $QCD$ in a range of temperatures, from $T_c$ to around $4T_c$, and their results for the topological susceptibility differ strongly, both in the size and in the temperature dependence, from the dilute instanton gas prediction, giving rise to a shift of the axion dark-matter window of almost one order of magnitude with respect to the instanton computation.

The authors of Ref. [36], however, observed in the same model very distinct temperature dependences of the topological susceptibility in the ranges above and below 250 MeV; while, for temperatures above 250 MeV, the dependence is found to be consistent with the dilute instanton gas approximation, at lower temperatures, the falloff of topological susceptibility is milder.

On the other hand, a novel approach is proposed in [37], i.e., the fixed $Q$ integration, based on the computation of the mean value of the gauge action and chiral condensate at fixed topological charge $Q$; the authors found a topological susceptibility many orders of magnitude smaller than that of Ref. [35] in the cosmologically relevant temperature region.

A more recent lattice calculation [39] of the topological properties of $N_f = 2 + 1$ QCD with physical quark masses and temperatures around 500 MeV gives as a result a small but non-vanishing topological susceptibility, although with large error bars in the continuum limit extrapolations, pointing that the effects of the $U_A(1)$ axial anomaly still persist at these temperatures.

The JLQCD collaboration [43] also reported results for the topological susceptibility in two-flavor $QCD$, in the temperature range 195–330 MeV, for several quark masses, and their data show a suppression of $\chi_T(m)$ near the chiral limit. The authors claimed that their results are not accurate enough to determine whether $\chi_T(m)$ vanishes at a finite quark mass.

In short, we see how, despite the great effort devoted to investigating the fate of the axial anomaly in the chirally symmetric phase of $QCD$, the current situation on this issue is far from satisfactory.

## 3. Physical Effects of the $U_A(1)$ Anomaly in Models with Exact $SU_A(N_f)$ Chiral Symmetry

We devote the rest of this article mainly to analyze the physical effects of the $U_A(1)$ anomaly in a fermion-gauge theory with two or more flavors, which exhibits an exact $SU_A(N_f)$ chiral symmetry in the chiral limit. However, we also give a quick look to the one-flavor model and to the multi-flavor model with spontaneous non-abelian chiral symmetry breaking. Although many of the results presented here can be found in [18,44,45], we make the rest of this article self-contained for ease of reading.

We show in this section that a gauge-fermion quantum field theory, with $U_A(1)$ axial anomaly, and in which the scalar condensate vanishes in the chiral limit because of an exact non-abelian $SU_A(2)$ chiral symmetry, should exhibit a singular quark-mass dependence of the vacuum energy density and a divergent correlation length in the correlation function of the scalar condensate, if the $U_A(1)$ symmetry is effectively broken. On the contrary, if we assume that all correlation lengths are finite, and hence the vacuum energy density is an analytical function of the quark mass, we show that the vacuum energy density becomes, at least up to second order in the quark masses, $\theta$-independent. In the former case, the non-anomalous Ward–Takahashi (W-T) identities tell us that several pseudoscalar correlation functions, those of the $SU_A(2)$ chiral partners of the flavor singlet scalar meson, should exhibit a divergent correlation length too. We also argue that this result can be generalized for any number of flavors $N_f > 2$.

### 3.1. Some Background

To begin, let us write the continuum Euclidean action for a vector-like gauge theory with global $U_A(1)$ anomaly in the presence of a $\theta$-vacuum term

$$S = \int \mathrm{d}^d x \left\{ \sum_f^{N_f} \bar{\psi}_f(x) \left( \gamma_\mu D_\mu(x) + m_f \right) \psi_f(x) + \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) + i\theta Q(x) \right\} \tag{1}$$

where $d$ is the space-time dimensionality, $D_\mu(x)$ is the covariant derivative, $N_f$ is the number of flavors, and $Q(x)$ is the density of topological charge of the gauge configuration. The topological charge $Q$ is the integral of the density of topological charge $Q(x)$ over the space-time volume, and it is an integer number which in the case of $QCD$ reads as follows

$$Q = \frac{g^2}{64\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x). \tag{2}$$

To keep mathematical rigor, we avoid ultraviolet divergences with the help of a lattice regularization and use Ginsparg-Wilson (G-W) fermions [46], the overlap fermions [47,48] being an explicit realization of them. The motivation to use G-W fermions is that they share with the continuum formulation all essential ingredients. Indeed, G-W fermions show an explicit $U_A(1)$ anomalous symmetry [49], good chiral properties, a quantized topological charge, and allow us to establish and exact index theorem on the lattice [50].

The lattice fermionic action for a massless G-W fermion can be written in a compact form as

$$S_F = a^d \bar{\psi} D \psi = a^d \sum_{v,w} \bar{\psi}(v) D(v,w) \psi(w) \tag{3}$$

where $v$ and $w$ contain site, Dirac, and color indices, and $D$, the Dirac–Ginsparg–Wilson operator, obeys the essential anticommutation equation

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \tag{4}$$

$a$ being the lattice spacing.

Action (3) is invariant under the following lattice $U_A(1)$ chiral rotation

$$\psi \to e^{i\alpha\gamma_5\left(I - \frac{1}{2}aD\right)}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{i\alpha\left(I - \frac{1}{2}aD\right)\gamma_5} \tag{5}$$

which for $a \to 0$ reduces to the standard continuum chiral transformation. However, the integration measure of Grassmann variables is not invariant, and the change of variables (5) induces a Jacobian

$$e^{-i2\alpha \frac{a}{2} tr(\gamma_5 D)} \tag{6}$$

where

$$\frac{a}{2} tr(\gamma_5 D) = n_- - n_+ = Q \tag{7}$$

is an integer number, the difference between left-handed and right-handed zero modes, which can be identified with the topological charge $Q$ of the gauge configuration. Equations (6) and (7) show us how Ginsparg–Wilson fermions reproduce the $U_A(1)$ axial anomaly.

We can also add a symmetry breaking mass term, $m\bar{\psi}(1 - \frac{a}{2}D)\psi$ to action (3), so G-W fermions with mass are described by the fermion action

$$S_F = a^d \bar{\psi} D \psi + a^d m \bar{\psi}\left(1 - \frac{a}{2}D\right)\psi \tag{8}$$

and it can also be shown that the scalar and pseudoscalar condensates

$$S = \bar{\psi}\left(1 - \frac{a}{2}D\right)\psi \qquad P = i\bar{\psi}\gamma_5\left(1 - \frac{a}{2}D\right)\psi \tag{9}$$

transform, under the chiral $U_A(1)$ rotations (5), as a vector, just in the same way as $\bar{\psi}\psi$ and $i\bar{\psi}\gamma_5\psi$ do in the continuum formulation.

In what follows, we use dimensionless fermion fields and a dimensionless Dirac–Ginsparg–Wilson operator. In such a case, the fermion action for the $N_f$-flavor model is

$$S_F = \sum_f^{N_f} \left\{ \bar{\psi}_f D \psi_f + m_f \bar{\psi}\left(1 - \frac{1}{2}D\right)\psi_f \right\} \tag{10}$$

where $m_f$ is the mass of flavor $f$ in lattice units. The partition function of this model, in the presence of a $\theta$-vacuum term, can be written as the sum over all topological sectors, $Q$, of the partition function in each topological sector times a $\theta$-phase factor,

$$Z = \sum_Q Z_Q e^{i\theta Q} \tag{11}$$

where $Q$, which takes integer values, is bounded at finite volume by the number of degrees of freedom. At large lattice volume, the partition function should behave as

$$Z\left(\beta, m_f, \theta\right) = e^{-VE\left(\beta, m_f, \theta\right)} \tag{12}$$

where $E\left(\beta, m_f, \theta\right)$ is the vacuum energy density, $\beta$ is the inverse gauge coupling, $m_f$ is the $f$-flavor mass, and $V = V_s \times L_t$ is the lattice volume in units of the lattice spacing.

### 3.2. $Q = 0$ *Topological Sector. The One-Flavor Model and the Multi-Flavor Model with Spontaneous Chiral Symmetry Breaking*

In our analysis of the physical phenomena induced by the topological properties of the theory, the $Q = 0$ topological sector plays an essential role, and because of that we devote this subsection to review some results concerning the relation between vacuum expectation values of local and non-local operators computed in the $Q = 0$ sector, with their corresponding values in the full theory, which takes into account the contribution of all topological sectors. In particular, we show that the vacuum energy density, as well as the vacuum expectation value of any finite operator, as for instance local or intensive operators, computed in the $Q = 0$ topological sector, is equal, in the infinite volume limit, to its corresponding value in the full theory. We also show that this property is in general not true for non-local operators, the flavor-singlet pseudoscalar susceptibility being a paradigmatic example of this. However, there are non-local operators, for instance the second-order fermion-mass derivatives of the vacuum energy density, the values of which in the $Q = 0$ sector match their corresponding values in the full theory, in the infinite lattice volume limit.

We also analyze in this subsection the one-flavor case, as well as the multi-flavor case with spontaneous chiral symmetry breaking, and show how, although the aforementioned properties imply that the $U_A(1)$ symmetry is spontaneously broken in the $Q = 0$ topological sector, the Goldstone theorem is not realized because the divergence of the flavor-singlet pseudoscalar susceptibility, in this sector, does not originate from a divergent correlation length [18].

The partition function and the mean value of any operator $O$, for instance the scalar and pseudoscalar condensates, or any correlation function, in the $Q = 0$ topological sector, can be computed, respectively, as

$$Z_{Q=0} = \frac{1}{2\pi} \int \mathrm{d}\theta Z(\beta, m_f, \theta) \tag{13}$$

$$\langle O \rangle^{Q=0} = \frac{\int \mathrm{d}\theta \langle O \rangle_\theta Z(\beta, m_f, \theta)}{\int \mathrm{d}\theta Z(\beta, m_f, \theta)} \tag{14}$$

where $\langle O \rangle_\theta$, which is the mean value of $O$ computed with the lattice regularized integration measure (1), is a function of the inverse gauge coupling $\beta$, flavor masses $m_f$, and $\theta$, and we restrict ourselves to the case in which it takes a finite value in the infinite lattice volume limit. Since the vacuum energy density (12), as a function of $\theta$, has its absolute minimum at $\theta = 0$ for non-vanishing fermion masses, the following relations hold in the infinite volume limit

$$E_{Q=0}\left(\beta, m_f\right) = E\left(\beta, m_f, \theta\right)_{\theta=0} \tag{15}$$

$$\langle O \rangle^{Q=0} = \langle O \rangle_{\theta=0} \tag{16}$$

where $E_{Q=0}\left(\beta, m_f\right)$ is the vacuum energy density of the $Q = 0$ topological sector.

Taking in mind these results, let us start with the analysis of the one-flavor model at zero temperature. The results that follow apply, for instance, to one-flavor $QCD$ in four dimensions or to the one-flavor Schwinger model.

In the one flavor model, the only axial symmetry is an anomalous $U_A(1)$ symmetry. The standard wisdom on the vacuum structure of this model in the chiral limit is that it is unique at each given value of $\theta$, the $\theta$-vacuum. Indeed, the only plausible reason to have a degenerate vacuum in the chiral limit would be the spontaneous breakdown of chiral symmetry, but, since it is anomalous, actually there is no symmetry. Furthermore, due to the chiral anomaly, the model shows a mass gap in the chiral limit, and therefore all correlation lengths are finite in physical units. Since the model is free from infrared divergences, the vacuum energy density can be expanded in powers of the fermion mass $m_u$, treating the quark mass term as a perturbation [51]. This expansion is then an ordinary Taylor series

$$E(\beta, m_u, \theta) = E_0(\beta) - \Sigma(\beta) m_u \cos\theta + O(m_u^2), \tag{17}$$

giving rise to the following expansions for the scalar and pseudoscalar condensates

$$\langle S_u \rangle = -\Sigma(\beta)\cos\theta + O(m_u) \tag{18}$$

$$\langle P_u \rangle = -\Sigma(\beta)\sin\theta + O(m_u) \tag{19}$$

where $S_u$ and $P_u$ are the scalar and pseudoscalar condensates (9) normalized by the lattice volume

$$S_u = \frac{1}{V}\bar{\psi}\left(1 - \frac{1}{2}D\right)\psi \qquad P_u = \frac{i}{V}\bar{\psi}\gamma_5\left(1 - \frac{1}{2}D\right)\psi \tag{20}$$

The topological susceptibility $\chi_T$ is given, on the other hand, by the following expansion

$$\chi_T = \Sigma(\beta) m_u \cos\theta + O(m_u^2) \tag{21}$$

The resolution of the $U_A(1)$ problem is obvious if we set down the W-T identity which relates the pseudoscalar susceptibility $\chi_\eta = \sum_x \langle P_u(x) P_u(0)\rangle$, the scalar condensate $\langle S_u \rangle$, and the topological susceptibility $\chi_T$

$$\chi_\eta = -\frac{\langle S_u \rangle}{m_u} - \frac{\chi_T}{m_u^2}. \tag{22}$$

Indeed, the divergence in the chiral limit of the first term in the right-hand side of (22) is canceled by the divergence of the second term in this equation, giving rise to a finite pseudoscalar susceptibility, and a finite non-vanishing mass for the pseudoscalar $\eta$ boson.

In what concerns the $Q = 0$ topological sector, we want to notice two relevant features:

1. The global $U_A(1)$ axial symmetry is not anomalous in the $Q = 0$ topological sector.
2. If we apply Equation (16) to the computation of the vacuum expectation value of the scalar condensate, we get that the $U_A(1)$ symmetry is spontaneously broken in the $Q = 0$ sector because the chiral limit of the infinite volume limit of the scalar condensate, the limits taken in this order, does not vanish.

Equation (14) allows us to write for the infinite volume limit of the two-point pseudoscalar correlation function, $\langle P_u(x) P_u(0)\rangle$, the following relation

$$\langle P_u(x) P_u(0)\rangle^{Q=0} = \langle P_u(x) P_u(0)\rangle_{\theta=0}. \tag{23}$$

This equation implies that the mass of the pseudoscalar boson, $m_\eta$, which can be extracted from the long distance behavior of the two-point correlation function, computed in the $Q = 0$ sector, is equal to the value we should get in the full theory, taking into account the contribution of all topological sectors. On the other hand, the topological susceptibility,

$\chi_T$, vanishes in the $Q=0$ sector, and hence the W-T identity (22 ) in this sector reads as follows

$$\chi_\eta^{Q=0} = -\frac{\langle S_u \rangle^{Q=0}}{m_u}. \tag{24}$$

This identity tells us that, due to the spontaneous breaking of the $U_A(1)$ symmetry in the $Q=0$ sector, the pseudoscalar susceptibility diverges in the chiral limit, $m_u \to 0$, in this topological sector. This is a very surprising result because it suggests that the pseudoscalar boson would be a Goldstone boson, and therefore its mass, $m_\eta$, would vanish in the $m_u \to 0$ limit.

The loophole to this paradoxical result is that the divergence of the susceptibility does not necessarily implies a divergent correlation length. The susceptibility is the infinite volume limit of the integral of the correlation function over all distances, in this order, and the infinite volume limit and the space-time integral of the correlation function do not necessarily commute [18]. The infinite range interaction Ising model is a paradigmatic example of the non-commutativity of the two limits.

Let us see with some detail what actually happens. The $\langle P_u(x)P_u(0)\rangle^{Q=0}$ correlation function at any finite space-time volume $V$ verifies the following equation

$$\langle P_u(x)P_u(0)\rangle^{Q=0} = \frac{\int \mathrm{d}\theta \langle P_u(x)P_u(0)\rangle_{c,\theta} e^{-VE(\beta,m,\theta)}}{\int \mathrm{d}\theta e^{-VE(\beta,m,\theta)}} + \frac{\int \mathrm{d}\theta \langle P_u(0)\rangle_\theta^2 e^{-VE(\beta,m,\theta)}}{\int \mathrm{d}\theta e^{-VE(\beta,m,\theta)}} \tag{25}$$

where $\langle P_u(x)P_u(0)\rangle_{c,\theta}$ is the connected pseudoscalar correlation function at a given $\theta$. The first term in the right-hand side of Equation (25) converges in the infinite lattice volume limit to $\langle P_u(x)P_u(0)\rangle_{\theta=0}$, the pseudoscalar correlation function at $\theta=0$. To compute the large lattice volume behavior of the second term on the right-hand side of (25), we can expand $\langle P_u(0)\rangle_\theta^2$, and the vacuum energy density in powers of the $\theta$ angle as follows

$$\langle P_u(0)\rangle_\theta^2 = \left(m_u\chi_\eta + \langle S_u\rangle\right)^2\theta^2 + O(\theta^4). \tag{26}$$

$$E(\beta,m_u,\theta) = E_0(\beta,m_u) - \frac{1}{2}\chi_T(\beta,m_u)\theta^2 + O(\theta^4) \tag{27}$$

and making an expansion around the saddle point solution we get, for the dominant contribution to the second term of the right hand side of (25) in the large lattice volume limit,

$$\frac{\int \mathrm{d}\theta \langle P_u(0)\rangle_\theta^2 e^{-VE(\beta,m,\theta)}}{\int \mathrm{d}\theta e^{-VE(\beta,m,\theta)}} = \frac{1}{V}\frac{\left(m_u\chi_\eta + \langle S_u\rangle\right)^2}{\chi_T}. \tag{28}$$

Since the topological susceptibility $\chi_T$ is linear in $m_u$, for small fermion mass (21), and the scalar condensate $\langle S_u\rangle$ is finite in the chiral limit, this contribution is singular at $m_u=0$.

Equations (25) and (28) show that indeed the pseudoscalar correlation function in the zero-charge topological sector converges, in the infinite volume limit, to the pseudoscalar correlation function in the full theory at $\theta=0$. These equations also show what we can call a *cluster violation* at finite volume for the pseudoscalar correlation function, in the $Q=0$ topological sector, which disappears in the infinite volume limit. This *cluster violation* at finite volume is therefore irrelevant in what concerns the pseudoscalar correlation function, but, conversely, it plays a fundamental role when computing the pseudoscalar susceptibility in the $Q=0$ topological sector. In fact, if we sum up in Equation (25) over all lattice points, and take the infinite volume limit, just in this order, we get for the pseudoscalar susceptibility in the $Q=0$ topological sector

$$\chi_\eta^{Q=0} = \chi_\eta + \frac{\left(m_u\chi_\eta + \langle S_u\rangle\right)^2}{\chi_T}. \tag{29}$$

This equation shows that the pseudoscalar susceptibility, in the $Q=0$ sector, diverges in the chiral limit due to the finite contribution of (28) to this susceptibility. Hence, it is shown that, although the $Q = 0$ topological sector breaks spontaneously the $U_A(1)$ axial symmetry to give account of the anomaly, the Goldstone theorem is not fulfilled because the divergence of the pseudoscalar susceptibility in this sector does not come from a divergent correlation length.

The multi-flavor model with spontaneous non-abelian chiral symmetry breaking, as for instance $QCD$ in the low temperature phase, shows some important differences with respect to the one-flavor case. The model also suffers from the chiral anomaly and has a spontaneously broken $SU_A(N_f)$ chiral symmetry. Because of the Goldstone theorem, there are $N_f^2 - 1$ massless pseudoscalar bosons in the chiral limit, and, in contrast to the one-flavor case, the infinite-volume limit and the chiral limit do not commute. However, in what concerns the flavor-singlet pseudoscalar susceptibility, the essential features previously described for the one-flavor model still work in the several-flavors case.

Let us consider the simplest case of two degenerate flavors, $m_u = m_d = m$. The anomalous W-T identity (22) for the flavor-singlet pseudoscalar susceptibility reads now

$$\chi_\eta = -\frac{\langle S \rangle}{m} - \frac{4\chi_T}{m^2} \tag{30}$$

while the non-anomalous identity for the pion susceptibility is

$$\chi_{\vec{\pi}} = -\frac{\langle S \rangle}{m} \tag{31}$$

where $S = S_u + S_d$. The $Q = 0$ sector breaks spontaneously the $U_A(2)$ symmetry, and the W-T identities for this sector are

$$\chi_\eta^{Q=0} = \chi_{\vec{\pi}}^{Q=0} = -\frac{\langle S \rangle^{Q=0}}{m} - \tag{32}$$

The analysis done in this subsection allows to conclude that, although $\chi_{\vec{\pi}}$ is a non-local operator, it takes, in the infinite lattice volume limit, the same value in the $Q = 0$ sector as in the full theory. Conversely, that is not true for the flavor-singlet pseudoscalar susceptibility, which diverges in the chiral limit in the $Q = 0$ sector, while remaining finite in the full theory. A straightforward analysis, as the one done for the one-flavor case, shows that, again, the divergence of $\chi_\eta^{Q=0}$ does not come from a divergent correlation length.

The case in which the $SU(N_f)$ chiral symmetry is fulfilled in the vacuum is discussed in detail in the next subsections.

### *3.3. Two Flavors and Exact $SU_A(2)$ Chiral Symmetry*

There are several relevant physical theories, as for instance the two-flavor Schwinger model or $QCD$ in the high temperature phase, that suffer from the $U_A(1)$ axial anomaly, and in which the non-abelian chiral symmetry is fulfilled in the vacuum. We discuss in what follows what are the physical expectations in these theories. It is argued that a theory which verifies the aforementioned properties should show, in the chiral limit, a divergent correlation length, and a rich spectrum of massless chiral bosons. To this end, we start with the assumption that all correlation lengths are finite and show that, in such a case, the axial $U_A(1)$ symmetry is effectively restored.

We consider a fermion-gauge model with two flavors, up and down, with masses $m_u$ and $m_d$, exact $SU_A(2)$ chiral symmetry, and global $U_A(1)$ axial anomaly. The Euclidean fermion-gauge action (10) is

$$S_F = m_u \bar{\psi}_u \left(1 - \frac{1}{2}D\right)\psi_u + m_d \bar{\psi}_d \left(1 - \frac{1}{2}D\right)\psi_d + \bar{\psi}_u D \psi_u + \bar{\psi}_d D \psi_d \tag{33}$$

where $D$ is the Dirac–Ginsparg–Wilson operator.

If we assume, as in the one-flavor model, that all correlation lengths are finite, and the model shows a mass gap in the chiral limit, the vacuum energy density can also be expanded, as in the one-flavor case, in powers of the fermion masses $m_u, m_d$, as an ordinary Taylor series

$$E(\beta, m_u, m_d) = E(\beta, 0, 0) - \frac{1}{2}m_u^2 \chi_{s_{u,u}}(\beta) - \frac{1}{2}m_d^2 \chi_{s_{d,d}}(\beta) - m_u m_d \chi_{s_{u,d}}(\beta) + \ldots \quad (34)$$

The linear terms in (34) vanish because the $SU_A(2)$ symmetry is fulfilled in the vacuum, and $\chi_{s_{u,u}}, \chi_{s_{d,d}}$, and $\chi_{s_{u,d}}$ are the scalar up, down, and up–down susceptibilities, respectively,

$$\begin{aligned} \chi_{s_{u,u}}(\beta) &= V\left\langle S_u^2 \right\rangle_{m_u=m_d=0} \\ \chi_{s_{d,d}}(\beta) &= V\left\langle S_d^2 \right\rangle_{m_u=m_d=0} \\ \chi_{s_{u,d}}(\beta) &= V\langle S_u S_d \rangle_{m_u=m_d=0} \end{aligned} \quad (35)$$

where $S_u$ and $S_d$ are the scalar up and down condensates (20), normalized by the lattice volume. The disconnected contributions are absent in (35) because the $SU_A(2)$ chiral symmetry constrains $\langle S_u \rangle_{m_u=m_d=0}$ and $\langle S_d \rangle_{m_u=m_d=0}$ to vanish, and $\chi_{s_{u,u}}(\beta) = \chi_{s_{d,d}}(\beta)$ because of flavor symmetry. Moreover, we know that the vacuum energy density of the $Q = 0$ topological sector, in the infinite volume limit, is also be given by (34). (In [45], it is implicitly assumed that the vacuum energy density of the $Q = 0$ sector is also a $C^2$ function of $m_u$ and $m_d$. We show here that this assumption is justified.)

In the presence of a $\theta$-vacuum term, expansion (34) becomes

$$E(\beta, m_u, m_d) = E(\beta, 0, 0) - \frac{1}{2}m_u^2 \chi_{s_{u,u}}(\beta) - \frac{1}{2}m_d^2 \chi_{s_{d,d}}(\beta) - m_u m_d \cos\theta \chi_{s_{u,d}}(\beta) + \ldots \quad (36)$$

The scalar up and down susceptibilities for massless fermions get all their contribution from the $Q = 0$ topological sector, and therefore we can write

$$\chi_{s_{u,u}}(\beta) = \chi_{s_{d,d}}(\beta) = \chi_{s_{u,u}}^{Q=0}(\beta) = \chi_{s_{d,d}}^{Q=0}(\beta)$$

Since the $SU_A(2)$ chiral symmetry is fulfilled in the vacuum, the vacuum expectation value of any local order parameter for this symmetry vanishes in the chiral limit. It is also shown that any local operator takes, in the thermodynamic limit, the same vacuum expectation value in the $Q = 0$ topological sector as in the full theory. Therefore, the $SU(2)_A$ chiral symmetry of the $Q = 0$ sector should also be fulfilled in the vacuum of this sector.

The scalar up and down susceptibilities, in the $Q = 0$ sector, for non-vanishing quark masses, also agree with their corresponding values in the full theory, in the infinite volume limit (the simplest way to see that is true is to take into account that these susceptibilities can be obtained as second-order mass derivatives of the free energy density, and the free energy density of the $Q = 0$ sector and of the full theory agree if both quark masses are of the same sign)

$$\chi_{s_{u,u}}^{Q=0}(\beta, m_u, m_d) = \chi_{s_{u,u}}(\beta, m_u, m_d) \qquad \chi_{s_{d,d}}^{Q=0}(\beta, m_u, m_d) = \chi_{s_{d,d}}(\beta, m_u, m_d).$$

Therefore, these quantities can be obtained from (34)

$$\begin{aligned} \chi_{s_{u,u}}^{Q=0}(\beta, m_u, m_d) &= \chi_{s_{u,u}}(\beta) + \ldots \\ \chi_{s_{d,d}}^{Q=0}(\beta, m_u, m_d) &= \chi_{s_{d,d}}(\beta) + \ldots \end{aligned}$$

where the dots indicate terms that vanish in the chiral limit.

The pseudoscalar up and down susceptibilities, in the $Q=0$ sector, $\chi^{Q=0}_{p_{u,u}}(\beta, m_u, m_d) = V\langle P_u^2\rangle^{Q=0}$, $\chi^{Q=0}_{p_{d,d}}(\beta, m_u, m_d) = V\langle P_d^2\rangle^{Q=0}$, can be obtained from the W-T identities in this sector (24) beside (34)

$$\chi^{Q=0}_{p_{u,u}}(\beta, m_u, m_d) = \chi_{s_{u,u}}(\beta) + \frac{|\, m_d \,|}{|\, m_u \,|}\chi_{s_{u,d}}(\beta) + \dots$$

$$\chi^{Q=0}_{p_{d,d}}(\beta, m_u, m_d) = \chi_{s_{d,d}}(\beta) + \frac{|\, m_u \,|}{|\, m_d \,|}\chi_{s_{u,d}}(\beta) + \dots \tag{37}$$

where the absolute value of the quark masses is due to the fact that these susceptibilities are even functions of the quark masses, and again the dots indicate terms that vanish in the chiral limit.

The difference of the scalar and pseudoscalar susceptibilities for the up or down quarks, $\chi_{s_{u,u}} - \chi_{p_{u,u}}$, $\chi_{s_{d,d}} - \chi_{p_{d,d}}$, is an order parameter for both the $U_A(1)$ axial symmetry and the $SU_A(2)$ chiral symmetry. We can compute this quantity, in the full theory, making use of (34), the W-T identities (22), and the topological susceptibility

$$\chi_T(\beta, m_u, m_d) = m_u m_d \chi_{s_{u,d}}(\beta) + \dots$$

the last obtained from (36), and we get

$$\chi_{p_{u,u}}(\beta, m_u, m_d) = -\frac{\langle S_u\rangle}{m_u} - \frac{\chi_T}{m_u^2} = \chi_{s_{u,u}}(\beta) + \dots$$

$$\chi_{p_{d,d}}(\beta, m_u, m_d) = -\frac{\langle S_d\rangle}{m_d} - \frac{\chi_T}{m_d^2} = \chi_{s_{d,d}}(\beta) + \dots \tag{38}$$

where, also in this case, the dots indicate terms that vanish in the chiral limit. We see from Equation (38) that, indeed, and despite the $U_A(1)$ anomaly, $\chi_{s_{u,u}} - \chi_{p_{u,u}}$ and $\chi_{s_{d,d}} - \chi_{p_{d,d}}$, which are also order parameters for the $SU_A(2)$ chiral symmetry, vanish in the chiral limit, as it should be. (A non-local order parameter for a given symmetry, which is fulfilled in the vacuum, can diverge if the correlation length diverges, for instance the non-linear susceptibility $\chi_{nl}(h) = \frac{\partial^2 m(h)}{\partial h^2}$ in the Ising model at the critical temperature. However, we assume here that all correlation lengths are finite, and hence the non-local order parameter should vanish.)

Conversely, if we compute this order parameter in the $Q=0$ topological sector, we get

$$\chi^{Q=0}_{s_{u,u}}(\beta, m_u, m_d) - \chi^{Q=0}_{p_{u,u}}(\beta, m_u, m_d) = -\frac{|\, m_d \,|}{|\, m_u \,|}\chi_{s_{u,d}}(\beta) + \dots$$

$$\chi^{Q=0}_{s_{d,d}}(\beta, m_u, m_d) - \chi^{Q=0}_{p_{d,d}}(\beta, m_u, m_d) = -\frac{|\, m_u \,|}{|\, m_d \,|}\chi_{s_{u,d}}(\beta) + \dots$$

and therefore this order parameter for the non-abelian chiral symmetry only vanishes in the chiral limit if $\chi_{s_{u,d}}(\beta) = 0$. Thus, it is shown that, under the assumption that all correlation lengths are finite, an exact $SU_A(2)$ chiral symmetry in the $Q=0$ sector requires a $\theta$-independent vacuum energy density (36), which implies, among other things, that the axial susceptibility $\chi_\pi - \chi_\delta$, an order parameter that has been used to test the effectiveness of the $U_A(1)$ anomaly, vanishes in the chiral limit.

Note, on the other hand, that a non-vanishing value of $\chi_{s_{u,d}}(\beta)$ implies that not only the $SU_A(2)$ chiral symmetry of the $Q=0$ sector is spontaneously broken, but also the $SU_V(2)$ flavor symmetry, as follows from (37). Moreover, a simple calculation of the sum of the flavor singlet scalar $\chi_\sigma$ and pseudoscalar $\chi_\eta$ susceptibilities for massless quarks give us

$$\chi^{Q=0}_{\sigma_{m_u=m_d=0}} + \chi^{Q=0}_{\eta_{m_u=m_d=0}} = 2\chi_{s_{u,u}}(\beta) + 2\chi_{s_{d,d}}(\beta) = 4\chi_{s_{u,u}}(\beta) \tag{39}$$

while if, according to standard Statistical Mechanics, we decompose our degenerate vacuum, or Gibbs state, into the sum of pure states [52], and calculate $\chi_\sigma + \chi_\eta$ in each one of these pure states, we get

$$\chi^{Q=0}_{\sigma_{m_u=m_d=0}} + \chi^{Q=0}_{\eta_{m_u=m_d=0}} = 4\chi_{s_{u,u}}(\beta) + \frac{1+\lambda^2}{2\lambda}\chi_{s_{u,d}}(\beta) \tag{40}$$

with $\lambda = \frac{|m_d|}{|m_u|}$. We see that the consistency between Equations (39) and (40) requires again that $\chi_{s_{u,d}}(\beta) = 0$.

Therefore, even if one accepts that the $Q = 0$ sector spontaneously breaks the $SU_A(2)$ axial and $SU_V(2)$ flavor symmetries, even though all local order parameters for these symmetries vanish, we find that the consistency of the vacuum structure with the theoretical prejudices about the Gibbs state of a statistical system requires, once more, that $\chi_{s_{u,u}}(\beta) = 0$, and hence a $\theta$-independent vacuum energy density in the full theory.

In the one-flavor model, we do not find inconsistencies between the assumption that the correlation length is finite, and the physics of the $Q = 0$ topological sector. The chiral condensate takes a non vanishing value in the chiral limit, and hence the $U_A(1)$ axial symmetry is spontaneously broken in the $Q = 0$ sector, giving account in this way of the $U_A(1)$ axial anomaly of the full theory. In the two-flavor model, and under the same assumption of finiteness of the correlation length, we would need a non-vanishing value of $\chi_{s_{u,d}}(\beta)$ to have an effective $U_A(1)$ axial symmetry breaking which, also in this case, would imply the spontaneous breaking of the global $U_A(1)$ symmetry in the $Q = 0$ sector. However, in such a case, we find strong inconsistencies that lead us to conclude that, either $\chi_{s_{u,d}}(\beta) = 0$, and hence the $U_A(1)$ symmetry is effectively restored, or a divergent correlation length is imperative if the $U_A(1)$ symmetry is not effectively restored.

### *3.4. Landau Approach*

We argue that the two-flavor theory with exact $SU_A(2)$ chiral symmetry and axial $U_A(1)$ symmetry violation should exhibit a divergent correlation length in the scalar sector, in the chiral limit. In this subsection, we give a qualitative but powerful argument which strongly supports this result. To this end, we explore the expected phase diagram of the model in the $Q = 0$ topological sector [44] and apply the Landau theory of phase transitions to it.

Since the $SU_A(2)$ chiral symmetry is assumed to be fulfilled in the vacuum, and the flavor singlet scalar condensate is an order parameter for this symmetry, its vacuum expectation value $\langle S\rangle = \langle S_u\rangle + \langle S_d\rangle = 0$ vanishes in the limit in which the fermion mass $m \to 0$. However, if we consider two non-degenerate fermion flavors, up and down, with masses $m_u$ and $m_d$, respectively, and take the limit $m_u \to 0$ keeping $m_d \neq 0$ fixed, the up condensate $S_u$ reaches a non-vanishing value

$$\lim_{m_u\to 0}\langle S_u\rangle = s_u(m_d) \neq 0 \tag{41}$$

because the $U(1)_u$ axial symmetry, which exhibits our model when $m_u = 0$, is anomalous, and the $SU_A(2)$ chiral symmetry, which would enforce the up condensate to be zero, is explicitly broken if $m_d \neq 0$.

Obviously, the same argument applies if we interchange $m_u$ and $m_d$, and we can therefore write

$$\lim_{m_d\to 0}\langle S_d\rangle = s_d(m_u) \neq 0 \tag{42}$$

and since the $SU_A(2)$ chiral symmetry is recovered and fulfilled in the vacuum when $m_u, m_d \to 0$, we get

$$\lim_{m_d\to 0} s_u(m_d) = \lim_{m_u\to 0} s_d(m_u) = 0 \tag{43}$$

Let us consider now our model, with two non-degenerate fermion flavors, restricted to the $Q = 0$ topological sector. As discussed above, the mean value of any local or intensive operator in the $Q = 0$ topological sector is equal, if we restrict ourselves to the region in which both $m_u > 0$, and $m_d > 0$, to its mean value in the full theory, in the infinite volume limit (since the two flavor model with $m_u < 0$ and $m_d < 0$ at $\theta = 0$ is equivalent to the same model with $m_u > 0$ and $m_d > 0$, this result is also true if both $m_u < 0$ and $m_d < 0$). We can hence apply this result to $\langle S_u \rangle$ and $\langle S_d \rangle$ and write the following equations

$$\lim_{m_u \to 0} \langle S_u \rangle^{Q=0} = s_u(m_d) \neq 0$$

$$\lim_{m_d \to 0} \langle S_d \rangle^{Q=0} = s_d(m_u) \neq 0 \tag{44}$$

The global $U(1)_u$ axial symmetry of our model at $m_u = 0$, and the $U(1)_d$ symmetry at $m_d = 0$, are not anomalous in the $Q = 0$ sector, and Equation (44) tells us that both the $U(1)_u$ symmetry at $m_u = 0, m_d \neq 0$ and the $U(1)_d$ symmetry at $m_u \neq 0, m_d = 0$ are spontaneously broken in this sector. This is not surprising at all since the present situation is similar to what happens in the one-flavor model discussed above.

Figure 1 is a schematic representation of the phase diagram of the two-flavor model, in the $Q = 0$ topological sector, and in the $(m_u, m_d)$ plane, which emerges from the previous discussion. The two coordinate axis show first-order phase transition lines. If we cross perpendicularly the $m_d = 0$ axis, the mean value of the down condensate jumps from $s_d(m_u)$ to $-s_d(m_u)$, and the same is true if we interchange up and down. All first-order transition lines end however at a common point, the origin of coordinates $m_u = m_d = 0$, where all condensates vanish because at this point we recover the $SU_A(2)$ chiral symmetry, which is assumed to also be a symmetry of the vacuum. Notice that, if the $SU_A(2)$ chiral symmetry is spontaneously broken, as happens for instance in the low temperature phase of $QCD$, the phase diagram in the $(m_u, m_d)$ plane would be the same as that of Figure 1 with the only exception that the origin of coordinates is not an end point.

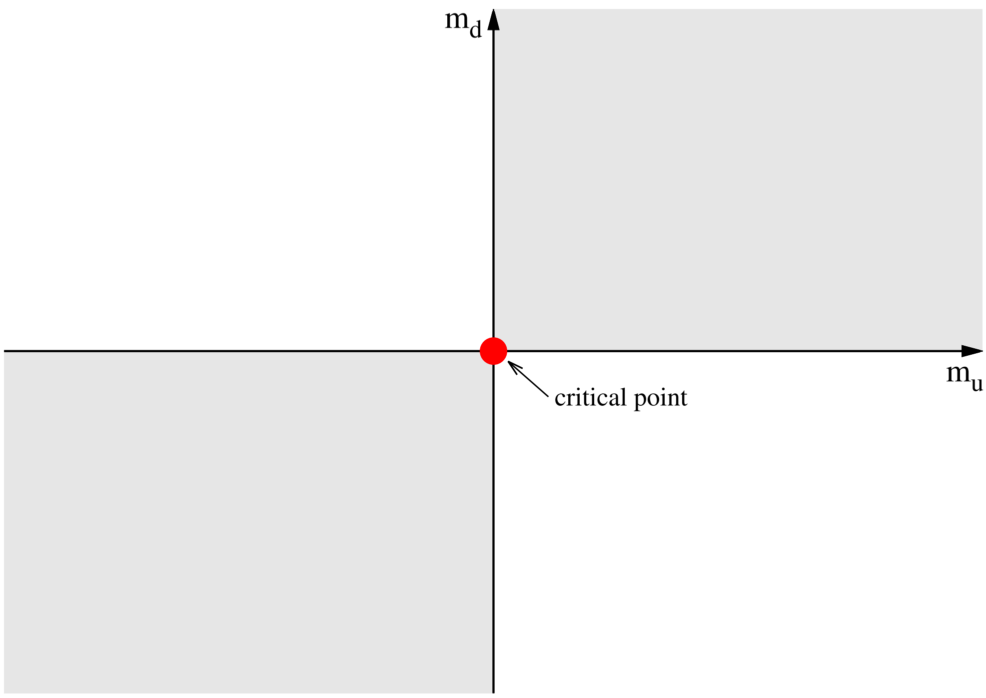


**Figure 1.** Phase diagram of the two-flavor model in the $Q = 0$ topological sector. The coordinate axis in the $(m_u, m_d)$ plane are first-order phase transition lines. The origin of coordinates is the end point of all first-order transition lines. The vacuum energy density, its derivatives, and expectation values of local operators of the two-flavor model at $\theta = 0$ only agree with those of the $Q = 0$ sector in the first $(m_u > 0, m_d > 0)$ and third $(m_u < 0, m_d < 0)$ quadrants (the darkened areas).

Landau's theory of phase transitions predicts that the end point placed at the origin of coordinates in the $(m_u, m_d)$ plane is a critical point, the scalar condensate should show a non-analytic dependence on the fermion masses $m_u$ and $m_d$ as we approach the critical point, and hence the scalar susceptibility should diverge. However, since the vacuum energy density in the $Q = 0$ topological sector, as well as its fermion mass derivatives, matches the vacuum energy density and fermion mass derivatives in the full theory, and the same is true for the critical equation of state, Landau's theory of phase transitions predicts a non-analytic dependence of the flavor singlet scalar condensate on the fermion mass, and a divergent correlation length in the chiral limit of our full theory, in which we take into account the contribution of all topological sectors.

More precisely, we can apply the Landau approach to analyze the critical behavior around the two first-order transition lines in Figure 1 near the end point, or critical point. In the analysis of the $m_d = 0$ transition line, we consider $m_d$ as an external "magnetic field" and $m_u$ as the "temperature", and vice versa for the analysis of the $m_u = 0$ line. Then, the standard Landau approach tells us that the up and down condensates verify the two following equations of state

$$-m_u \langle S_u \rangle^{-3} = -2C_1 m_d \langle S_u \rangle^{-2} + 4C_2$$
$$- m_d \langle S_d \rangle^{-3} = -2C_1 m_u \langle S_d \rangle^{-2} + 4C_2 \tag{45}$$

where $C_1$ and $C_2$ are two positive constants. If we fix the ratio of the up and down masses $\frac{m_u}{m_d} = \lambda$, the equations of state (45) allow us to write the following expansions for de up and down condensates

$$\langle S_u \rangle = -m_u^{\frac{1}{3}} \left( \left( \frac{1}{4C_2} \right)^{\frac{1}{3}} + \frac{C_1}{3(2C_2^2)^{\frac{1}{3}} \lambda} m_u^{\frac{1}{3}} + \ldots \right)$$
$$\langle S_d \rangle = -m_d^{\frac{1}{3}} \left( \left( \frac{1}{4C_2} \right)^{\frac{1}{3}} + \frac{C_1 \lambda}{3(2C_2^2)^{\frac{1}{3}}} m_d^{\frac{1}{3}} + \ldots \right) \tag{46}$$

Equation (46) shows explicitly the non analytical behavior of the up and down condensates. In the degenerate flavor case, $m_u = m_d = m$, the scalar condensate and the flavor-singlet scalar susceptibility near the critical point scale as

$$\langle S \rangle = \langle S_u \rangle + \langle S_d \rangle = -\left( \frac{2}{C_2} \right)^{\frac{1}{3}} m^{\frac{1}{3}} + \ldots$$
$$\chi_\sigma(m) = \frac{1}{3} \left( \frac{2}{C_2} \right)^{\frac{1}{3}} m^{-\frac{2}{3}} + \ldots \tag{47}$$

showing up explicitly the divergence of the flavor singlet scalar susceptibility in the chiral limit.

We see that the critical behavior of the chiral condensate in the Landau approach (47) is described by the mean field critical exponent $\delta = 3$. Mean field critical exponents are expected to be correct in high dimensions, while, in low dimensions, the effect of fluctuations can change their mean field values. This means that, in the latter case, the Landau approach give us a good qualitative description of the phase diagram but fails in its quantitative predictions of critical exponents.

To finish the Landau approach analysis, we want to point out that all these results can be generalized in a straightforward way to a number of flavors $N_f > 2$.

### 3.5. Critical Behavior of the Two-Flavor Model with an Isospin Breaking Term

Beyond the Landau approach, we can parameterize the critical behavior of the flavor singlet scalar condensate and of the mass-dependent contribution to the vacuum energy density, in the two degenerate flavor model, with a critical exponent $\delta > 1$

$$\langle S \rangle_{m \to 0} \simeq -Cm^{\frac{1}{\delta}}. \tag{48}$$

$$E(\beta, m) - E(\beta, 0) \simeq -\frac{C\delta}{\delta + 1} m^{\frac{\delta+1}{\delta}}. \tag{49}$$

where $C$ is a dimensionless positive constant that depends on the inverse gauge coupling $\beta$. Equation (48) gives us a divergent scalar susceptibility, $\chi_\sigma(m) \sim \frac{C}{\delta} m^{\frac{1-\delta}{\delta}}$, and hence a massless scalar boson as $m \to 0$.

If, on the other hand, we write the W-T identity for the isotriplet of "pions" which follows from the $SU_A(2)$ non-anomalous chiral symmetry

$$\chi_{\vec{\pi}}(m) = -\frac{\langle S \rangle}{m}, \tag{50}$$

we get that also $\chi_{\vec{\pi}}(m)$ diverges when $m \to 0$ as $Cm^{\frac{1-\delta}{\delta}}$, and a rich spectrum of massless bosons $(\sigma, \vec{\pi})$ emerges in the chiral limit. The susceptibility of the flavor singlet pseudoscalar condensate fulfills the anomalous W-T identity (30), and, because of the $U_A(1)$ axial anomaly, the $\eta$-boson mass is expected to remain finite (non-vanishing) in the chiral limit.

The hyperscaling hypothesis, which arises as a natural consequence of the block-spin renormalization group approach, says that the only relevant length near the critical point of a *magnetic* system, in what concerns the singular part $E_s(\beta, m)$ of the free or vacuum energy density, is the correlation length $\xi$. Since Equation (49) contains only the singular contribution to the vacuum energy density, we can write

$$E_s(\beta, m) \simeq -\frac{C\delta}{\delta + 1} m^{\frac{\delta+1}{\delta}} \sim \xi^{-d} \tag{51}$$

and the following relationship between the correlation length and the fermion mass

$$\xi \sim m^{-\frac{\delta+1}{d\delta}} \tag{52}$$

which implies that the pion and sigma-meson masses scale with the fermion mass as follows

$$m_{\vec{\pi}}, m_\sigma \sim m^{\frac{\delta+1}{d\delta}} \tag{53}$$

In the presence of an isospin breaking term, the fermion action can be written in a compact form as

$$S_F = \left(\frac{m_u + m_d}{2}\right) \bar{\psi}\left(1 - \frac{1}{2}D\right)\psi - \left(\frac{m_d - m_u}{2}\right) \bar{\psi}\left(1 - \frac{1}{2}D\right)\tau_3\psi + \bar{\psi}D\psi \tag{54}$$

where $\psi$ is a Grassmann field carrying site, Dirac, color, and flavor indices and $\tau_3$ is the third Pauli matrix acting in flavor space.

If we also include a $\theta$-vacuum term in the action, this $\theta$-term can be removed through a chiral $U_A(1)$ transformation, which leaves the $\bar{\psi}D\psi$ interaction term invariant. If next we also perform a suitable non-anomalous chiral transformation, we get the effective fermion action that follows

$$S_F = M(m_u, m_d, \theta)\bar{\psi}\left(1 - \frac{1}{2}D\right)\psi + A(m_u, m_d, \theta)i\bar{\psi}\gamma_5\left(1 - \frac{1}{2}D\right)\psi$$

$$+ B(m_u, m_d, \theta)\bar{\psi}\left(1 - \frac{1}{2}D\right)\tau_3\psi + \bar{\psi}D\psi \tag{55}$$

where $M(m_u, m_d, \theta)$, $A(m_u, m_d, \theta)$ and $B(m_u, m_d, \theta)$ are given by

$$M(m_u, m_d, \theta) = \frac{1}{2}\left(m_u^2 + m_d^2 + 2m_u m_d \cos\theta\right)^{\frac{1}{2}} \tag{56}$$

$$A(m_u, m_d, \theta) = \frac{2m_u m_d \sin\frac{\theta}{2}}{(m_u + m_d)\left(1 + \frac{m_u^2 + m_d^2 - 2m_u m_d}{m_u^2 + m_d^2 + 2m_u m_d}\tan^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \tag{57}$$

$$B(m_u, m_d, \theta) = -\frac{m_d - m_u}{2\cos\frac{\theta}{2}\left(1 + \frac{m_u^2 + m_d^2 - 2m_u m_d}{m_u^2 + m_d^2 + 2m_u m_d}\tan^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \tag{58}$$

Since we do not expect singularities at non-vanishing fermion masses, the vacuum energy density $E(\beta, M, A, B)$ can be expanded in powers of $A$ and $B$ as an ordinary Taylor series, and, taking into account the symmetries of the effective action (55), we can write the following equation for this expansion up to second order

$$E(\beta, m_u, m_d, \theta) \equiv E(\beta, M, A, B) = E(\beta, M, 0, 0) + \frac{1}{2}A^2\chi_\eta(\beta, M) + \frac{1}{2}B^2\chi_\delta(\beta, M) + \ldots \tag{59}$$

where $\chi_\eta(\beta, M)$ and $\chi_\delta(\beta, M)$ are the flavor singlet pseudoscalar susceptibility and the $\delta$-meson susceptibility in the theory with two degenerate flavors of mass $M(m_u, m_d, \theta)$, respectively. Note that this expansion should have a good convergence if $\theta$ and $m_d - m_u$ are small.

The vacuum energy density, to the lowest order of the expansion (59), is that of the model with two degenerate flavors of mass $M(m_u, m_d, \theta)$, in the absence of a $\theta$-vacuum term. We show above that this model should show a critical behavior (48) and (49), around the chiral limit, and hence we get, to the lowest order of this expansion,

$$E(\beta, m_u, m_d, \theta) - E(\beta, 0, 0, 0) = -\frac{C}{2^{\frac{\delta+1}{\delta}}}\frac{\delta}{\delta + 1}\left(m_u^2 + m_d^2 + 2m_u m_d \cos\theta\right)^{\frac{\delta+1}{2\delta}} + \ldots \tag{60}$$

The free energy density depends on $m_u$, $m_d$ and $\theta$ through $\left(m_u^2 + m_d^2 + 2m_u m_d \cos\theta\right)^{\frac{1}{2}}$, and its dominant contribution in the chiral limit is given by the power-law behavior of Equation (60) (note that if we apply this expansion of the vacuum energy density to two-flavor $QCD$ at $T = 0$, where chiral symmetry is spontaneously broken, and hence $\delta = \infty$ in (48) and (49), we get the vacuum energy density of the low energy chiral effective Lagrangian model [51]).

The flavor-singlet pseudoscalar susceptibility, $\chi_\eta(\beta, M)$, fulfills the anomalous W-T identity (30), and hence it is expected to remain finite in the chiral limit. Since the $SU_A(2)$ chiral symmetry is exact in this limit, the same holds true for $\chi_\delta(\beta, M)$. In such conditions, the relevance of the second-order correction to the zero-order contribution to the vacuum energy density (59), for two degenerate flavors, turns out to be

$$\frac{A^2(m, \theta)}{E(\beta, M, 0) - E(\beta, 0, 0)} \sim m^{1-\frac{1}{\delta}}\frac{\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}\right)^{1+\frac{1}{\delta}}} \tag{61}$$

while, in the isospin breaking case, and for small $\theta$ values, we have

$$\frac{A^2(m_u, m_d, \theta)}{E(\beta, M, 0, 0) - E(\beta, 0, 0, 0)} \sim \frac{m_u^2 m_d^2 \theta^2}{(m_u + m_d)^{3+\frac{1}{\delta}}}$$

$$\frac{B^2(m_u, m_d, \theta)}{E(\beta, M, 0, 0) - E(\beta, 0, 0, 0)} \sim \frac{(m_d - m_u)^2}{(m_u + m_d)^{1+\frac{1}{\delta}}} \tag{62}$$

Since $\delta > 1$ ($\delta = 3$ in the mean field model), we see that the critical behavior of the model, which describes the low energy theory, is fully controlled in both cases by the zero-order contribution to the vacuum energy density (63), and the second-order contribution can be neglected in what concerns the chiral limit of the theory.

Let us now look at some interesting physical consequences that can be obtained from Equation (60). In the degenerate flavor case, $m_u = m_d = m$, Equations (52), (53), and (60) become

$$E(\beta, m, \theta) - E(\beta, 0, 0) = -\frac{C\delta}{\delta+1}\left(m\cos\frac{\theta}{2}\right)^{\frac{\delta+1}{\delta}} + \ldots \tag{63}$$

$$\xi \sim \left(m\cos\frac{\theta}{2}\right)^{-\frac{\delta+1}{d\delta}} \tag{64}$$

$$m_{\vec{\pi}}, m_\sigma \sim \left(m\cos\frac{\theta}{2}\right)^{\frac{\delta+1}{d\delta}} \tag{65}$$

For non-degenerate flavors, the vacuum energy density (60) at $\theta = 0$ is a function of $m_u + m_d$; hence, the vacuum expectation values of the up and down condensates are equal, and the same holds true for their susceptibilities:

$$\langle S_u \rangle = \langle S_u \rangle = -\frac{C}{2^{\frac{\delta+1}{\delta}}}(m_u + m_d)^{\frac{1}{\delta}}$$

$$\sum_x (\langle S_u(x) S_u(0) \rangle - \langle S_u(x) \rangle \langle S_u(0) \rangle) = \sum_x (\langle S_d(x) S_d(0) \rangle - \langle S_d(x) \rangle \langle S_d(0) \rangle) =$$

$$\sum_x (\langle S_u(x) S_d(0) \rangle - \langle S_u(x) \rangle \langle S_d(0) \rangle) = \frac{C}{2^{\frac{\delta+1}{\delta}}}\frac{1}{\delta}(m_u + m_d)^{\frac{1-\delta}{\delta}}. \tag{66}$$

We do not see any dependency on $m_d - m_u$, and isospin breaking effects are therefore absent in these quantities, which on the other hand show a singular behavior in the chiral limit. The normalized flavor singlet scalar susceptibility, $\chi_\sigma$,

$$\chi_\sigma = \frac{C}{2^{\frac{1}{\delta}}}\frac{1}{\delta}(m_u + m_d)^{\frac{1-\delta}{\delta}}. \tag{67}$$

diverges in the chiral limit, while the $\delta$-meson susceptibility, $\chi_\delta$, vanishes in the zero-order approximation to the vacuum energy density, indicating that it is a good approximation when the ratio of the $\sigma$ and $\delta$ meson masses is small, $\frac{m_\sigma}{m_\delta} \ll 1$.

The topological susceptibility is given by

$$\chi_T = \frac{C}{2^{\frac{\delta+1}{\delta}}}(m_u + m_d)^{\frac{1-\delta}{\delta}} m_u m_d \tag{68}$$

showing that this quantity is sensitive to the isospin breakdown.

The W-T identity for the charged pions, $\pi^\pm$, reads

$$\chi_{\pi^\pm} = -\frac{\langle S_u \rangle + \langle S_d \rangle}{m_u + m_d} \tag{69}$$

and hence we get

$$\chi_{\pi^\pm} = \frac{C}{2^{\frac{1}{\delta}}}(m_u + m_d)^{\frac{1-\delta}{\delta}}. \tag{70}$$

Similar to the $\sigma$-susceptibility, the charged pions susceptibility diverges in the chiral limit.

To calculate the susceptibility of the neutral pion, we use the following $W-T$ identities

$$\sum_x \langle P_u(x)P_u(0)\rangle = -\frac{\langle S_u\rangle}{m_u} - \frac{\chi_T}{m_u^2}$$
$$\sum_x \langle P_d(x)P_d(0)\rangle = -\frac{\langle S_d\rangle}{m_d} - \frac{\chi_T}{m_d^2}$$
$$\sum_x \langle P_u(x)P_d(0)\rangle = -\frac{\chi_T}{m_u m_d} \tag{71}$$

which give us

$$\sum_x \langle P_u(x)P_u(0)\rangle = \sum_x \langle P_d(x)P_d(0)\rangle = -\sum_x \langle P_u(x)P_d(0)\rangle = \frac{C}{2^{\frac{\delta+1}{\delta}}}(m_u+m_d)^{\frac{1-\delta}{\delta}} \tag{72}$$

and, for the normalized neutral pion susceptibility, we get

$$\chi_{\pi^0} = \frac{C}{2^{\frac{1}{\delta}}}(m_u+m_d)^{\frac{1-\delta}{\delta}}. \tag{73}$$

Equations (70) and (73) show that the $\pi^{\pm}$ and $\pi^0$susceptibilities are equal and independent of $m_d - m_u$. Again, isospin breaking effects are absent in these quantities, and, even though $m_d - m_u \neq 0$, the three pions have the same mass. In what concerns the flavor-singlet pseudoscalar susceptibility, $\chi_\eta$, Equation (72) shows that it vanishes.

Finally, if for simplicity we consider two degenerate flavors, Equations (53) and (68) imply that the pion mass $m_{\tilde{\pi}}$ (or the $\sigma$-meson mass) and the topological susceptibility $\chi_T$ verify the following relation

$$\frac{m_{\tilde{\pi}}}{(\chi_T)^{\frac{1}{d}}} = k(\beta, L_t) \tag{74}$$

where $k$ is a dimensionless quantity that depends on the inverse gauge coupling $\beta$, and eventually, at finite temperature $T$, on the lattice temporal extent $L_t$, but that is independent of the fermion mass $m$.

In summary, it is shown that, in the zero-order approximation to the vacuum energy density, which accounts for the chiral critical behavior of the theory, isospin breaking effects only manifest in the topological susceptibility. The three pions have the same mass, the ratio of the pion (73) and $\sigma$-meson (67) susceptibilities is equal to the critical exponent $\delta$, and the pion (or $\sigma$-meson) mass is related with the topological susceptibility, as shown in Equation (74).

## 4. Two-Flavor Schwinger Model as a Test Bed

Quantum Electrodynamics in $(1+1)$-dimensions is a good laboratory to test the results reported in the previous section. The model is confining [53], exactly solvable at zero fermion mass, has non-trivial topology, and shows explicitly the $U_A(1)$ axial anomaly [54]. Besides that, the Schwinger model does not require infinite renormalization, and this means that, if we use a lattice regularization, the bare parameters remain finite in the continuum limit.

On the other hand, the $SU_A(N_f)$ non-anomalous axial symmetry in the chiral limit of the multi-flavor Schwinger model is fulfilled in the vacuum, and this property makes this model a perfect candidate to check our predictions on the existence of quasi-massless scalar and pseudoscalar bosons in the spectrum of the model, the mass of which vanishes in the chiral limit.

The Euclidean continuum action for the two-flavor theory is

$$S = \int d^2x \{\bar{\psi}_u(x)\gamma_\mu(\partial_\mu + iA_\mu(x))\psi_u(x) + \bar{\psi}_d(x)\gamma_\mu(\partial_\mu + iA_\mu(x))\psi_d(x)\} +$$

$$\int \mathrm{d}^2x\{m_u\bar{\psi}_u(x)\psi_u(x) + m_d\bar{\psi}_d(x)\psi_d(x) + \frac{1}{4e^2}F^2_{\mu\nu}(x) + i\theta Q(x)\} \tag{75}$$

where $m_u$ and $m_d$ are the fermion masses and $e$ is the electric charge or gauge coupling, which has mass dimensions. $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, and $\gamma_\mu$ are $2 \times 2$ matrices satisfying the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{76}$$

At the classic level this theory has an internal $SU_V(2) \times SU_A(2) \times U_V(1) \times U_A(1)$ symmetry in the chiral limit. However, the $U_A(1)$-axial symmetry is broken at the quantum level because of the axial anomaly. The divergence of the axial current is

$$\partial_\mu J^A_\mu(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x), \tag{77}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric tensor, and hence does not vanish. The axial anomaly induces the topological $\theta$-term $i\theta Q = i\theta \int \mathrm{d}^2xQ(x)$ in the action, where

$$Q(x) = \frac{1}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x) \tag{78}$$

is the density of topological charge, the topological charge $Q$ being an integer number.

The Schwinger model was analyzed years ago by Coleman [7], computing some quantitative properties of the theory in the continuum for both, weak coupling $\frac{e}{m} \ll 1$, and strong coupling or chiral limit $\frac{e}{m} \gg 1$.

For the one-flavor case, Coleman computed the particle spectrum of the model, which shows a mass gap in the chiral limit, and conjectured the existence of a phase transition at $\theta = \pi$ and some intermediate fermion mass $m$ separating a weak coupling phase ($\frac{e}{m} \ll 1$), where the $Z_2$ symmetry of the model at $\theta = \pi$ is spontaneously broken, from a strong coupling phase ($\frac{e}{m} \gg 1$), in which the $Z_2$ symmetry is fulfilled in the vacuum. This qualitative result has recently been confirmed by numerical simulations of the Euclidean-lattice version of the model [55].

What is however more interesting for the content of this article is the Coleman analysis of the two-flavor model. As stated above, the theory (75) has an internal $SU_V(2) \times SU_A(2) \times U_V(1) \times U_A(1)$ symmetry in the chiral limit, and the $U_A(1)$ axial symmetry is anomalous. Since continuous internal symmetries cannot be spontaneously broken in a local field theory in two dimensions [56], the $SU_A(2)$ symmetry has to be fulfilled in the vacuum, and the scalar condensate, which is an order parameter for this symmetry, vanishes in the chiral limit. Hence, the two-flavor Schwinger model verifies all the conditions assumed in Section 3.

We summarize here the main Coleman's findings for the two-flavor model with degenerate masses $m_u = m_d = m$:

1. For weak coupling, $\frac{e}{m} \ll 1$, the results on the particle spectrum are almost the same as for the massive Schwinger model.
2. For strong coupling, $\frac{e}{m} \gg 1$, the low-energy effective theory depends only on one mass parameter, $m^{\frac{2}{3}}e^{\frac{1}{3}}\cos^{\frac{2}{3}}\frac{\theta}{2}$; the vacuum energy density is then proportional to

$$E(m,e,\theta) \propto e^{\frac{2}{3}}\left(m\cos\frac{\theta}{2}\right)^{\frac{4}{3}}; \tag{79}$$

and the chiral condensate, at $\theta = 0$, is therefore

$$\langle\bar{\psi}\psi\rangle \propto m^{\frac{1}{3}}e^{\frac{2}{3}} \tag{80}$$

3. The lightest particle in the theory is an isotriplet, and the next lightest is an isosinglet. The isosinglet/isotriplet mass ratio is $\sqrt{3}$. If there are other stable particles in the

model, they must be $O\left(\left[\frac{e}{m}\right]^{\frac{2}{3}}\right)$ times heavier than these. The light boson mass, $M$, has a fractional power dependence on the fermion mass $m$:

$$M \propto e^{\frac{1}{3}} \left( m \cos \frac{\theta}{2} \right)^{\frac{2}{3}} \tag{81}$$

Many of these results have been corroborated by several authors both in the continuum [57–61] and using the lattice approach [62,63]. Coleman concluded his paper [7] by asking some questions concerning things he did not understand, and we cite here two of them:

1. Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other?
2. Why does the next-lightest particle has $I^{PG} = 0^{++}$, rather than $0^{--}$?

The results of Section 3 allow us to qualitatively understand the main Coleman's findings for the two-flavor model with degenerate masses in the strong coupling limit, as well as to give a reliable answer to the previous questions.

In Section 3.5, we predict, from the interplay between the $U_A(1)$ anomaly and the exact $SU_A(2)$ chiral symmetry, a singular behavior of the vacuum energy density (49) and (63) in the chiral limit limit as $E \sim C\left(m \cos \frac{\theta}{2}\right)^{\frac{\delta+1}{\delta}}$. In the Schwinger model, a simple dimensional analysis tell us that $C$ must be proportional to $e^{\frac{\delta-1}{\delta}}$. Therefore, our result matches perfectly Coleman's result (79) if we choose $\delta = 3$.

In what concerns the masses of the light bosons, our prediction (65), $m_{\bar{\pi}}, m_\sigma \sim \left(m \cos \frac{\theta}{2}\right)^{\frac{\delta+1}{d\delta}}$ matches, for $\delta = 3$, Colemans's result (81) too.

In Section 3.5, we also predict that the flavor-singlet scalar susceptibility (67) and the "pion" susceptibility (70) and (73), should diverge in the chiral limit as $\frac{K}{\delta} m^{\frac{1-\delta}{\delta}} e^{\frac{\delta-1}{\delta}}$ and $K m^{\frac{1-\delta}{\delta}} e^{\frac{\delta-1}{\delta}}$, respectively (the factor $e^{\frac{\delta-1}{\delta}}$ comes again from dimensional analysis in the Schwinger model), and for $\delta = 3$ we have

$$\chi_{\sigma\, m\to 0} = \frac{K}{3} m^{-\frac{2}{3}} e^{\frac{2}{3}} \sim \frac{|\langle 0 \mid \hat{O}_\sigma \mid \sigma \rangle|^2}{m_\sigma} \qquad \chi_{\pi^0\, m\to 0} = K m^{-\frac{2}{3}} e^{\frac{2}{3}} \sim \frac{|\langle 0 \mid \hat{O}_{\pi^0} \mid \pi^0 \rangle|^2}{m_{\pi^0}} \tag{82}$$

where $K$ is a dimensionless constant.

We also show that the $\sigma$ and $\bar{\pi}$ meson masses, in the strong-coupling limit, scale with the quark mass as

$$m_{\bar{\pi}}, m_\sigma \sim m^{\frac{2}{3}} e^{\frac{1}{3}}. \tag{83}$$

Taking into account that the $SU_A(2)$ symmetry is exact in the chiral limit, Equations (82) and (83) imply that

$$\lim_{m\to 0} |\langle 0 \mid \hat{O}_\sigma \mid \sigma \rangle|^2 = \lim_{m\to 0} |\langle 0 \mid \hat{O}_{\pi^0} \mid \pi^0 \rangle|^2 \sim e \tag{84}$$

and therefore we have

$$\lim_{m\to 0} \frac{\chi_{\pi^0}(m,e)}{\chi_\sigma(m,e)} = \lim_{m\to 0} \frac{m_\sigma(m,e)}{m_{\pi^0}(m,e)} = 3 \tag{85}$$

These results show that indeed the lightest particle in the theory is an isotriplet, and the next lightest is an isosinglet $I^{PG} = 0^{++}$. However, our result for the ratio $\frac{m_\sigma}{m_{\bar{\pi}}} = 3$ [45] is in disagreement with Coleman's result $\frac{m_\sigma}{m_{\bar{\pi}}} = \sqrt{3}$ [7].

In what concerns the first Coleman's question, we argue in Section 3.5 that the strong coupling limit performed by Coleman corresponds to the zero-order contribution to the vacuum energy density expansion (59). This zero-order contribution depends on the quark masses only through the combination $m_u + m_d$, and we show that in such a case only the topological susceptibility is sensitive to isospin breaking effects. The three pion susceptibilities (70) and (73) and masses are equal, and to see isospin breaking effects we should go to the second-order contribution. The relevance of the second-order correction to the zero-order contribution to the vacuum energy density is also estimated (62), and for $\theta = 0$ turns out to be of the order of $\frac{(m_d - m_u)^2}{(m_u + m_d)^{\frac{4}{3}} e^{\frac{2}{3}}}$, a result that justifies the validity of the zero-order approximation in the strong-coupling ($\frac{e}{m_{u,d}} \gg 1$) limit (Georgi recently argued [64] that isospin breaking effects are exponentially suppressed in the two-flavor Schwinger model as a consequence of conformal coalescence).

The analysis done in this section strongly suggests that the existence of quasi-massless chiral bosons in the spectrum of the two-flavor Schwinger model, near the chiral limit, does not originates in some uninteresting peculiarities of two-dimensional models, but it should be a consequence of the interplay between exact non-abelian chiral symmetry, and an effectively broken $U_A(1)$ anomalous symmetry. What is a two-dimensional peculiarity is the fact that, in the chiral limit, when all fermion masses vanish, these quasi-massless bosons become unstable, and the low-energy spectrum of the model reduces to a massless non-interacting boson, in accordance with Coleman's theorem [56] which forbids the existence of massless interacting bosons in two dimensions.

## 5. Conclusions and Discussion

Thanks to massive $QCD$ simulations on the lattice, we have at present a good qualitative and quantitative understanding of the vacuum realization of the non-abelian $SU_A(N_f)$ chiral symmetry, as a function of the physical temperature. As far as the $U_A(1)$ anomaly and its associated $\theta$ parameter are concerned, and especially in the high temperature phase, the current situation is however far from satisfactory. With the aim of clarifying the current status concerning this issue, we devote the first part of this article to analyzing the present status of the investigations on the effectiveness of the $U_A(1)$ axial anomaly in $QCD$, at temperatures around and above the non-abelian chiral transition critical temperature. We show that theoretical predictions require assumptions whose validity is not always proven, and lattice simulations using different discretization schemes lead to apparently contradictory conclusions in several cases. Hence, despite the great effort devoted to investigating the fate of the axial anomaly in the chirally symmetric phase of $QCD$, we still do not have a clear answer to this question.

In the second part of the article we analyze some recently suggested [45] interesting physical implications of the $U_A(1)$ anomaly, in systems where the non-abelian axial symmetry is fulfilled in the vacuum. The standard wisdom on the origin of massless bosons in the spectrum of a Quantum Field Theory, describing the interaction of gauge fields coupled to matter fields, is based on two well known features: gauge symmetry and spontaneous symmetry breaking of continuous symmetries. We show that the topological properties of the theory can be the basis of an alternative mechanism, other than Goldstone's theorem, to generate massless bosons in the chiral limit, if the $U_A(1)$ symmetry remains effectively broken, and the non-abelian $SU_A(N_f)$ chiral symmetry is fulfilled in the vacuum.

The two-flavor Schwinger model, or Quantum Electrodynamics in two space-time dimensions, is a good test-bed for our predictions. Indeed, the Schwinger model shows a non-trivial topology, which induces the $U_A(1)$ axial anomaly. Moreover, in the two-flavor case, the non-abelian $SU_A(2)$ chiral symmetry is fulfilled in the vacuum, as required by Coleman's theorem [56] on the impossibility to break spontaneously continuous symmetries in two dimensions. This model was analyzed by Coleman long ago in [7], where he computed some quantitative properties of the theory in the continuum for both weak coupling, $\frac{e}{m} \ll 1$, and strong coupling $\frac{e}{m} \gg 1$. In what concerns the strong-coupling results, the main Coleman findings are qualitatively in agreement with our predictions. The vacuum energy density, and the chiral condensate shows a singular dependence on the fermion mass, $m$, in the chiral limit, and the flavor singlet scalar susceptibility diverges when $m \to 0$. In addition, our results provide a reliable answer to some questions that Coleman asked himself.

It is worth wondering if the reason for the rich spectrum of light chiral bosons near the chiral limit, found in the Schwinger [7] and $U(N)$ [65] models, lies in some uninteresting peculiarities of two-dimensional models, or if there is a deeper and general explanation for this phenomenon. We want to remark, concerning this, that the analysis done in Section 4 strongly suggests that the existence of quasi-massless chiral bosons in the spectrum of the two-flavor Schwinger model, near the chiral limit, does not originates in some uninteresting peculiarities of two-dimensional models but it should be a consequence of the interplay between exact non-abelian chiral symmetry, and an effectively broken $U_A(1)$ anomalous symmetry. What is a two-dimensional peculiarity is the fact that, in the chiral limit, when all fermion masses vanish, these quasi-massless bosons become unstable, and the low-energy spectrum of the model reduces to a massless non-interacting boson [66,67], in accordance with Coleman's theorem [56] which forbids the existence of massless interacting bosons in two dimensions.

In what concerns $QCD$, the analysis of the effects of the $U_A(1)$ axial anomaly in its high temperature phase, in which the non-abelian chiral symmetry is restored in the ground state, has aroused much interest in recent time because of its relevance in axion phenomenology. Moreover, the way in which the $U_A(1)$ anomaly manifests itself in the chiral symmetry restored phase of $QCD$ at high temperature could be tested when probing the $QCD$ phase transition in relativistic heavy ion collisions.

We argue in Section 3 that a quantum field theory, with an exact non-abelian $SU_A(2)$ symmetry, and in which the $U_A(1)$ axial symmetry is effectively broken, should exhibit a singular quark-mass dependence in the vacuum energy density and a divergent correlation length in the correlation function of the scalar condensate, in the chiral limit. On the contrary, if all correlation lengths are finite, and hence the vacuum energy density is an analytical function of the quark mass, we show that the vacuum energy density becomes, at least up to second order in the quark masses, $\theta$-independent. The topological susceptibility either vanish or is at least of fourth order in the quark masses and, in such a case, all typical effects of the $U_A(1)$ anomaly are lost. $QCD$ in the chirally symmetric phase, $T \gtrsim T_c$, shows an exact non-abelian axial symmetry, and, hence, either the vacuum energy density is an analytical function of the quark masses and $QCD$ becomes $\theta$-independent or the screening mass spectrum of the model shows several quasi-massless chiral bosons, whose masses vanish in the chiral limit. Which of the two aforementioned possibilities actually happens in the high temperature phase of $QCD$ is a difficult question, as follows from the current status of lattice simulations reported in this article.

A recent lattice calculation [39] of the topological properties of three-flavor $QCD$ with physical quark masses and temperatures around 500 MeV gives as a result a small but non-vanishing topological susceptibility, although with large error bars in the continuum limit extrapolations, suggesting that the effects of the $U_A(1)$ axial anomaly still persist at these temperatures. If we assume this to be true, and hence that there is a temperature interval in the high temperature phase where the $U_A(1)$ anomalous symmetry remains effectively broken, we can apply to this temperature interval the main conclusions of Section 3.

Taking into account lattice determination of the light quark masses [68] ($m_u \simeq 2$ MeV, $m_d \simeq 5$ MeV, $m_s \simeq 94$ MeV), we can consider $QCD$ with two quasi-massless quarks as a good approach. The results of Section 3 predict then a spectrum of light $\sigma$ and $\bar{\pi}$ mesons at $T \gtrsim T_c$. The presence of these light scalar and pseudoscalar mesons in the chirally symmetric high temperature phase of $QCD$ could, on the other hand, significantly influence the dilepton and photon production observed in the particle spectrum [69] at heavy-ion collision experiments.

Lattice calculations of mesonic screening masses in two- [34] and three-flavor [41] $QCD$, around and above the critical temperature, give results that are unfortunately not enough to allow a good check of our spectrum prediction. However, the results of [41] show a small change of the pion screening-mass when crossing the critical temperature and a decreasing screening mass, at $T \gtrapprox T_c$, when going from the $\bar{u}s$ to the $\bar{u}d$ channel.

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