symmetry

MDPI

Article

Generalized Fibonacci Numbers, Cosmological Analogies, and an Invariant

Valerio Faraoni * and Farah Atieh

Department of Physics & Astronomy, Bishop's University, 2600 College Street, Sherbrooke, QC J1M 1Z7, Canada; fatieh19@ubishops.ca

* vfaraoni@ubishops.ca; Tel.: +1-819-822-9600; Fax: +1-819-822-9611

Abstract: Continuous generalizations of the Fibonacci sequence satisfy ODEs that are formal analogues of the Friedmann equation describing a spatially homogeneous and isotropic cosmology in general relativity. These analogies are presented together with their Lagrangian and Hamiltonian formulations and with an invariant of the Fibonacci sequence.

Keywords: Fibonacci sequence; cosmology; analogy; general relativity

1. Introduction

Fibonacci, also known as Leonardo Pisano or Leonardo Bonacci, introduced Hindu–Arabic numerals to Europe with his book *Liber Abaci* in 1202 [1]. He posed and solved a well-known problem involving the growth of a population of rabbits in idealized situations. The solution, now known as the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, ...$, is written as

$$F_n = F_{n-1} + F_{n-2}\,, \tag{1}$$

where F_n is the n-th Fibonacci number and $F_0 = 0, F_1 = 1$. Furthermore, the ratio of two consecutive terms F_{n+1}/F_n approaches the golden ratio

$$\varphi \equiv \frac{1+\sqrt{5}}{2} \approx 1.61803398\,... \tag{2}$$

as $n \to +\infty$.

The Fibonacci sequence can be generalized to the continuum using Binet's formula

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}\,. \tag{3}$$

Furthermore, the analytic function

$$F_{(e)}(x) = \frac{\varphi^x - \varphi^{-x}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\,\sinh(x\ln\varphi) \tag{4}$$

reproduces part of the Fibonacci numbers, $F_n = F_{(e)}(n)$, for even $x = n \in \mathbb{N}$, while

$$F_{(o)}(x) = \frac{\varphi^x + \varphi^{-x}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\,\cosh(x\ln\varphi) \tag{5}$$

reproduces the other Fibonacci numbers for odd $x = n \in \mathbb{N}$. The function

$$F(x) = \frac{\varphi^x - \cos(\pi x)\varphi^{-x}}{\sqrt{5}} \tag{6}$$

check for updates

Citation: Faraoni, V.; Atieh, F. Generalized Fibonacci Numbers, Cosmological Analogies, and an Invariant. *Symmetry* **2021**, *13*, 200. https://doi.org/10.3390/sym13020200

Academic Editor: Abraham Ungar
Received: 12 January 2021
Accepted: 26 January 2021
Published: 26 January 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

reproduces the entire Fibonacci sequence for $x = n \in \mathbb{N}$. Here, we focus on $F_{(e,o)}(x)$, which admit analogies with relativistic cosmology, while no such analogy exists for $F(x)$.

The functions $F_{(e,o)}$ satisfy the dual relations (where a prime denotes differentiation with respect to x)

$$F'_{(e)}(x) = (\ln \varphi) F_{(o)}(x)\,, \tag{7}$$

$$F'_{(o)}(x) = (\ln \varphi) F_{(e)}(x)\,, \tag{8}$$

and the second-order ODE

$$F''_{(e,o)} - \left(\ln^2 \varphi\right) F_{(e,o)} = 0\,, \tag{9}$$

of which $F_{(e)}$ and $F_{(o)}$ are two linearly independent solutions. In physics, this equation describes the one-dimensional motion of a particle of position F in the inverted harmonic oscillator potential $V(F) = -kF^2/2$ (with $K = 2\ln^2 \varphi$), which is used as an example of an unstable mechanical system (this property corresponds to the fact that the Fibonacci numbers F_n increase without bound as $n \to \infty$).

We will also use

$$f(x) = \varphi^x\,, \tag{10}$$

which is a Fibonacci function according to the definition in [2]; i.e.,

$$f(x+2) = f(x+1) + f(x) \qquad \forall x \in \mathbb{R}\,. \tag{11}$$

However, $F_{(e,o)}(x)$ are not Fibonacci functions since they contain φ^{-x}, which is not a Fibonacci function (it is easy to prove [2] that, among power-law functions $y(x) = b^x$, the only Fibonacci function is the one with a base equal to the golden ratio, $b = \varphi$).

In the following section, we briefly recall the basics of spatially homogeneous and isotropic cosmology in general relativity, and then we present the formal analogy with Equations (7)–(9) in Section 3. We then deduce Lagrangian and Hamiltonian formulations for the Fibonacci ODEs and derive from these an invariant of the (discrete) Fibonacci sequence.

2. FLRW Cosmology

In the context of Einstein's relativistic theory of gravity, the most basic assumption of cosmology is the Copernican principle stating that, on average (i.e., on scales larger than a few tens of Megaparsecs), the universe is spatially homogeneous and isotropic [3–8]. The stringent symmetry requirements of spatial homogeneity and isotropy force the geometry to have a constant spatial curvature [9]. The spacetime metric is necessarily given by the Friedmann–Lemaître–Robertson–Walker (FLRW) line element, written in polar comoving coordinates (t, r, ϑ, ϕ) as [3,4]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2_{(2)}\right] \tag{12}$$

where $d\Omega^2_{(2)} = d\vartheta^2 + \sin^2\vartheta\, d\phi^2$ is the line element on the unit 2-sphere and the sign of the constant curvature index K classifies the three-dimensional spatial sections $t = \text{const}$. $K > 0$ corresponds to positively curved three-spheres, $K = 0$ to Euclidean flat sections, and $K < 0$ to hyperbolic three-sections. The dynamics of FLRW cosmology is contained in the scale factor $a(t)$. In FLRW cosmology, the matter source causing spacetime to curve is usually (but not necessarily) taken to be a perfect fluid with energy density $\rho(t)$ and pressure $P(t) = w\rho(t)$, where w is a constant equation of the state parameter. The evolution

of $a(t)$ and $\rho(t)$ is ruled by the Einstein equations adapted to a high degree of symmetry: the Einstein–Friedmann equations. They comprise [3–5] the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \tag{13}$$

(a first order constraint), the acceleration or Raychaudhuri equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}\,, \tag{14}$$

and the covariant conservation equation,

$$\dot{\rho} + 3H(P + \rho) = 0 \tag{15}$$

expressing energy conservation for the cosmic fluid. Here, G is Newton's gravitational constant, Λ is Einstein's famous cosmological constant, an overdot denotes differentiation with respect to the cosmological time t, $H \equiv \dot{a}/a$ is the Hubble function [4,5], and units are used in which the speed of light is unity we follow the notations used in [4,5]. For a generic cosmic fluid, only two of the three Equations (13)–(15) are independent. When there is no cosmic fluid and the cosmological constant Λ (which can be treated as an effective fluid with energy density $\rho = \frac{\Lambda}{8\pi G} = -P$) is the only energy content, the conservation Equation (15) is satisfied identically.

3. The Cosmological Analogy

Let us consider first the function $F_{(e)}(x)$, which satisfies

$$F'_{(e)} = \frac{2}{\sqrt{5}}\,\ln\varphi\cosh(x\ln\varphi) = \frac{2}{\sqrt{5}}\,\ln\varphi\sqrt{1 + \frac{5F^2_{(e)}}{4}}\,. \tag{16}$$

Dividing this equation by $F_{(e)}$ and squaring, one obtains

$$\left(\frac{F'_{(e)}}{F_{(e)}}\right)^2 = \frac{4\ln^2\varphi}{5} + \frac{4\ln^2\varphi}{5F^2_{(e)}}\,, \tag{17}$$

which is formally analogous to the Friedmann equation (Equation (13)), provided that $\left(x, F_{(e)}(x)\right) \to (t, a(t))$ and

$$\rho = P = 0\,, \tag{18}$$

$$\Lambda = 3\ln^2\varphi\,, \tag{19}$$

$$K = -\frac{4\ln^2\varphi}{5} \tag{20}$$

(contrary to FLRW cosmology, these quantities are dimensionless in the Fibonacci side of the analogy). A priori, the formal equivalence with the Friedmann Equation is not sufficient for the analogy to hold, and one must check that Equations (14) and (15) are also satisfied: this is straightforward to do using Equation (9), while the conservation equation is trivially satisfied with $\rho = P = 0$ (and also by the Λ-effective fluid with $P = -\rho = \text{const.}$). The universe with a scale factor analogous to the function $F_{(e)}(x)$ has hyperbolic 3D spatial sections and is empty but expands due to the repulsive cosmological constant.

By squaring Equation (16), one has introduced the possibility of solutions with $F'_{(e)} < 0$, corresponding to a contracting, instead of expanding, universe. In fact, Equation (13) then gives

$$\dot{a} = \pm\sqrt{\frac{\Lambda a^2}{3} + |K|}\,, \tag{21}$$

which integrates to

$$\ln\left[C\left(\sqrt{\Lambda^2 a^2 + 3\Lambda|K|} + \Lambda a\right)\right] = \pm\sqrt{\frac{\Lambda}{3}}(t - t_0) \tag{22}$$

where C and t_0 are integration constants (C serves the purpose of making the argument of the logarithm dimensionless since, in the units used [3–5], the scale factor a carries the dimensions of a length and Λ those of an inverse length squared). By exponentiating both sides, squaring and collecting similar terms, one is left with

$$a(t) = a_0\left[\mathrm{e}^{\pm\sqrt{\frac{\Lambda}{3}}(t-t_0)} - 3|K|\Lambda C^2\,\mathrm{e}^{\mp\sqrt{\frac{\Lambda}{3}}(t-t_0)}\right], \tag{23}$$

with a_0 constant. In practice, in FLRW cosmology, the dimensionless radial coordinate r can be rescaled to normalize the curvature index K (usually to ± 1); here, we can rescale to obtain $|K|\Lambda C^2 = 1/3$ and

$$a(t) = \pm a_0 \sinh\left[\sqrt{\frac{\Lambda}{3}}(t - t_0)\right] \tag{24}$$

where, in order to keep the scale factor non-negative, the upper sign applies for $t \geq t_0$ and the lower one for $t \leq t_0$. At late times $t \to +\infty$, this metric is asymptotic to the metric of de Sitter spacetime $a_{(dS)}(t) = a_0\,\mathrm{e}^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}$.

Let us focus now on the function $F_{(o)}(x)$: proceeding as we did for $F_{(e)}(x)$, one obtains

$$\left(\frac{F'_{(o)}}{F_{(o)}}\right)^2 = -\ln^2\varphi + \frac{4\ln^2\varphi}{5F^2_{(o)}}\,, \tag{25}$$

which is formally analogous to Equation (13) with

$$\rho = P = 0\,, \tag{26}$$

$$\Lambda = 3\ln^2\varphi\,, \tag{27}$$

$$K = \frac{4\ln^2\varphi}{5}\,. \tag{28}$$

The analogous FLRW universe is again empty and propelled by the positive cosmological constant, but the spatial sections are now closed three-spheres.

Again, by squaring, one introduces the possibility of a negative sign for $\dot{a}$. Proceeding in parallel with what has been done for $F_{(e)}$, one obtains

$$a(t) = a_0\left[\mathrm{e}^{\pm\sqrt{\frac{\Lambda}{3}}(t-t_0)} + 3K\Lambda C^2\,\mathrm{e}^{\mp\sqrt{\frac{\Lambda}{3}}(t-t_0)}\right] \tag{29}$$

and normalizing K suitably,

$$a(t) = a_0\cosh\left[\sqrt{\frac{\Lambda}{3}}(t - t_0)\right]. \tag{30}$$

This scale factor describes a universe contracting from an infinite size, bouncing at the minimum value a_0, and then expanding forever and asymptoting to the de Sitter space—a behavior sought for in quantum cosmology to avoid the classical Big Bang singularity $a = 0$.

Finally, we can consider the Fibonacci function $f(x) = \varphi^x$, which trivially satisfies

$$\left(\frac{f'}{f}\right)^2 = \ln^2 \varphi \tag{31}$$

and is analogous to an empty ($\rho = P = 0$), spatially flat ($K = 0$) universe expanding exponentially, $a_{(dS)} = a_0\, \mathrm{e}^{\sqrt{\frac{\Lambda}{3}}\, t}$ due to the positive cosmological constant $\Lambda = 3 \ln^2 \varphi$. This is the maximally symmetric de Sitter universe, which is an attractor in inflationary models of the early universe [6,7] and in dark energy-dominated models of the late (present-day) accelerating universe [10].

4. Lagrangian and Hamiltonian

Both the mechanical analogy of Equation (9) and the cosmological analogy suggest the Lagrangian for $F_{(e,o)}(x)$

$$L_{(e,o)}\left(F_{(e,o)}(x), F'_{(e,o)}(x)\right) = \frac{1}{2}\,(F'_{(e,o)})^2 + \frac{\ln^2 \varphi}{2}\, F^2_{(e,o)}\,. \tag{32}$$

The corresponding Euler–Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L_{(e,o)}}{\partial F'_{(e,o)}}\right) - \frac{\partial L_{(e,o)}}{\partial F_{(e,o)}} = 0 \tag{33}$$

reproduces Equation (9).

The associated Hamiltonian, expressed in terms of the canonical variables $q_{(e,o)} \equiv F_{(e.o)}$ and $p_{(e,o)} \equiv \partial L_{(e,o)}/\partial F'_{(e,o)} = F'_{(e,o)}$, is

$$\mathcal{H}_{(e,o)} = p_{(e,o)} F'_{(e,o)} - L_{(e,o)} = \frac{\left(p_{(e,o)}\right)^2}{2} - \frac{\ln^2 \varphi}{2}\, F^2_{(e,o)}\,, \tag{34}$$

and the Hamilton equations are

$$q'_{(e,o)} = \frac{\partial \mathcal{H}_{(e,o)}}{\partial p_{(e,o)}} = p_{(e,o)} = F'_{(e,o)} = \ln \varphi\, F_{(o,e)} \tag{35}$$

(which reproduces Equations (7) and (8)) and

$$p'_{(e,o)} = -\frac{\partial \mathcal{H}_{(e,o)}}{\partial q_{(e,o)}} = \ln^2 \varphi\, F_{(e,o)}\,. \tag{36}$$

Since $L_{(e,o)}$ does not depend explicitly on x, the Hamiltonian $\mathcal{H}_{(e,o)}$ is conserved,

$$\frac{\left(p_{(e,o)}\right)^2}{2} - \frac{\ln^2 \varphi}{2}\, F^2_{(e,o)} = E_{(e,o)}\,, \tag{37}$$

where the constants $E_{(e,o)}$ have the meaning of energy of the system. Writing them explicitly, we have

$$E_{(e,o)} = \frac{\ln^2 \varphi}{2}\left(F_{(o,e)} + F_{(e,o)}\right)\left(F_{(o,e)} - F_{(e,o)}\right) = \pm\frac{2 \ln^2 \varphi}{5}\,, \tag{38}$$

which gives the first integral $\left(F^2_{(o,e)} - F^2_{(e,o)}\right)$ on the Fibonacci side of the analogy. In terms of the discrete Fibonacci sequence, we therefore present the following proposition.

Proposition 1. *The quantity*

$$\begin{aligned} I &= \left[(F_{2m} + F_{2m-1})^2 - (F_{2m-1} + F_{2m-2})^2\right] \\ &= F^2_{2m} + F^2_{2m-1} - F^2_{2m-1} - F^2_{2m-2} + 2(F_{2m}F_{2m-1} - F_{2m-1}F_{2m-2}) \end{aligned} \tag{39}$$

does not depend on m and is an invariant of the Fibonacci sequence.

To the best of our knowledge, this invariant is not related to known invariants (for example, those of the Fibonacci convolution sequences [11]).

5. Conclusions

The functions $F_{(e,o)}$ associated with the continuum generalization of the Fibonacci sequence exhibit analogies with certain spatially homogeneous and isotropic universes in FLRW cosmology, as well as a mechanical analogy with an inverted harmonic oscillator, which we have presented. With the help of these analogies, it is easy to derive a Lagrangian and a Hamiltonian associated with the ODEs satisfied by the functions $F_{(e,o)}$. Then, the conservation of the Hamiltonian (or energy) yields a quantity I, given by Equation (39) that is constant across the Fibonacci sequence; i.e., it does not depend on the index n. This invariant may turn out to be useful in view of the many applications of the Fibonacci numbers in mathematics and in the natural sciences (e.g., [12–18]).

Author Contributions: Both authors contributed to the design and implementation of this research. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2016-03803 to V.F.) and by Bishop's University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: There are no data associated with this work because of its theoretical nature.

Acknowledgments: We thank two referees for useful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sigler, L.E. *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*; Springer: New York, NY, USA, 2002.
2. Han, J.S.; Kim, H.S.; Neggers, J. On Fibonacci functions with Fibonacci numbers. *Adv. Differ. Equ.* **2012**, *2012*, 126. [CrossRef]
3. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure Of Space-Time*; Cambridge University Press: Cambridge, CA, USA, 1973.
4. Wald, R.M. *General Relativity*; Chicago University Press: Chicago, IL, USA, 1984.
5. Carroll, S.M. *Spacetime and Geometry: An Introduction to General Relativity*; Addison-Wesley: San Francisco, CA, USA, 2004.
6. Kolb, E.W.; Turner, M.S. *The Early Universe*; Addison-Wesley: Redwood City, CA, USA, 1990.
7. Liddle, A.R.; Lyth, D.H. *Cosmological Inflation and Large-Scale Structure*; Cambridge University Press: Cambridge, CA, USA, 2000.
8. Liddle, A.R. *An Introduction to Modern Cosmology*; Wiley: New York, NY, USA, 2015.
9. Eisenhart, L.P. *Riemannian Geometry*; Princeton University Press: Princeton, NJ, USA, 1949.
10. Amendola, L.; Tsujikawa, S. *Dark Energy, Theory and Observations*; Cambridge University Press: Cambridge, MA, USA, 2010.
11. Hoggatt, V.E., Jr.; Bicknell-Johnson, M. Fibonacci Convolution Sequences. *Fibonacci Quart* **1977**, *15*, 117.
12. Baringhaus, L. Fibonacci numbers, Lucas numbers and integrals of certain Gaussian processes. *Proc. Am. Math. Soc.* **1996**, *124*, 3875. [CrossRef]
13. Gardiner, J. Fibonacci, quasicrystals and the beauty of flowers. *Plant Signal. Behav.* **2012**, *7*, 1721. [CrossRef]

14. Jacquod, P.; Silvestrov, P.G.; Beenakker, C.W.J. Golden rule decay versus Lyapunov decay of the quantum Loschmidt echo. *Phys. Rev. E* **2001**, *64*, 055203(R). [CrossRef] [PubMed]
15. Affleck, I. Golden mean seen in a magnet. *Nature* **2010**, *464*, 362. [CrossRef] [PubMed]
16. Luminet, J.-P.; Weeks, J.; Riazuelo, A.; Ehoucq, R.; Uzan, J.-P. Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background. *Nature* **2003**, *425*, 593. [CrossRef] [PubMed]
17. Boman, B.M.; Dinh, T.N.; Decker, K.; Emerick, B.; Raymond, C.; Schleiniger, G. Why do Fibonacci numbers appear in patterns of growth in nature? A model for tissue renewal based on asymmetric cell division. *Fibonacci Quart* **2017**, *55*, 30.
18. Boman, B.M.; Ye, Y.; Decker, K.; Raymond, C.; Schleiniger, G. Geometric Branching Patterns based on p-Fibonacci Sequences: Self-similarity Across Different Degrees of Branching and Multiple Dimensions. *Fibonacci Quart* **52019**, *7*, 29.