Article

Ostrowski Type Inequalities Involving Harmonically Convex Functions and Applications

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Abstract: The main objective of this paper is to derive some new generalizations of Ostrowski type inequalities for the functions whose first derivatives absolute value are harmonically convex. We also discuss some special cases of the obtained results. In the last section, we present some applications of the obtained results.

Keywords: convex; harmonic; derivative; ostrowski; inequality

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1. Introduction and Preliminaries

A function $\Lambda : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex, if

$$\Lambda((1-\tau)a_1 + \tau a_2) \leq (1-\tau)\Lambda(a_1) + \tau\Lambda(a_2), \quad \forall a_1, a_2 \in I, \tau \in [0, 1].$$

Iscan [1] introduced the notion of harmonically convex functions as follows:

A function $\Lambda : I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$\Lambda\left(\frac{a_1 a_2}{\tau a_1 + (1-\tau)a_2}\right) \leq (1-\tau)\Lambda(a_1) + \tau\Lambda(a_2), \quad \forall a_1, a_2 \in I, \tau \in [0, 1].$$

It is worth mentioning here that the harmonic property has played a significant role in different fields of pure and applied sciences. In [2], the authors have discussed the important role of the harmonic mean in Asian stock options. Interestingly, harmonic means are applied in electric circuit theory. More specifically, the total resistance of a set of parallel resistors is just half of the harmonic mean of the total resistors. For example, if R_1 and R_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula:

$$R_T = \frac{R_1 R_2}{R_1 + R_2} = \frac{1}{2} H(R_1, R_2),$$

which is the half of the harmonic mean.

Noor [3] showed that the harmonic mean also plays a crucial role in the development of parallel algorithms for solving nonlinear problems. Several authors have used harmonic means and harmonic convex functions to propose some iterative methods for solving linear and nonlinear system of equations.

Convexity played a vital role in the development of the theory of inequalities. Several inequalities constitute direct consequences of applications of convexity. In this regard, Hermite–Hadamard’s inequality is one of the most extensively as well as intensively studied result. It provides us with an estimate of the (integral) mean value of a continuous convex function. It reads as:

Let $\Lambda : I = [a_1, a_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, then

$$\Lambda\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \Lambda(x) dx \leq \frac{\Lambda(a_1) + \Lambda(a_2)}{2}.$$

Iscan [4] extended the classical version of Hermite–Hadamard’s inequality using the harmonic convexity property of the function.

Let $\Lambda : I = [a_1, a_2] \subset (0, \infty) \rightarrow \mathbb{R}$ be a harmonically convex function, then

$$\Lambda\left(\frac{2a_1a_2}{a_1 + a_2}\right) \leq \frac{a_1a_2}{a_2 - a_1} \int_{a_1}^{a_2} \Lambda(x) dx \leq \frac{\Lambda(a_1) + \Lambda(a_2)}{2}.$$

An interesting problem related to Hermite–Hadamard’s inequality is its precision. Note that the left Hermite–Hadamard inequality can be estimated by the following inequality:

$$\left| \Lambda(x) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \Lambda(x) dx \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a_1 + a_2}{2}}{a_2 - a_1} \right)^2 \right] M(a_2 - a_1),$$

where M is the Lipschitz constant which is equal to $\sup \left\{ \left| \frac{\Lambda(x) - \Lambda(y)}{x - y} \right|; x \neq y \right\}$. This above inequality is known in the literature as Ostrowski’s inequality.

Recently, several research articles have been written on different generalizations of Ostrowski’s inequality using different techniques. For example, Alomari et al. [5] obtained Ostrowski type inequalities using the class of s -convex functions. Ardic et al. [6] obtained Ostrowski type inequalities using the class of GG -convex and GA -convex functions. Budak and Sarikaya [7] obtained generalized Ostrowski type inequalities for functions whose first derivatives’ absolute values are convex. Budak and Sarikaya [8] also obtained some new weighted Ostrowski type inequalities for functions of two variables with bounded variation. Iscan [4] obtained some Ostrowski type inequalities using the class of harmonically s -convex functions. Khurshid et al. [9] obtained a conformable fractional version of Ostrowski type inequalities using preinvex functions. Koroglu [10] obtained some more generalized Ostrowski type inequalities using harmonically convex functions. Mohsin et al. [11] obtained new generalizations of Ostrowski type inequalities using harmonically h -convex functions. Set [12] obtained some generalized fractional refinements of Ostrowski type inequalities using the class of s -convex functions. Recently, Sun [13] obtained some more local fractional versions of Ostrowski type inequalities and discussed its applications. For more details on Ostrowski type inequalities and its applications, cf. [14].

The aim of this paper is to obtain some new generalizations of Ostrowski’s inequality essentially utilising the harmonic convexity property of the functions. We first derive a new auxiliary result which will play a significant role in the development of these results. We also discuss some special cases that can be deduced from the main results of the paper. In the last section, we present some applications of the obtained results. We hope that the ideas and techniques presented within this paper will inspire interested readers.

2. Main Results

In this section, we discuss our main results.

Let $L[a_1, a_2]$ be the space of Lebesgue integrable functions on the interval $[a_1, a_2]$. Our first result is an auxiliary lemma. This result will play a significant role in the development of the next results.

Lemma 1. Let $\Lambda : [a_1, a_2] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a_1, a_2) with $a_1 < a_2$. If Λ' belongs to $L[a_1, a_2]$, then, for all $\varrho \in (0, 1)$ and for all $x \in [a_1, a_2]$, the following equality holds true:

$$\begin{aligned} & \int_0^1 h(\tau, \varrho) \Lambda' \left(\frac{a_1 a_2}{\tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)} \right) \\ & \quad \times \frac{a_1 a_2 (a_2 - a_1)(1-2\zeta)}{[\tau((1-\zeta)a_1 + \zeta a_2) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)]^2} d\tau \\ & = \varrho \frac{a_1(a_2 - x) \Lambda \left(\frac{a_1 a_2}{\zeta a_1 + (1-\zeta)a_2} \right) + a_2(x - a_1) \Lambda \left(\frac{a_1 a_2}{(1-\zeta)a_1 + \zeta a_2} \right)}{x(a_2 - a_1)} \\ & \quad + (1-\varrho) \Lambda \left(\frac{x a_1 a_2}{(1-\zeta)(x(a_1 + a_2) - a_1 a_2) + \zeta a_1 a_2} \right) - \frac{a_1 a_2}{(a_2 - a_1)(1-2\zeta)} \int_{\frac{a_1 a_2}{\zeta a_1 + (1-\zeta)a_2}}^{\frac{a_1 a_2}{(1-\zeta)a_1 + \zeta a_2}} \frac{\Lambda(s)}{s^2} ds \\ & = T(\Lambda, \varrho, \zeta, x). \end{aligned} \quad (1)$$

for $\zeta \in [0, 1] \setminus \{\frac{1}{2}\}$, where

$$h(\tau, \varrho) = \begin{cases} \tau - \varrho \frac{a_1(a_2 - x)}{x(a_2 - a_1)}, & \tau \in \left[0, \frac{a_1(a_2 - x)}{x(a_2 - a_1)}\right], \\ \tau - 1 + \varrho \frac{a_2(x - a_1)}{x(a_2 - a_1)}, & \tau \in \left(\frac{a_1(a_2 - x)}{x(a_2 - a_1)}, 1\right]. \end{cases}$$

Proof. For the sake of brevity, we write $A_\tau = \tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)$. Let

$$\begin{aligned} I &= \int_0^1 h(\tau, \varrho) \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1)(1-2\zeta)}{A_\tau^2} d\tau \\ &= \int_0^{\frac{a_1(a_2 - x)}{x(a_2 - a_1)}} \left[\tau - \varrho \frac{a_1(a_2 - x)}{x(a_2 - a_1)} \right] \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1)(1-2\zeta)}{A_\tau^2} d\tau \\ & \quad + \int_{\frac{a_1(a_2 - x)}{x(a_2 - a_1)}}^1 \left[\tau - 1 + \varrho \frac{a_2(x - a_1)}{x(a_2 - a_1)} \right] \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1)(1-2\zeta)}{A_\tau^2} d\tau \\ &= I_1 + I_2. \end{aligned}$$

Now, an integration by parts yields

$$\begin{aligned} I_1 &= \int_0^{\frac{a_1(a_2 - x)}{x(a_2 - a_1)}} \left[\tau - \varrho \frac{a_1(a_2 - x)}{x(a_2 - a_1)} \right] \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1)(1-2\zeta)}{A_\tau^2} d\tau \\ &= (1-\varrho) \frac{a_1(a_2 - x)}{x(a_2 - a_1)} \Lambda \left(\frac{x a_1 a_2}{(1-\zeta)(x(a_1 + a_2) - a_1 a_2) + \zeta a_1 a_2} \right) \\ & \quad + \frac{\varrho a_1(a_2 - x)}{x(a_2 - a_1)} \Lambda \left(\frac{a_1 a_2}{\zeta a_1 + (1-\zeta)a_2} \right) - \int_0^{\frac{a_1(a_2 - x)}{x(a_2 - a_1)}} \Lambda \left(\frac{a_1 a_2}{A_\tau} \right) d\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left[\tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right] \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\zeta)}{A_\tau^2} d\tau \\ &= \frac{\varrho a_2 (x - a_1)}{x(a_2 - a_1)} \Lambda \left(\frac{a_1 a_2}{(1 - \zeta)a_1 + \zeta a_2} \right) \\ &\quad + (1 - \varrho) \frac{a_2 (x - a_1)}{x(a_2 - a_1)} \Lambda \left(\frac{x a_1 a_2}{(1 - \zeta)(x(a_1 + a_2) - a_1 a_2) + \zeta a_1 a_2} \right) - \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \Lambda \left(\frac{a_1 a_2}{A_\tau} \right) d\tau. \end{aligned}$$

Summing I_1 and I_2 , we have

$$\begin{aligned} I &= \varrho \frac{a_1(a_2 - x) \Lambda \left(\frac{a_1 a_2}{\zeta a_1 + (1 - \zeta)a_2} \right) + a_2(x - a_1) \Lambda \left(\frac{a_1 a_2}{(1 - \zeta)a_1 + \zeta a_2} \right)}{x(a_2 - a_1)} \\ &\quad + (1 - \varrho) \Lambda \left(\frac{x a_1 a_2}{(1 - \zeta)(x(a_1 + a_2) - a_1 a_2) + \zeta a_1 a_2} \right) - \int_0^1 \Lambda \left(\frac{a_1 a_2}{A_\tau} \right) d\tau. \end{aligned}$$

This implies

$$\begin{aligned} I &= \varrho \frac{a_1(a_2 - x) \Lambda \left(\frac{a_1 a_2}{\zeta a_1 + (1 - \zeta)a_2} \right) + a_2(x - a_1) \Lambda \left(\frac{a_1 a_2}{(1 - \zeta)a_1 + \zeta a_2} \right)}{x(a_2 - a_1)} \\ &\quad + (1 - \varrho) \Lambda \left(\frac{x a_1 a_2}{(1 - \zeta)(x(a_1 + a_2) - a_1 a_2) + \zeta a_1 a_2} \right) - \frac{a_1 a_2}{(a_2 - a_1)(1 - 2\zeta)} \int_{\frac{a_1 a_2}{\zeta a_1 + (1 - \zeta)a_2}}^{\frac{a_1 a_2}{(1 - \zeta)a_1 + \zeta a_2}} \frac{\Lambda(s)}{s^2} ds. \end{aligned}$$

This completes the proof. $\square$

We now discuss some special cases of Lemma 1. To the best of our knowledge, these special cases are also new in the literature.

I. If we set $\zeta = 1$ in Lemma 1, then the following equality holds:

$$\begin{aligned} &\int_0^1 h(\tau, \varrho) \Lambda' \left(\frac{a_1 a_2}{tb + (1 - \tau)a_1} \right) \frac{a_1 a_2 (a_1 - a_2)}{(tb + (1 - \tau)a_1)^2} d\tau \\ &= \varrho \frac{a_1(a_2 - x) \Lambda(a_2) + a_2(x - a_1) \Lambda(a_1)}{x(a_2 - a_1)} + (1 - \varrho) \Lambda(x) - \frac{a_1 a_2}{(a_2 - a_1)} \int_{a_1}^{a_2} \frac{\Lambda(s)}{s^2} ds. \end{aligned} \quad (2)$$

where

$$h(\tau, \varrho) = \begin{cases} \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)}, & \tau \in \left[0, \frac{a_1(a_2-x)}{x(a_2-a_1)}\right], \\ \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)}, & \tau \in \left(\frac{a_1(a_2-x)}{x(a_2-a_1)}, 1\right]. \end{cases}$$

In addition, if we set $\varrho = 0$ in (3), then the following equality holds:

$$\begin{aligned} &\int_0^1 h(\tau) \Lambda' \left(\frac{a_1 a_2}{tb + (1 - \tau)a_1} \right) \frac{a_1 a_2 (a_1 - a_2)}{(tb + (1 - \tau)a_1)^2} d\tau \\ &= \Lambda(x) - \frac{a_1 a_2}{(a_2 - a_1)} \int_{a_1}^{a_2} \frac{\Lambda(s)}{s^2} ds. \end{aligned} \quad (3)$$

where

$$h(\tau) = \begin{cases} \tau, & \tau \in \left[0, \frac{a_1(a_2-x)}{x(a_2-a_1)}\right], \\ \tau - 1, & \tau \in \left(\frac{a_1(a_2-x)}{x(a_2-a_1)}, 1\right]. \end{cases}$$

Before proceeding to our next result, let us recall the integral form of the hypergeometric function:

$${}_2F_1(x, y; c; z) = \frac{1}{B(y, c-y)} \int_0^1 \tau^{y-1} (1-\tau)^{c-y-1} (1-z\tau)^{-x} d\tau,$$

for $|z| < 1, c > y > 0$.

We now derive our first refinement of Ostrowski type inequality using the class of harmonically convex functions.

Theorem 1. Let the expression $T(\Lambda, \varrho, \varsigma, x)$ be as defined in (1) of Lemma 1. Suppose $\Lambda : I \rightarrow \mathbb{R}$ is a differentiable mapping on I° , where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|\Lambda'|$ is harmonically convex, then, for $\varrho \in [0, 1]$, $x \in [a_1, a_2]$ and $\varsigma \in [0, 1] \setminus \{\frac{1}{2}\}$, the following inequality holds true:

$$|T(\Lambda, \varrho, \varsigma, x)| \leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left\{ \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| B_1 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| B_2 \right. \\ \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| B_3 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| B_4 \right\},$$

where

$$B_1 = \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{\tau}{A_{a_2}^2} d\tau \\ = (\varsigma a_1 + (1-\varsigma)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2, 2, 4, \left(\frac{a_1 \varrho (a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \right. \\ \left. + \frac{1}{3} {}_2F_1 \left(2, 3, 4, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2, 2, 3, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \right\}, \\ B_2 = \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{1-\tau}{A_{a_2}^2} d\tau \\ = (\varsigma a_1 + (1-\varsigma)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2, 1, 3, \left(\frac{a_1 \varrho (a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \right. \\ - \frac{\varrho a_1(a_2-x)}{6x(a_2-a_1)} {}_2F_1 \left(2, 2, 4, \left(\frac{\varrho a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \\ + \frac{1}{2} \left(1 + \frac{\varrho a_1(a_2-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2, 2, 3, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \\ \left. - \frac{\varrho a_1(a_2-x)}{3x(a_2-a_1)} {}_2F_1 \left(2, 3, 4, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) - \varrho {}_2F_1 \left(2, 1, 2, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)} \right) \right) \right\},$$

$$\begin{aligned}
B_3 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{\tau}{A_\tau^2} d\tau \\
&= (\varsigma a_2 + (1-\varsigma)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2, 1, 3, \left(\frac{a_2 \varrho (x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right. \\
&\quad - \frac{\varrho a_2(x-a_1)}{6x(a_2-a_1)} {}_2F_1 \left(2, 2, 4, \left(\frac{\varrho a_2(a_1-x)(1-2\varsigma)}{x(ub+(1-\varsigma)a_1)} \right) \right) \\
&\quad + \frac{1}{2} \left(1 + \frac{\varrho a_2(a_1-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2, 2, 3, \left(\frac{a_2(a_1-x)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \\
&\quad \left. - \frac{\varrho a_2(x-a_1)}{3x(a_2-a_1)} {}_2F_1 \left(2, 3, 4, \left(\frac{a_2(x-a_1)(1-2\varsigma)}{x(ub+(1-\varsigma)a_1)} \right) \right) - \varrho {}_2F_1 \left(2, 1, 2, \left(\frac{a_2(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right\}, \\
B_4 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{1-\tau}{A_\tau^2} d\tau \\
&= (\varsigma a_2 + (1-\varsigma)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2, 2, 4, \left(\frac{a_2 \varrho (x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right. \\
&\quad \left. + \frac{1}{3} {}_2F_1 \left(2, 3, 4, \left(\frac{a_2(a_1-x)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2, 2, 3, \left(\frac{a_2(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right\}.
\end{aligned}$$

Proof. For the sake of brevity, we write $A_{a_1} = \varsigma a_1 + (1-\varsigma)a_2$ and $A_{a_2} = \varsigma a_2 + (1-\varsigma)a_1$. Using Lemma 1, we have

$$\begin{aligned}
|T(\Lambda, \varrho, \varsigma, x)| &\leq \left\{ \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right\| d\tau \right. \\
&\quad \left. + \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right\| d\tau \right\} \\
&\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left\{ \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right\| d\tau \right. \\
&\quad \left. + \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right\| d\tau \right\} \\
&= a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) K_1.
\end{aligned}$$

Using the harmonic convexity of $|\Lambda'|$, we get

$$\begin{aligned}
 K_1 &\leq \left\{ \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{\tau}{A_\tau} d\tau \right. \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{(1-\tau)}{A_\tau} d\tau \right) \right. \\
 &\quad \left. + \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{\tau}{A_\tau} d\tau \right. \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{(1-\tau)}{A_\tau} d\tau \right) \right\} \\
 &= \left\{ \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| B_1 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| B_2 \right) + \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right| B_3 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right| B_4 \right) \right\}.
 \end{aligned}$$

This completes the proof. $\square$

We now derive a second refinement of Ostrowski type inequalities involving harmonically convex functions. We derive this result using the power mean inequality.

Theorem 2. Let $\Lambda : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° , where $a_1, a_2 \in I$ with $a_1 < a_2$. If, for $q \geq 1$, $|\Lambda'|^q$ is harmonically convex, for $\varrho \in [0, 1]$, $x \in [a_1, a_2]$ and $\varsigma \in [0, 1] \setminus \{\frac{1}{2}\}$, the following inequality holds true:

$$\begin{aligned}
 |T(\Lambda, \varrho, \varsigma, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\phi_1)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_1^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_2^* \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\phi_2)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_3^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_4^* \right)^{\frac{1}{q}} \right), \quad (4)
 \end{aligned}$$

where

$$\begin{aligned}
 \phi_1 &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| d\tau = \left(\frac{(2\varrho^2 - 2\varrho + 1)}{2} \right) \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2, \\
 \phi_2 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| d\tau = \left(\frac{(2\varrho^2 - 2\varrho + 1)}{p+1} \right) \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2,
 \end{aligned}$$

$$\begin{aligned}
B_1^* &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{\tau}{(A_\tau)^{2q}} d\tau \\
&= (\varsigma a_1 + (1-\varsigma)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2q, 2, 4, \left(\frac{a_1 \varrho(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \right. \\
&\quad \left. + \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \right\}, \\
B_2^* &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{1-\tau}{(A_\tau)^{2q}} d\tau \\
&= (\varsigma a_1 + (1-\varsigma)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2q, 1, 3, \left(\frac{a_1 \varrho(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \right. \\
&\quad - \frac{\varrho a_1(a_2-x)}{6x(a_2-a_1)} {}_2F_1 \left(2q, 2, 4, \left(\frac{\varrho a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \\
&\quad + \frac{1}{2} \left(1 + \frac{\varrho a_1(a_2-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \\
&\quad \left. - \frac{\varrho a_1(a_2-x)}{3x(a_2-a_1)} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) - \varrho {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1+(1-\varsigma)a_2)} \right) \right) \right\}, \\
B_3^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{\tau}{(A_\tau)^{2q}} d\tau \\
&= (\varsigma a_2 + (1-\varsigma)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2q, 1, 3, \left(\frac{a_2 \varrho(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right. \\
&\quad - \frac{\varrho a_2(x-a_1)}{6x(a_2-a_1)} {}_2F_1 \left(2q, 2, 4, \left(\frac{\varrho a_2(a_1-x)(1-2\varsigma)}{x(ub+(1-\varsigma)a_1)} \right) \right) \\
&\quad + \frac{1}{2} \left(1 + \frac{\varrho a_2(a_1-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \left(\frac{a_2(a_1-x)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \\
&\quad \left. - \frac{\varrho a_2(x-a_1)}{3x(a_2-a_1)} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_2(x-a_1)(1-2\varsigma)}{x(ub+(1-\varsigma)a_1)} \right) \right) - \varrho {}_2F_1 \left(2q, 1, 2, \left(\frac{a_2(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right\}, \\
B_4^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{1-\tau}{(A_\tau)^{2q}} d\tau \\
&= (\varsigma a_2 + (1-\varsigma)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2q, 2, 4, \left(\frac{a_2 \varrho(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right. \\
&\quad \left. + \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_2(a_1-x)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_2(x-a_1)(2\varsigma-1)}{x(ub+(1-\varsigma)a_1)} \right) \right) \right\}.
\end{aligned}$$

Proof. For the sake of brevity, we write $A_{a_1} = \varsigma a_1 + (1 - \varsigma)a_2$ and $A_{a_2} = \varsigma a_2 + (1 - \varsigma)a_1$. We suppose that $q > 1$. Taking modulus in Lemma 1 and using the power mean inequality, we have

$$\begin{aligned}
 |T(\Lambda, \varrho, \varsigma, x)| &\leq \left\{ \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right\| d\tau \right. \\
 &\quad \left. + \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right\| d\tau \right\} \\
 &\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| d\tau \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right\|^q d\tau \right)^{\frac{1}{q}} \\
 &\quad \left. + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| d\tau \right)^{1-\frac{1}{q}} \right. \\
 &\quad \left. \times \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \left\| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right\|^q d\tau \right)^{\frac{1}{q}} \right\} \\
 &= a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) K_2.
 \end{aligned}$$

Using the harmonic convexity of $|\Lambda'|^q$, we get

$$\begin{aligned}
 K_2 &\leq \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| d\tau \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{\tau}{(A_\tau)^{2q}} d\tau \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{(1-\tau)}{(A_\tau)^{2q}} d\tau \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| d\tau \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{\tau}{(A_\tau)^{2q}} d\tau \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{(1-\tau)}{(A_\tau)^{2q}} d\tau \right)^{\frac{1}{q}} \right\} \\
 &= \left((\phi_1)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_1^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_2^* \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\phi_2)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_3^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_4^* \right)^{\frac{1}{q}} \right).
 \end{aligned}$$

This completes the proof. $\square$

Remark 1. If $q = 1$, then Theorem 2 reduces to Theorem 1.

Corollary 1. Under the assumptions of Theorem 2, if we choose $\varsigma = 0$ in (4), then we have

$$\begin{aligned}
 |T(\Lambda, \varrho, 0, x)| &\leq a_1 a_2 (a_2 - a_1) \left((\phi_1)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_1^{**} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_2^{**} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\phi_2)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_3^{**} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_4^{**} \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where ϕ_1, ϕ_2 are given in Theorem 2 and

$$\begin{aligned}
 B_1^{**} &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{\tau}{(\tau a_1 + (1-\tau)a_2)^{2q}} d\tau \\
 &= a_2^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2q, 2, 4, \left(\frac{a_1 \varrho(a_2-x)}{xb} \right) \right) \right. \\
 &\quad \left. + \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_1(a_2-x)}{xb} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)}{xb} \right) \right) \right\}, \\
 B_2^{**} &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \frac{1-\tau}{(\tau a_1 + (1-\tau)a_2)^{2q}} d\tau \\
 &= a_2^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2q, 1, 3, \left(\frac{a_1 \varrho(a_2-x)}{xb} \right) \right) \right. \\
 &\quad - \frac{\varrho a_1(a_2-x)}{6x(a_2-a_1)} {}_2F_1 \left(2q, 2, 4, \left(\frac{\varrho a_1(a_2-x)}{xa} \right) \right) \\
 &\quad + \frac{1}{2} \left(1 + \frac{\varrho a_1(a_2-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)}{xb} \right) \right) \\
 &\quad \left. - \frac{\varrho a_1(a_2-x)}{3x(a_2-a_1)} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_1(a_2-x)}{xb} \right) \right) - \varrho {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)}{xb} \right) \right) \right\}, \\
 B_3^{**} &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \frac{\tau}{(\tau a_1 + (1-\tau)a_2)^{2q}} d\tau \\
 &= a_1^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2 \left\{ \varrho^2 {}_2F_1 \left(2q, 1, 3, \left(\frac{a_2 \varrho(a_1-x)}{xa} \right) \right) \right. \\
 &\quad - \frac{\varrho a_2(x-a_1)}{6x(a_2-a_1)} {}_2F_1 \left(2q, 2, 4, \left(\frac{\varrho a_2(a_1-x)}{xa} \right) \right) \\
 &\quad + \frac{1}{2} \left(1 + \frac{\varrho a_2(a_1-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \left(\frac{a_2(a_1-x)}{xa} \right) \right) \\
 &\quad \left. - \frac{\varrho a_2(x-a_1)}{3x(a_2-a_1)} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_2(a_1-x)}{xa} \right) \right) - \varrho {}_2F_1 \left(2q, 1, 2, \left(\frac{a_2(a_1-x)}{xa} \right) \right) \right\}, \\
 B_4^{**} &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(a_1-x)}{x(a_2-a_1)} \right| \frac{1-\tau}{(\tau a_1 + (1-\tau)a_2)^{2q}} d\tau \\
 &= a_1^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^3 \left\{ \frac{\varrho^3}{3} {}_2F_1 \left(2q, 2, 4, \left(\frac{a_2 \varrho(a_1-x)}{xa} \right) \right) \right. \\
 &\quad \left. + \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_2(a_1-x)}{xa} \right) \right) - \frac{\varrho}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_2(a_1-x)}{xa} \right) \right) \right\}.
 \end{aligned}$$

Corollary 2. Under the assumptions of Theorem 2, if we choose $\varrho = 0$ in (4), then

$$\begin{aligned}
 |T(\Lambda, 0, \varsigma, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\phi_1^{**})^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_1^{***} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_2^{***} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\phi_2^{**})^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q B_3^{***} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q B_4^{***} \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 \phi_1^{**} &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} |\tau| d\tau = \frac{1}{2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2, \\
 \phi_2^{**} &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 |\tau-1| d\tau = \frac{1}{2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2, \\
 B_1^{***} &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} |\tau| \frac{\tau}{(A_\tau)^{2q}} d\tau \\
 &= (\zeta a_1 + (1-\zeta)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^3 \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right), \\
 B_2^{***} &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} |\tau| \frac{1-\tau}{(A_\tau)^{2q}} d\tau \\
 &= (\zeta a_1 + (1-\zeta)a_2)^{-2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 \frac{1}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right), \\
 B_3^{***} &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 |\tau-1| \frac{\tau}{(A_\tau)^{2q}} d\tau \\
 &= (\zeta a_2 + (1-\zeta)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^2 \frac{1}{2} {}_2F_1 \left(2q, 2, 3, \left(\frac{a_2(a_1-x)(2\zeta-1)}{x(ub + (1-\zeta)a_1)} \right) \right), \\
 B_4^{***} &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 |\tau-1| \frac{1-\tau}{(A_\tau)^{2q}} d\tau \\
 &= (\zeta a_2 + (1-\zeta)a_1)^{-2} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^3 \frac{1}{3} {}_2F_1 \left(2q, 3, 4, \left(\frac{a_2(a_1-x)(2\zeta-1)}{x(ub + (1-\zeta)a_1)} \right) \right).
 \end{aligned}$$

Corollary 3. Under the assumptions of Theorem 2, if we choose $q = 0$ and $x = \frac{2a_1a_2}{a_1+a_2}$ in Corollary 2, then

$$\begin{aligned}
 \left| T \left(\Lambda, 0, \zeta, \frac{2a_1a_2}{a_1+a_2} \right) \right| &\leq a_1a_2(a_2-a_1)(1-2\zeta) \left((\phi_1^\diamond)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1a_2}{A_{a_2}} \right) \right|^q B_1^\diamond + \left| \Lambda' \left(\frac{a_1a_2}{A_{a_1}} \right) \right|^q B_2^\diamond \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\phi_2^\diamond)^{1-\frac{1}{q}} \left(\left| \Lambda' \left(\frac{a_1a_2}{A_{a_2}} \right) \right|^q B_3^\diamond + \left| \Lambda' \left(\frac{a_1a_2}{A_{a_1}} \right) \right|^q B_4^\diamond \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where

$$\phi_1^\diamond = \frac{1}{8}, \quad \phi_2^\diamond = -\frac{1}{8}$$

$$\begin{aligned}
B_1^\diamond &= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2} \frac{1}{24} {}_2F_1\left(2q, 3, 4, \frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ua_1 + (1 - \varsigma)a_2)}\right), \\
B_2^\diamond &= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2} \frac{1}{8} {}_2F_1\left(2q, 2, 3, \frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ua_1 + (1 - \varsigma)a_2)}\right), \\
B_3^\diamond &= (\varsigma a_2 + (1 - \varsigma)a_1)^{-2} \frac{1}{8} {}_2F_1\left(2q, 2, 3, \frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ub + (1 - \varsigma)a_1)}\right), \\
B_4^\diamond &= (\varsigma a_2 + (1 - \varsigma)a_1)^{-2} \frac{1}{24} {}_2F_1\left(2q, 3, 4, \frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ub + (1 - \varsigma)a_1)}\right).
\end{aligned}$$

In order to establish our next refinement of Ostrowski's inequality, we use Hölder's inequality.

Theorem 3. Let $\Lambda : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° , where $a_1, a_2 \in I$ with $a_1 < a_2$. Let $q \geq 1$ and define p by the equality $\frac{1}{p} + \frac{1}{q} = 1$. If the function $|\Lambda'|^q$ is harmonically convex, then, for $\varrho \in [0, 1]$, $x \in [a_1, a_2]$, and $\varsigma \in [0, 1] \setminus \{\frac{1}{2}\}$, the following inequality is valid:

$$\begin{aligned}
|T(\Lambda, \varrho, \varsigma, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\psi_1^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^* \right)^{\frac{1}{q}} \right. \\
&\quad \left. + (\psi_2^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^* \right)^{\frac{1}{q}} \right), \quad (5)
\end{aligned}$$

where

$$\begin{aligned}
\psi_1^* &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \frac{1}{A_\tau^{2p}} d\tau \\
&= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2p} \frac{a_1(a_2-x)}{x(a_2-a_1)} {}_2F_1\left(2q, 1, 2, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)}\right)\right), \\
\psi_2^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \frac{1}{A_\tau^{2p}} d\tau \\
&= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2p} \left({}_2F_1\left(2q, 1, 2, \left(\frac{(a_2-a_1)(1-2\varsigma)}{2(ua_1 + (1-\varsigma)a_2)}\right)\right) \right. \\
&\quad \left. - \frac{a_1(a_2-x)}{x(a_2-a_1)} {}_2F_1\left(2q, 1, 2, \left(\frac{a_1(a_2-x)(1-2\varsigma)}{x(ua_1 + (1-\varsigma)a_2)}\right)\right) \right), \\
C_1^* &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \tau \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^q d\tau \\
&= \frac{1}{(q+1)(q+2)} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{q+2} \left(\varrho^{q+2} - (1-\varrho)^{q+2} + (1-\varrho)^{q+1}(q+2) \right), \\
C_2^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^q d\tau \\
&= \frac{1}{(q+1)(q+2)} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{q+1} \\
&\quad \times \left\{ \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) \left((1-\varrho)^{q+2} - \varrho^{q+2} - (q+2)(1-\varrho)^{q+1} \right) + (q+2)(\varrho^{q+1} + (1-\varrho)^{q+1}) \right\},
\end{aligned}$$

$$\begin{aligned}
C_3^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \tau \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^q d\tau \\
&= \frac{1}{(q+1)(q+2)} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{q+1} \\
&\quad \times \left\{ \left(\left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right) ((1-\varrho)^{q+2} - \varrho^{q+2} - (q+2)(1-\varrho)^{q+1}) \right) + (q+2)(\varrho^{q+1} + (1-\varrho)^{q+1}) \right\}, \\
C_4^* &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^q d\tau \\
&= \frac{1}{(q+1)(q+2)} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{q+2} \left(\varrho^{q+2} - (1-\varrho)^{q+2} + (1-\varrho)^{q+1}(q+2) \right).
\end{aligned}$$

Proof. For brevity, we denote $A_{a_1} = \varsigma a_1 + (1-\varsigma)a_2$ and $A_{a_2} = \varsigma a_2 + (1-\varsigma)a_1$. Taking modulus in Lemma 1 and using Hölder's inequality, we have

$$\begin{aligned}
&|T(\Lambda, \varrho, \varsigma, x)| \\
&\leq \left\{ \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right| d\tau \right. \\
&\quad \left. + \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\varsigma)}{A_\tau^2} \right| d\tau \right\} \\
&\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \frac{1}{(A_\tau)^{2p}} d\tau \right)^{\frac{1}{p}} \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^q \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \right|^q d\tau \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \frac{1}{(A_\tau)^{2p}} d\tau \right)^{\frac{1}{p}} \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^q \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \right|^q d\tau \right)^{\frac{1}{q}} \right\} \\
&= a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) K_3.
\end{aligned}$$

Using the harmonic convexity of $|\Lambda'|^q$, we get

$$\begin{aligned}
 K_3 &\leq \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \frac{1}{(A_\tau)^{2p}} d\tau \right)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \tau \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^q d\tau \right. \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} (1-\tau) \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^q d\tau \right)^{\frac{1}{q}} \right. \\
 &\quad + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \frac{1}{(A_\tau)^{2p}} d\tau \right)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \tau \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^q d\tau \right. \\
 &\quad \left. \left. + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^q d\tau \right)^{\frac{1}{q}} \right\} \\
 &= \left((\psi_1^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^* \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\psi_2^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^* \right)^{\frac{1}{q}} \right)
 \end{aligned}$$

This completes the proof. $\square$

Corollary 4. Under the assumptions of Theorem 3, if we choose $\varsigma = 0$ in 5, then

$$\begin{aligned}
 |T(\Lambda, \varrho, 0, x)| &\leq a_1 a_2 (a_2 - a_1) \left((\psi_1^{**})^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^* \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\psi_2^{**})^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^* + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^* \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 \psi_1^{**} &= a_2^{-2p} \frac{a_1(a_2-x)}{x(a_2-a_1)} {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)}{xb} \right) \right), \\
 \psi_2^{**} &= a_2^{-2p} \left({}_2F_1 \left(2q, 1, 2, \left(\frac{(a_2-a_1)}{a_2} \right) \right) - \frac{a_1(a_2-x)}{x(a_2-a_1)} {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)}{xb} \right) \right) \right),
 \end{aligned}$$

and where C_1^*, C_2^*, C_3^* and C_4^* are given in Theorem 3.

Corollary 5. Under the assumptions of Theorem 3, if we choose $q = 0$ in (5), then

$$|T(\Lambda, 0, \varsigma, x)| \leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\psi_1^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^{**} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^{**} \right)^{\frac{1}{q}} \right. \\ \left. + (\psi_2^*)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^{**} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^{**} \right)^{\frac{1}{q}} \right),$$

where ψ_1^*, ψ_2^* are given in Theorem 3 and

$$C_1^{**} = \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \tau^{q+1} d\tau = \frac{1}{(q+2)} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{q+2}, \\ C_2^{**} = \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} (1-\tau) \tau^q d\tau \\ = \frac{1}{(q+1)(q+2)} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{q+1} \left\{ \left(\left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) (-q-1) \right) + (q+2) \right\}, \\ C_3^{**} = \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \tau |\tau-1|^q d\tau \\ = \frac{1}{(q+1)(q+2)} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{q+1} \left\{ \left(\left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right) (-q-1) \right) + (q+2) \right\}, \\ C_4^{**} = \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) |\tau-1|^q d\tau = \frac{1}{(q+2)} \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{q+2}.$$

Corollary 6. Under the assumptions of Theorem 3, if we choose $q = 0$ and $x = \frac{2a_1 a_2}{a_1 + a_2}$ in Corollary 5, then

$$\left| T \left(\Lambda, 0, \varsigma, \frac{2a_1 a_2}{a_1 + a_2} \right) \right| \leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\psi_1^{**})^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^{***} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^{***} \right)^{\frac{1}{q}} \right. \\ \left. + (\psi_2^{**})^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^{***} + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^{***} \right)^{\frac{1}{q}} \right),$$

where

$$\begin{aligned}
 \psi_1^{**} &= \int_0^{\frac{1}{2}} \frac{1}{A_\tau^{2p}} d\tau \\
 &= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2p} \frac{1}{2} {}_2F_1\left(2q, 1, 2, \left(\frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ua_1 + (1 - \varsigma)a_2)}\right)\right), \\
 \psi_2^{**} &= \int_{\frac{1}{2}}^1 \frac{1}{A_\tau^{2p}} d\tau \\
 &= (\varsigma a_1 + (1 - \varsigma)a_2)^{-2p} \left({}_2F_1\left(2q, 1, 2, \left(\frac{(a_2 - a_1)(1 - 2\varsigma)}{(ua_1 + (1 - \varsigma)a_2)}\right)\right) \right. \\
 &\quad \left. - \frac{1}{2} {}_2F_1\left(2q, 1, 2, \left(\frac{(a_2 - a_1)(1 - 2\varsigma)}{2(ua_1 + (1 - \varsigma)a_2)}\right)\right) \right), \\
 C_1^{***} &= \int_0^{\frac{1}{2}} \tau^{q+1} d\tau = \frac{1}{(q+2)} \left(\frac{1}{2}\right)^{q+2}, \\
 C_2^{***} &= \int_0^{\frac{1}{2}} (1 - \tau)\tau^q d\tau = \frac{q+3}{2(q+1)(q+2)} \left(\frac{1}{2}\right)^{q+1}, \\
 C_3^{***} &= \int_{\frac{a_1}{a_1+a_2}}^1 \tau|\tau-1|^q d\tau = \frac{q+3}{2(q+1)(q+2)} \left(\frac{1}{2}\right)^{q+1}, \\
 C_4^{***} &= \int_{\frac{a_1}{a_1+a_2}}^1 (1 - \tau)|\tau-1|^q d\tau = \frac{1}{(q+2)} \left(\frac{1}{2}\right)^{q+2}.
 \end{aligned}$$

Theorem 4. Let the notions and hypothesis be as in Theorem 3. Then, the following inequality is true for $q \in [0, 1]$, $x \in [a_1, a_2]$ and $\varsigma \in [0, 1] \setminus \{\frac{1}{2}\}$:

$$\begin{aligned}
 |T(\Lambda, \varrho, \varsigma, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\varsigma) \left((\psi_1)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2 \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\psi_2)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4 \right)^{\frac{1}{q}} \right), \quad (6)
 \end{aligned}$$

where

$$\begin{aligned}
 \psi_1 &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^p d\tau = \left(\frac{\varrho^{p+1} + (1-\varrho)^{p+1}}{p+1} \right) \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{p+1}, \\
 \psi_2 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^p d\tau = \left(\frac{\varrho^{p+1} + (1-\varrho)^{p+1}}{p+1} \right) \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{p+1},
 \end{aligned}$$

$$\begin{aligned}
C_1 &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \tau [\tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)]^{-2q} d\tau \\
&= \frac{(\zeta a_1 + (1-\zeta)a_2)^{-2q}}{2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right), \\
C_2 &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} (1-\tau) [\tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)]^{-2q} d\tau \\
&= (\zeta a_1 + (1-\zeta)a_2)^{-2q} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) \left({}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right. \\
&\quad \left. - \left(\frac{a_1(a_2-x)}{2x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right), \\
C_3 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \tau [\tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)]^{-2q} d\tau \\
&= \frac{(\zeta a_1 + (1-\zeta)a_2)^{-2q}}{2} \left({}_2F_1 \left(2q, 2, 3, \left(\frac{(1-2\zeta)(a_2-x)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right. \\
&\quad \left. - \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right), \\
C_4 &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) [\tau(\zeta a_2 + (1-\zeta)a_1) + (1-\tau)(\zeta a_1 + (1-\zeta)a_2)]^{-2q} d\tau \\
&= (\zeta a_1 + (1-\zeta)a_2)^{-2q} \left(\frac{1}{2} {}_2F_1 \left(2q, 1, 3, \left(\frac{(1-2\zeta)(a_2-x)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right. \\
&\quad \left. - \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \left(\frac{a_1(a_2-x)(1-2\zeta)}{x(ua_1 + (1-\zeta)a_2)} \right) \right) \right)
\end{aligned}$$

Proof. For simplicity, we assume that $A_{a_1} = \zeta a_1 + (1-\zeta)a_2$ and $A_{a_2} = \zeta a_2 + (1-\zeta)a_1$. Taking modulus in Lemma 1 and using Hölder's inequality, we have

$$\begin{aligned}
|T(\Lambda, \varrho, \zeta, x)| &\leq \left\{ \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right| \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\zeta)}{A_\tau^2} \right| d\tau \right. \\
&\quad \left. + \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right| \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{a_1 a_2 (a_2 - a_1) (1 - 2\zeta)}{A_\tau^2} \right| d\tau \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq a_1 a_2 (a_2 - a_1) (1 - 2\zeta) \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^p d\tau \right)^{\frac{1}{p}} \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right|^q d\tau \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^p d\tau \right)^{\frac{1}{p}} \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \Lambda' \left(\frac{a_1 a_2}{A_\tau} \right) \frac{1}{A_\tau^2} \right|^q d\tau \right)^{\frac{1}{q}} \right\} \\
&= a_1 a_2 (a_2 - a_1) (1 - 2\zeta) K_4.
\end{aligned}$$

Using the harmonic convexity of $|\Lambda'|^q$, we get

$$\begin{aligned}
K_4 &\leq \left\{ \left(\int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \left| \tau - \varrho \frac{a_1(a_2-x)}{x(a_2-a_1)} \right|^p d\tau \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \frac{\tau}{(A_\tau)^{2q}} d\tau + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \frac{(1-\tau)}{(A_\tau)^{2q}} d\tau \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \left| \tau - 1 + \varrho \frac{a_2(x-a_1)}{x(a_2-a_1)} \right|^p d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \frac{\tau}{(A_\tau)^{2q}} d\tau + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \frac{(1-\tau)}{(A_\tau)^{2q}} d\tau \right)^{\frac{1}{q}} \Bigg\} \\
&= \left((\psi_1)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2 \right)^{\frac{1}{q}} \right. \\
&\quad \left. + (\psi_2)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4 \right)^{\frac{1}{q}} \right).
\end{aligned}$$

This completes the proof. $\square$

Corollary 7. Under the assumptions of Theorem 4, if we choose $\zeta = 0$ in (6), then

$$\begin{aligned}
|T(\Lambda, \varrho, 0, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\zeta) \left((\psi_1)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1^\diamond + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2^\diamond \right)^{\frac{1}{q}} \right. \\
&\quad \left. + (\psi_2)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3^\diamond + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4^\diamond \right)^{\frac{1}{q}} \right)
\end{aligned}$$

where ψ_1, ψ_2 are given in Theorem 4 and

$$\begin{aligned}
 C_1^\diamond &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} \tau [\tau a_1 + (1-\tau)a_2]^{-2q} d\tau \\
 &= \frac{a_2^{-2q}}{2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \frac{a_1(a_2-x)}{xb} \right), \\
 C_2^\diamond &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} (1-\tau) [\tau a_1 + (1-\tau)a_2]^{-2q} d\tau \\
 &= a_2^{-2q} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) \left({}_2F_1 \left(2q, 1, 2, \frac{a_1(a_2-x)}{xb} \right) - \left(\frac{a_1(a_2-x)}{2x(a_2-a_1)} \right) {}_2F_1 \left(2q, 2, 3, \frac{a_1(a_2-x)}{xb} \right) \right), \\
 C_3^\diamond &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 \tau [\tau a_1 + (1-\tau)a_2]^{-2q} d\tau \\
 &= \frac{a_2^{-2q}}{2} \left({}_2F_1 \left(2q, 2, 3, \frac{(a_2-x)}{a_2} \right) - \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \frac{a_1(a_2-x)}{xb} \right) \right), \\
 C_4^\diamond &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 (1-\tau) [\tau a_1 + (1-\tau)a_2]^{-2q} d\tau \\
 &= a_2^{-2q} \left(\frac{1}{2} {}_2F_1 \left(2q, 1, 3, \frac{(a_2-x)}{a_2} \right) - \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right) {}_2F_1 \left(2q, 1, 2, \left(\frac{a_1(a_2-x)}{xb} \right) + \frac{1}{2} \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^2 {}_2F_1 \left(2q, 2, 3, \frac{a_1(a_2-x)}{xb} \right) \right) \right).
 \end{aligned}$$

Corollary 8. Under the assumptions of Theorem 4, if we choose $q = 0$ in (6), then

$$\begin{aligned}
 |T(\Lambda, 0, \zeta, x)| &\leq a_1 a_2 (a_2 - a_1) (1 - 2\zeta) \left((\psi_1^\diamond)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_1 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_2 \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + (\psi_2^\diamond)^{\frac{1}{p}} \left(\left| \Lambda' \left(\frac{a_1 a_2}{A_{a_2}} \right) \right|^q C_3 + \left| \Lambda' \left(\frac{a_1 a_2}{A_{a_1}} \right) \right|^q C_4 \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 \psi_1^\diamond &= \int_0^{\frac{a_1(a_2-x)}{x(a_2-a_1)}} |\tau|^p d\tau = \left(\frac{1}{p+1} \right) \left(\frac{a_1(a_2-x)}{x(a_2-a_1)} \right)^{p+1}, \\
 \psi_2^\diamond &= \int_{\frac{a_1(a_2-x)}{x(a_2-a_1)}}^1 |\tau-1|^p d\tau = \left(\frac{1}{p+1} \right) \left(\frac{a_2(x-a_1)}{x(a_2-a_1)} \right)^{p+1},
 \end{aligned}$$

where C_1, C_2, C_3 and C_4 are given in Theorem 4.

Corollary 9. Under the assumptions of Theorem 4, if we choose $\varsigma = 0$ and $x = \frac{2a_1a_2}{a_1+a_2}$ in Corollary 8, then

$$\left| T\left(\Lambda, 0, 0, \frac{2a_1a_2}{a_1+a_2}\right) \right| \leq a_1a_2(a_2-a_1)(1-2\varsigma) \left((\psi_1^{\diamond\diamond})^{\frac{1}{p}} \left(|\Lambda'(a_2)|^q C_1^{\diamond\diamond} + |\Lambda'(a_1)|^q C_2^{\diamond\diamond} \right)^{\frac{1}{q}} \right. \\ \left. + (\psi_2^{\diamond\diamond})^{\frac{1}{p}} \left(|\Lambda'(a_2)|^q C_3^{\diamond\diamond} + |\Lambda'(a_1)|^q C_4^{\diamond\diamond} \right)^{\frac{1}{q}} \right),$$

where

$$\psi_1^{\diamond\diamond} = \int_0^{\frac{a_1}{a_1+a_2}} |\tau|^p d\tau = \left(\frac{1}{p+1} \right) \left(\frac{1}{2} \right)^{p+1}, \\ \psi_2^{\diamond\diamond} = \int_{\frac{a_1}{a_1+a_2}}^1 |\tau-1|^p d\tau = \left(\frac{1}{p+1} \right) \left(\frac{1}{2} \right)^{p+1},$$

$$C_1^{\diamond\diamond} = \frac{1}{(a_2-a_1)^2} \left[\frac{a_2^{2-2q}}{1-2q} - \frac{a_2 \left(\frac{a_1+a_2}{2} \right)^{1-2q}}{1-2q} + \frac{\left(\frac{a_1+a_2}{2} \right)^{2-2q}}{2-2q} - \frac{a_2^{2-2q}}{2-2q} \right], \\ C_2^{\diamond\diamond} = \frac{1}{(a_2-a_1)^2} \left[\frac{a_2^{2-2q}}{2-2q} + \frac{a_1 \left(\frac{a_1+a_2}{2} \right)^{1-2q}}{1-2q} - \frac{\left(\frac{a_1+a_2}{2} \right)^{2-2q}}{2-2q} - \frac{a_1a_2^{1-2q}}{1-2q} \right], \\ C_3^{\diamond\diamond} = \frac{1}{(a_2-a_1)^2} \left[\frac{a_1^{2-2q}}{2-2q} + \frac{a_2 \left(\frac{a_1+a_2}{2} \right)^{1-2q}}{1-2q} - \frac{\left(\frac{a_1+a_2}{2} \right)^{2-2q}}{2-2q} - \frac{a_1^{1-2q}a_2}{1-2q} \right], \\ C_4^{\diamond\diamond} = \frac{1}{(a_2-a_1)^2} \left[\frac{a_1^{2-2q}}{1-2q} - \frac{a_1 \left(\frac{a_1+a_2}{2} \right)^{1-2q}}{1-2q} + \frac{\left(\frac{a_1+a_2}{2} \right)^{2-2q}}{2-2q} - \frac{a_1^{2-2q}}{2-2q} \right].$$

3. Applications

In this section, we present some applications to special means. We recall the following well known special means for two nonnegative real numbers:

1. The Arithmetic Mean: $A(a_1, a_2) := \frac{a_1+a_2}{2}$
2. The Geometric Mean: $G(a_1, a_2) := \sqrt{a_1a_2}$
3. The Harmonic Mean: $H(a_1, a_2) := \frac{2a_1a_2}{a_1+a_2}$
4. The Logarithmic Mean: $L(a_1, a_2) := \frac{a_2-a_1}{\ln a_2 - \ln a_1}$
5. The p -Logarithmic mean: $L_p(a_1, a_2) := \left(\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)} \right)^{\frac{1}{p}}, p \in \mathbb{R} \setminus \{-1, 0\}$.

We now discuss our results.

Proposition 1. For $0 < a_1 < a_2$, we have

$$\begin{aligned} & \left| H(a_1, a_2) - \frac{G^2(a_1, a_2)}{L(a_1, a_2)} \right| \\ & \leq a_1 a_2 (a_2 - a_1) \left[(\psi_1^{\diamond})^{\frac{1}{p}} \left(\frac{1}{(a_2 - a_1)^2} \left(\frac{a_2^{2-2q}}{2-2q} - \frac{a_1 a_2^{1-2q}}{1-2q} - \frac{(a_2 - a_1) A^{1-2q}(a_1, a_2)}{1-2q} \right) \right)^{\frac{1}{q}} \right. \\ & \quad \left. + (\psi_2^{\diamond})^{\frac{1}{p}} \left(\frac{1}{(a_2 - a_1)^2} \left(\frac{a_1^{2-2q}}{2-2q} - \frac{a_1^{1-2q} a_2}{1-2q} + \frac{(a_2 - a_1) A^{1-2q}(a_1, a_2)}{1-2q} \right) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The assertion follows from the Corollary 9, for $\Lambda(x) = x$. $\square$

Proposition 2. For $0 < a_1 < a_2$, we have

$$\begin{aligned} & \left| H^{p+2}(a_1, a_2) - G^2(a_1, a_2) L_p^p(a_1, a_2) \right| \\ & \leq a_1 a_2 (a_2 - a_1) \left[(\psi_1^{\diamond})^{\frac{1}{p}} \left(\frac{(p+2)a_2^{p+1}}{(a_2 - a_1)^2} \left(\frac{a_2^{2-2q}}{1-2q} - \frac{b A^{1-2q}(a_1, a_2)}{1-2q} \right) + \frac{(p+2)a_1^{p+1}}{(a_2 - a_1)^2} \left(\frac{a_1 a_2^{1-2q}}{1-2q} \right. \right. \right. \\ & \quad \left. \left. - \frac{a A^{1-2q}(a_1, a_2)}{1-2q} \right) + \frac{(p+1)(p+2) A^{2-2q}(a_1, a_2)}{(a_2 - a_1)(2-2q)} L_p^p(a_1, a_2) - \frac{(p+1)(p+2) a_2^{2-2q}}{(a_2 - a_1)(2-2q)} L_p^p(a_1, a_2) \right)^{\frac{1}{q}} \\ & \quad \left. + (\psi_2^{\diamond})^{\frac{1}{p}} \left(\frac{(p+2)a_2^{p+1}}{(a_2 - a_1)^2} \left(-\frac{a_1^{1-2q} a_2}{1-2q} + \frac{b A^{1-2q}(a_1, a_2)}{1-2q} \right) + \frac{(p+2)a_1^{p+1}}{(a_2 - a_1)^2} \left(\frac{a_1^{2-2q}}{1-2q} \right. \right. \right. \\ & \quad \left. \left. - \frac{a A^{1-2q}(a_1, a_2)}{1-2q} \right) - \frac{(p+1)(p+2) A^{2-2q}(a_1, a_2)}{(a_2 - a_1)(2-2q)} L_p^p(a_1, a_2) + \frac{(p+1)(p+2) a_1^{2-2q}}{(a_2 - a_1)(2-2q)} L_p^p(a_1, a_2) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The assertion follows from the Corollary 9, for $\Lambda(x) = x^{p+2}$. $\square$

4. Conclusions

We have derived several new refinements of Ostrowski type of integral inequalities using the class of harmonically convex functions. We have discussed several new special cases of the obtained results as well. This shows that our results are quite unifying. In order to show the significance of the main results, we have also presented some applications of our main results to special means. Recently, authors [15,16] have obtained Hermite–Hadamard’s inequality on higher dimensions. In addition, it will be an interesting problem for future research to consider the results obtained in this paper on higher dimensions.

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