

## Review

# Topological Doping and Superconductivity in Cuprates: An Experimental Perspective

John M. Tranquada 

Brookhaven National Laboratory, Condensed Matter Physics & Materials Science Division,  
Upton, NY 11973, USA; jtran@bnl.gov

**Abstract:** Hole doping into a correlated antiferromagnet leads to topological stripe correlations, involving charge stripes that separate antiferromagnetic spin stripes of opposite phases. The topological spin stripe order causes the spin degrees of freedom within the charge stripes to feel a geometric frustration with their environment. In the case of cuprates, where the charge stripes have the character of a hole-doped two-leg spin ladder, with corresponding pairing correlations, anti-phase Josephson coupling across the spin stripes can lead to a pair-density-wave order in which the broken translation symmetry of the superconducting wave function is accommodated by pairs with finite momentum. This scenario is now experimentally verified by recently reported measurements on  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  with  $x = 1/8$ . While pair-density-wave order is not common as a cuprate ground state, it provides a basis for understanding the uniform  $d$ -wave order that is more typical in superconducting cuprates.

**Keywords:** superconductivity; cuprates; stripe order; pair-density-wave order



**Citation:** Tranquada, J.M. Topological Doping and Superconductivity in Cuprates. *Symmetry* **2021**, *13*, 2365. <https://doi.org/10.3390/sym13122365>

Academic Editors: Serguei Brazovskii and Natasha Kirova

Received: 30 October 2021

Accepted: 3 December 2021

Published: 8 December 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

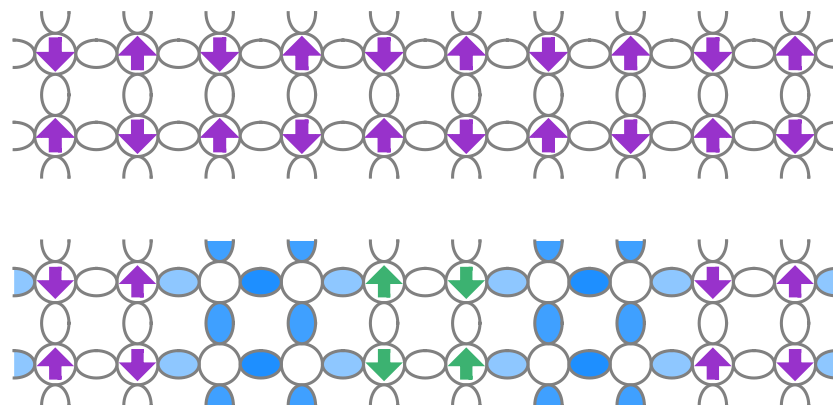
## 1. Introduction

Charge order has now been observed in virtually all hole-doped cuprate superconductor families [1–3]. In 214 cuprates, such as  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  (LSCO) and  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  (LBCO), the charge-stripe order is generally accompanied by spin-stripe order [4–8], as originally observed in Nd-doped  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  [9,10]; each of these orders breaks the translation symmetry of the square-lattice  $\text{CuO}_2$  planes. In a 1996 paper, Kivelson and Emery [11] pointed out the topological character of the combined spin and charge stripe orders. This corresponds to the fact that the period of the spin-stripe order is twice that of the charge-stripe order, as the antiferromagnetic phase flips by  $\pi$  across each charge stripe, as illustrated in Figure 1.

The topological character of stripes in cuprates is distinct from that of the topological insulators that have dominated attention more recently [12,13]. In the latter case, the focus is on Bloch states in which spin-orbit effects play a special role. In cuprates, in contrast, the effects of strong onsite Coulomb repulsion among Cu  $3d$  electrons tend to make Bloch states of questionable relevance. In a parent compound, such as  $\text{La}_2\text{CuO}_4$ , one has a single unpaired Cu  $3d_{x^2-y^2}$  electron on each Cu atom that acts as a local moment, with neighboring moments coupled antiferromagnetically by superexchange  $J$ , a local interaction. While the electronic band gap has a charge-transfer character due to O  $2p$  states that lie between the lower and upper Hubbard bands associated with the Cu  $3d_{x^2-y^2}$  orbital, it is the locally antiferromagnetic (AF) environment that limits the motion of doped holes.

It has taken quite some time to appreciate the significance of the topological order associated with spin stripes. Experimentally, the same antiphase relationship of spin stripes seen in superconducting cuprates also occurs in the case of insulating behavior in  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  [14] and in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  with  $0.02 \lesssim x \lesssim 0.05$  [15] (where the stripes run diagonally with respect to the Ni–O or Cu–O bonds). A theoretical analysis of interaction requirements for topological doping came to no firm conclusions [16]. Antiphase spin

stripes have been obtained from many different approaches: from Hartree–Fock calculations on the Hubbard model [17], from effective models that include long-range Coulomb interactions [18], and from advanced variational and quantum Monte Carlo evaluations of the  $t$ - $J$  [19] or Hubbard model [20,21].



**Figure 1.** Upper panel indicates the antiferromagnetic order of the undoped  $\text{CuO}_2$  planes, with spin direction (arrows) indicated on Cu atoms (circles), separated by O atoms (ellipses). Lower panel shows the spin configuration in the stripe-ordered phase at a doped-hole concentration of  $p = 1/8$ , with doped hole density indicated by blue shading; antiphase spin stripe indicated in green.

I argued recently [22] that the key feature of topological doping is that the spin degrees of freedom within the charge stripes feel a geometric frustration of their interactions with the neighboring spin stripes. This allows the charge stripes to develop quasi-one-dimensional spin correlations. In the case of cuprates with bond-parallel stripes, the charge stripes may be viewed as hole-doped, two-leg, spin  $S = 1/2$  ladders, which are established to have strong superconducting correlations [23,24]. This is a variation on the original proposal of superconducting charge stripes by Emery, Kivelson, and Zachar [25], who pointed out that a spin gap in a one-dimensional (1D) electron gas acts as a pairing amplitude; the difference is that they assumed that the spin gap would be transferred from the neighboring spin stripes, in which case one would never achieve superconductivity when the spin-stripe order is present. The advantage of the doped two-leg spin ladder is that it comes with its own spin gap.

To obtain superconducting order in the  $\text{CuO}_2$  planes, it is necessary to establish phase coherence, via Josephson coupling, between neighboring charge stripes [25]. Because of the conflict between local AF order and hole motion, this needs to be antiphase superconducting order, resulting in a pair-density-wave (PDW) state [26,27]. PDW order was initially proposed [28,29] to explain the experimental observation of two-dimensional superconductivity in  $\text{CuO}_2$  layers [30], with frustration of the usual Josephson coupling between planes [31].

While the initial case for PDW order was circumstantial, direct phase-sensitive evidence of PDW order in  $\text{LBCO } x = 1/8$  has now been reported [32]. This result is consistent with measurements of the Hall effect in high magnetic fields along the  $c$ -axis that suggest that the holes in the charge stripes remain paired, even in the absence of superconducting order [33]. Hence, there is now a solid case that charge stripes in cuprates are essential to pairing.

Of course, the superconducting ground state of most cuprates is the spatially uniform  $d$  wave, not PDW. This is still compatible with pairing correlations developing within charge stripes, but it requires disordered spin stripes with an energy gap [22,34]; uniform phase coherence can only be achieved at energies below the spin gap. The antiphase spin stripes play a critical role for the superconducting order: they either need to be ordered to allow the PDW phase order to be established, or gapped to enable a spatially uniform superconducting order. As a consequence, uniform superconductivity will not coexist with

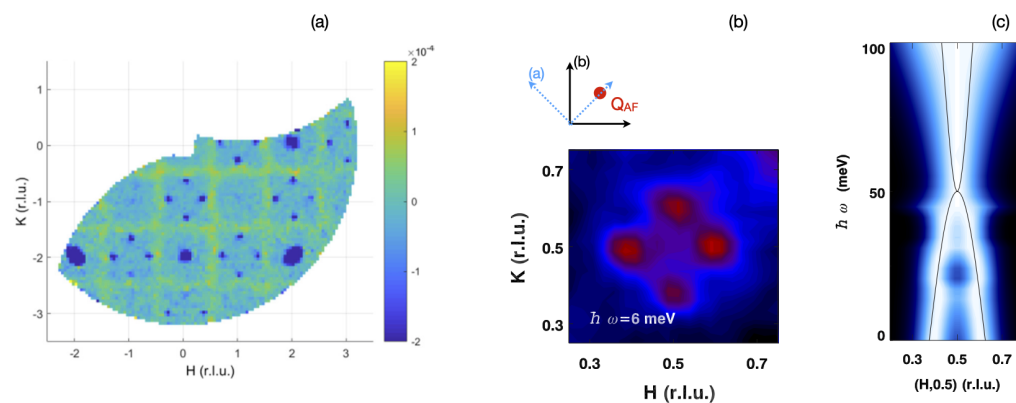
a PDW ground state. On the other hand, defects that require the superconducting order parameter to locally go to zero can favor the local PDW order without a spin order, as seen in studies by scanning tunneling spectroscopy [35,36].

In the following, I fill in details that provide support for the story laid out above.

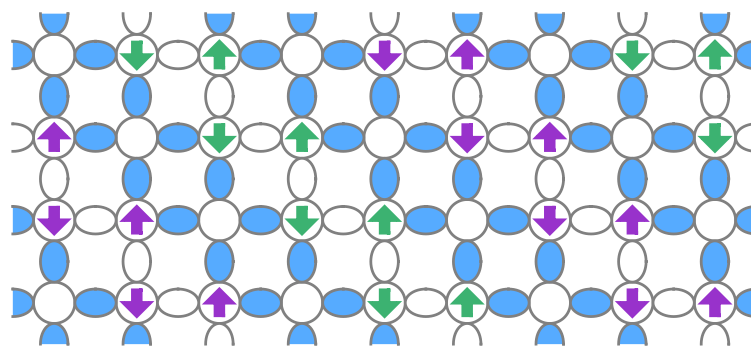
## 2. Stripe Order and Decoupling of Spin Excitations

The holes doped into the  $\text{CuO}_2$  planes tend to go into O  $2p$  states [37]. As pointed out by Emery and Reiter [38], if one could localize a single hole, it would cause the neighboring Cu moments to be parallel, which frustrates the AF order of the undoped system. In fact, it takes very few holes to kill the AF order. In LSCO, the commensurate AF order is gone by  $p = 0.02$ , and even before that, one has phase separation at low temperature [39]. This transition occurs at a hole density that is 20 times smaller than the limit for percolation due to substitution of nonmagnetic ions, as verified in LSCO with nonmagnetic Zn and Mg substitution for Cu [40].

Initially, the holes form diagonal stripes [15] and the system is insulating. This is similar to  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  (LSNO) and  $\text{La}_2\text{NiO}_{4+\delta}$  [41–43]. The case of LSNO with  $x = 1/3$  is of particular interest. Neutron scattering measurements of magnetic scattering are presented in Figure 2a; these can be understood in terms of the stripe order illustrated in Figure 3. Note that in contrast to the spin  $S = 1/2$  of  $\text{Cu}^{2+}$ , the  $\text{Ni}^{2+}$  sites have  $S = 1$ . The Ni moments on the spin stripes order [44] and exhibit well-defined spin waves [45,46]. Within the charge stripes, there is one hole per Ni site; a low-spin hybridization is expected to leave a net  $S = 1/2$  per Ni site along a charge stripe. The interaction of each such moment with the neighboring spin stripes is geometrically frustrated. It is still possible for the reduced Ni moments to couple antiferromagnetically along a charge stripe. For such a decoupled 1D spin chain, one would expect to see no order but spin excitations that disperse only along the stripe direction. Just such 1D spin excitations were first identified by Boothroyd et al. [47]; the role of the decoupling of interactions due to such site-centered charge stripes between antiphase spin stripes was recognized and confirmed in [48].



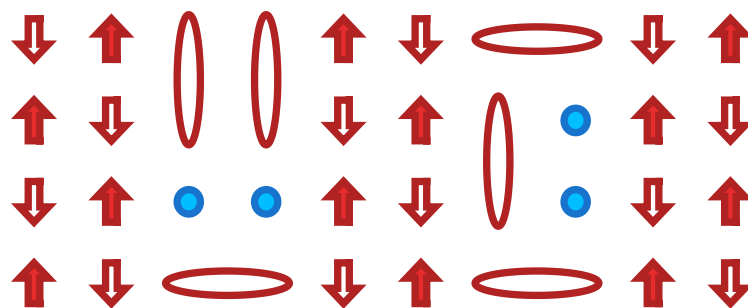
**Figure 2.** (a) Difference in neutron scattering intensity measured at 5 K and 70 K for  $\hbar\omega = 3 \pm 1$  meV in  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  with  $x = 1/3$ . Dark blue points at positions of the type  $(1 \pm \frac{1}{3}, 0, 0)$  and  $(1, \pm \frac{1}{3}, 0)$  correspond to spin waves associated with the spin stripe order, where the AF wave vector,  $\mathbf{Q}_{\text{AF}}$ , is  $(1, 0, 0)$ . Yellow lines correspond to cuts through 2D planes of scattering from 1D spin correlations in charge stripes. Note that twinning causes the measurement to include scattering from stripe domains rotated by  $90^\circ$ . Reprinted with permission from [48], © (2019) by the American Physical Society. (b) Neutron scattering intensity at  $\hbar\omega = 6$  meV and  $T = 10$  K for  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  with  $x = 1/8$ . Here,  $\mathbf{Q}_{\text{AF}} = (0.5, 0.5)$ ; inset shows relative orientations of axes in (a,b). (c) Fitted dispersion and  $Q$  widths of magnetic scattering in LBCO  $x = 1/8$  at 10 K along  $\mathbf{Q} = (H, 0.5)$ . Black line shows the hourglass dispersion often applied to such data. (b,c) Reprinted with permission from [49], ©(2007) by the American Physical Society.



**Figure 3.** Diagonal stripe order as observed in  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  with  $x = 1/3$ . Arrows indicate relative spin orientations on Ni sites (circles), with color change indicating antiphase domains. Blue shading indicates distribution of doped holes on O sites (ellipses).

In 214 cuprates, the stripe orientation rotates from diagonal to bond-parallel, and superconductivity appears for  $x \gtrsim 0.05$  [5,10,15]. The stripe order is stabilized by coupling to lattice anisotropy, with the strongest stripe order correlated with a strong suppression of three-dimensional superconducting order at  $x \approx 1/8$  [4,9]. The static spin order and the low-energy magnetic excitations correspond to the antiphase spin-stripe domains of Figure 1; an example is shown in Figure 2b. The absence of any other low-energy magnetic excitations indicates that the spin degrees of freedom on the charge stripes are gapped. The size of the gap at  $\mathbf{Q}_{\text{AF}}$ , apparent in Figure 2c, is  $\sim 50$  meV, above which commensurate AF excitations appear [50]; the effective correlation length for the high-energy excitations is only about one lattice spacing [49]. (A two-component picture of the magnetic excitations was also proposed in [51].)

We can reconcile the variations in the magnetic spectra through the model indicated schematically in Figure 4. If the charge stripes are centered on a row of bridging O atoms, then the charge stripes are effectively 2-leg spin ladders that are decoupled from the neighboring spin stripes due to frustration of the AF coupling [22]. An undoped spin ladder is a spin liquid [52], with a spin gap that can be as large as  $J/2$  [23]. The hole concentration in the 2-leg ladder picture of the charge stripes is 25%. With an effective  $J$  of  $\sim 100$  meV [50], the holes form pairs so as to avoid exciting the spins across the large spin gap. As illustrated in Figure 4, the spins can be viewed as forming a resonating-valence-bond (RVB) state of nearest-neighbor singlets. Theoretical analysis indicates that the singlet-triplet excitation energy is essentially the pairing scale for the doped holes, and the pairs have  $d$ -wave-like character [24,53].



**Figure 4.** Cartoon of cuprate spin stripe order at  $p = 1/8$ , with resulting pairing correlations within the charge stripes, as proposed in [22]. Here, only Cu sites are shown. Arrows indicate ordered spins; blue circles are doped holes; ellipses are spin singlets on pairs of Cu sites.

Note that the RVB state of the 2-leg ladder is a gapped spin liquid, in contrast to the gapless quantum spin liquid of Anderson's proposed RVB for the 2D square lattice [54]. It is closer to the short-range RVB of Kivelson, Rokhsar, and Sethna [55], in which a coupling to nearest-neighbor bond-length fluctuations (Peierls mechanism) stabilizes the singlet

correlations. In the stripe case, the charge segregation that enables the doped ladders is stabilized by soft phonons and a lattice distortion [9,56–59].

The distinction between the spin stripes and the doped ladders breaks down for excitations above the pairing scale. Such high-energy excitations can occur anywhere in the plane, and at such energies, the holes are no longer confined to pairs within charge stripes. The strong scattering between spins and holes leads to the short correlation length at high energies.

It is recognized that superconducting and charge-density-wave correlations compete with one another in 1D [60]. Recent calculations on 2-leg ladder models suggest that superconducting correlations survive in a 1-band Hubbard model [61] but not in a 3-band Hubbard model [62]. In the experimental case of interest, we do not have individual ladders; while the spin components are decoupled by the magnetic topology, the holes in neighboring ladders interact by long-range Coulomb repulsion and possibly other effects not considered in the calculations. Furthermore, superconducting order requires Josephson coupling between the charge stripes [25]. So the important question is, what happens regarding superconductivity in experiments?

### 3. PDW Order

In the case of optimal stripe order, LBCO with  $x = 1/8$ , 2D superconducting correlations were observed to set in together with the spin-stripe order at  $\sim 40$  K [30,63]; 2D superconducting order was established through a Berezinskii–Kosterlitz–Thouless transition at 16 K, with 3D superconductivity developing only at  $\sim 5$  K. Related behavior was observed in Nd-doped LSCO with  $x = 0.15$  [64], where the transition to the crystal structure that pins stripe order can be tuned with increasing Nd concentration [65]; measurements of *c*-axis optical conductivity demonstrated the loss of 3D superconductivity as the Nd concentration was tuned through the structural transition [31].

To explain the 2D superconductivity, a novel superconducting state was proposed: pair-density-wave order [28,29]. In the PDW state, the pair wave function oscillates from positive on one charge stripe to negative on the next, passing through zero in the spin stripes. Because the stripe order is pinned to a lattice anisotropy that rotates by  $90^\circ$  on passing from one layer to the next along the *c* axis [66], the interlayer Josephson coupling associated with PDW order should be frustrated.

The PDW order is characterized by a finite wave vector that matches that of the spin stripe order. This finite-momentum of pairs is shared with the concept of superconductivity in a strong uniform magnetic field proposed by Fulde and Farrel [67] and Larkin and Ochinnikov [68]; the difference is the absence of a net magnetic field. (Experimental evidence for a field-induced FFLO state in a layered organic superconductor was reported fairly recently [69,70].) It is also apt to note that there have been other proposals for pairing based on charge-density waves (CDWs) in cuprates. In particular, Castellani, Di Castro, Grilli, and coworkers [71–73] proposed that dynamical CDWs underlie the superconductivity of cuprates. The fluctuating CDWs would provide a pairing interaction between extended quasiparticles as in the general case of bosonic fluctuations near a quantum critical point [74]. From this perspective, static CDW order tends to compete with superconductivity.

The evidence of 2D superconductivity in LBCO, together with the PDW proposal, supported the alternative concept of intertwined order [75]. Here the idea is that the interactions that drive pairing and spin order actually work together, but benefit from spatial segregation. This approach builds on theoretical evidence that static spatial inhomogeneity can enhance pairing [76–78]. The concept of pairing within charge stripes has also had to evolve. The initial proposal for pairing in charge stripes relied on interacting with a spin gap in the neighboring spin stripes [25], which is not consistent with the presence of spin stripe order. The idea of charge stripes as doped 2-leg spin ladders resolves this problem [22].



While the proposed PDW state can explain the 2D superconductivity in LBCO, the story would be more compelling with direct evidence for PDW order in LBCO. Phase-sensitive evidence has now been reported [32]. Yang [79] predicted that one could (at least partially) restore the interlayer Josephson coupling by application of an in-plane magnetic field, and that the maximum effect would occur with the field at  $45^\circ$  to the in-plane Cu–O bonds. This angular dependence of the superconducting critical current density along the  $c$  axis is now confirmed by experiment [32]. Hence, PDW order coexisting with spin stripe order is experimentally verified. The conclusion that the charge stripes are the source of pairing seems unavoidable.

Further evidence of the last conclusion comes from transport measurements in a large magnetic field applied perpendicular to the planes. Such measurements on LBCO  $x = 1/8$  revealed, beyond a reentrant 2D superconducting phase at a field of 20 T, an ultra-quantum metal phase with a very large sheet resistance (twice the quantum of resistance for pairs) that appears to saturate at low temperature [33]. The Hall resistance in this phase, as in the 3D and 2D superconducting phases, is zero within the error bars. (Similar results were obtained for  $\text{La}_{1.7}\text{Eu}_{0.2}\text{Sr}_{0.10}\text{CuO}_4$  and  $\text{La}_{1.48}\text{Nd}_{0.4}\text{Sr}_{0.12}\text{CuO}_4$  [80]). A possible interpretation of the Hall resistance at high field is that the doped holes remain paired, even with the loss of PDW phase coherence between neighboring charge stripes. Theoretically, if one takes the disorder into account, this could be a Bose metal phase [81].

Incoherent stripe correlations are also present in the normal state of LSCO [7,8]. Intriguingly, a study of shot noise in tunnel junctions involving LSCO films found evidence for pairs in the the normal state of underdoped samples [82]. That result is at least compatible with the concept of pair correlations in the charge stripes, even at  $T > T_c$ .

#### 4. PDW vs. Uniform $d$ -Wave Superconductivity

Dynamic topological doping, in the form of incommensurate spin excitations, is a common feature of underdoped cuprates [83]. Spectroscopically, the differences between cuprates with PDW order [84–86] and those with uniform  $d$ -wave superconductivity [87,88] are small, while charge stripes, static or dynamic, are common [1–3]. Hence, it seems quite reasonable to propose that charge stripes are the common pairing centers.

The difference between the PDW and uniform superconducting states is associated with the presence or absence of static spin-stripe order. In the PDW state, the static spin-stripe order is essential for the pair correlations to develop (anti-)phase coherence between neighboring charge stripes; purely fluctuating spin stripes oppose superconducting phase order. We should take a moment to acknowledge that it is surprising that we can have such spin order at all. There is already a large tendency toward spin fluctuations in the undoped  $\text{CuO}_2$  planes [89], while 1D spin chains have no static order. There must be some degree of spin anisotropy present in order for the spin stripes to order. Besides being an open question, this represents a challenge for simulations using the Hubbard or  $t$ - $J$  models, as they lack any term that would tend to induce spin order. As a consequence, attempts to identify PDW order in numerical simulations have generally been unsuccessful [90].

While the spin stripes are good for isolating the doped spin ladders that yield pairing, they stand in the way of spatially uniform superconducting order. If they can be gapped, then it should be possible to develop a uniform superconducting phase among electronic states at energies below the spin gap. Indeed, an analysis of the available experimental results on cuprate families indicates that the energy gap for incommensurate spin excitations is an upper limit for the superconducting gap associated with long-range coherence [34]. Note that the local pairing scale within the charge stripes will be larger than the coherent gap of the uniform order. This is consistent with observations by angle-resolved photoemission [91–93] and Raman scattering [94,95] of antinodal gaps that are much larger than the scale of the coherent gap [22]. On the theory side, recent density-matrix-renormalization-group calculations of the Hubbard model (on a lattice of width 4 or 6 Cu sites, with boundaries joined to form a cylinder) found that a modulation of the hopping between neighboring sites in one direction (around the circumference of the cylinder),

as one might expect for charge stripes without spin order, enhances the superconducting correlations [78].

The close relationship between the PDW and uniform superconducting orders is illustrated by a study of the phase transitions in LBCO  $x = 0.115$  as a function of uniaxial strain [96]. In the absence of strain, bulk susceptibility measurements suggest an onset of 2D superconducting correlations, along with spin-stripe order, near 40 K; however, the spin-stripe order is weaker than at  $x = 1/8$  so that 3D superconducting order develops below  $\sim 12$  K [5]. Application of significant in-plane stress causes the bulk superconducting  $T_c$  to rise to 32 K, while muon-spin-rotation spectra indicate a reduction in the magnetically ordered volume fraction by more than 50%, consistent with a decrease in the volume of spin-stripe order and associated PDW order [96]. While the dominant character of the superconducting state changes under strain, the onset temperature of superconducting coherence changes relatively little.

Another connection is seen through the impact of proton irradiation on LBCO  $x = 1/8$  [97]. Protons create narrow tracks of structural defects, often used as pinning centers for magnetic vortices. In LBCO, moderate proton irradiation resulted in an increase in the bulk  $T_c$ , from 4 K to 6 K, while also reducing the correlation length of the charge stripes. It is difficult to see how the structural disorder induced by the bombardment would directly enhance pairing. Instead, the induced disorder must modify the coherent coupling among the correlated pairs already present.

Lee [98] proposed that PDW order is the dominant order in cuprates and that it explains the pseudogap behavior. While the proposal is interesting, there are a number of issues with it. For one thing, the PDW order in LBCO  $x = 1/8$  sets in at a temperature far below the  $T^*$  crossover temperature associated with pseudogap phenomena. While there are dynamic charge and spin stripes at higher temperatures [4,99], and there could be pairing correlations within those dynamic charge stripes, there is no evidence of coherence of pairs between neighboring charge stripes, which would be essential for a reasonable definition of PDW correlations, let alone PDW order. For another thing, the PDW order as defined in [26] is not generic to most cuprate families. For example, in  $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ , the charge-density-wave order develops together with a gap in the incommensurate spin excitations [100]. As discussed above, the spin gap is compatible with the uniform  $d$ -wave superconductivity that orders at lower temperatures.

Another distinction between different cuprates concerns the magnitude of the wave vector  $Q_{\text{co}}$  for the charge order and its variation with doping. In 214 cuprates,  $Q_{\text{co}}$  grows linearly with hole density  $p$  up to  $p \approx 1/8$ , where it saturates at  $\approx 1/8$  reciprocal lattice unit (rlu) [5,101]. Common contrasting behavior is typified by  $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ , where  $Q_{\text{co}}$  starts at  $\sim 0.34$  rlu for  $p \approx 0.08$ , and then decreases by about  $\sim 10\%$  with doping [100,102]. These distinct doping dependencies raise questions about the relationship between the orders in different compounds.

A new study of the doping and temperature dependence of the charge-stripe order in  $\text{La}_{1.8-x}\text{Eu}_{0.2}\text{Sr}_x\text{CuO}_4$  brings new insight to this issue [103]. At low temperature, where both charge- and spin-stripe orders are observed,  $Q_{\text{co}}$  follows the behavior identified in other 214 systems. With rising temperature, however,  $Q_{\text{co}}$  tends to grow in the disordered regime, especially at smaller hole density. A physically-inspired Landau–Ginzburg model, when fitted to the temperature-dependent  $Q_{\text{co}}$  measurements, provides an extrapolation that, at  $T \sim 400$  K, shows behavior of  $Q_{\text{co}}$  vs.  $p$  very similar to that found in YBCO [100,102]. Hence, it is plausible that the charge orders found in various cuprate families have a common origin.

## 5. PDW Around Defects

Scanning tunneling microscopy (STM) experiments have provided evidence for a local coexistence of PDW and uniform superconducting orders around defects in the superconducting order, such as magnetic vortex cores [35,104]. The detected signature corresponds to an induced charge modulation that results from the superimposed, locally

coexisting orders [26]. The main system studied by STM is  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ , which tends to have a large spin gap [105]. Local, short-range PDW correlations were also detected through spatial modulation of the superconducting gap [36].

The absence of spin-stripe order removes the conflict between PDW and uniform superconducting orders discussed above. At the same time, the PDW order detected by STM appears to be induced by local defects, which is distinct from the PDW ground state detected in LBCO, where spin-stripe order appears to be an intrinsic component. A defect such as a magnetic vortex or a Zn atom substituted for Cu causes the superconducting order to go to zero [106]; that may make PDW order energetically favored in the local environment [107].

This line of reasoning provides an interesting connection to the LBCO system. A crystal with  $x = 0.095$  shows a bulk  $T_c = 32$  K in zero magnetic field, along with weakened stripe order relative to  $x = 1/8$  [5]. Application of a  $c$ -axis magnetic field causes an enhancement of stripe order and a decoupling of the superconducting planes [108,109], presumably due to the dominance of PDW order. The regions of uniform superconductivity should act as pinning centers for the magnetic vortices since one can obtain an energy gain from inducing PDW order there.

If Zn defects act like magnetic vortices in terms of locally favoring PDW order, then enough Zn should, like the magnetic field, cause a decoupling of superconducting planes. Indeed, this effect was confirmed in a crystal of LBCO  $x = 0.095$  with 1% Zn [110]. Similar behavior was observed in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  with  $x = 0.13$  and 1% Fe [111]. One difference between Zn defects and magnetic vortices is that Zn is known to induce pinning of spin-stripe order [112]; however, it may lead to a reduction in the spin-stripe-ordering temperature when introduced to a system that already has strong spin-stripe order [113].

## 6. Relations to Other Superconductors

Topological doping is important in cuprates because it establishes regions of reduced dimensionality where pairing can develop in the presence of repulsive interactions. There is a natural connection with systems in which the lattice is formed from a coupling of lower-dimensional components. One example is alkali-doped  $\text{C}_{60}$  [114], where the dopants provide the charge carriers, while the  $\text{C}_{60}$  molecules provide the interactions. Chakravarty and Kivelson [115] proposed a model in which electrons could minimize repulsive interactions by hopping onto  $\text{C}_{60}$  molecules in pairs. In fact, they made a direct comparison to pairing in a doped 2-leg spin ladder. Another obvious parallel is with organic superconductors [116], where superconductivity tends to occur in proximity to spin-density-wave order [117,118].

The situation is different if we compare with other layered superconductors. In electron-doped cuprates, such as  $\text{Nd}_{2-x}\text{Ce}_x\text{CuO}_4$ , the carriers and the Cu moments do not spatially segregate. As a result, commensurate antiferromagnetic order survives to a higher carrier concentration, and superconductivity appears only when static order disappears [119]. There is a good deal of commensurate inelastic magnetic spectral weight at low energy. If this were a good thing for superconductivity, then one might expect to obtain a very high  $T_c$ ; instead, the highest  $T_c$  is lower than that in essentially all hole-doped families of cuprate superconductors. Neutron scattering studies on electron-doped superconductors show that the low-energy antiferromagnetic excitations become gapped on an energy scale comparable to the superconducting gap [120–122].

**Funding:** Work at Brookhaven is supported by the Office of Basic Energy Sciences, Materials Sciences and Engineering Division, U.S. Department of Energy (DOE) under Contract No. DE-SC0012704.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Conflicts of Interest:** The author declares no conflict of interest.



## References

- Frano, A.; Blanco-Canosa, S.; Keimer, B.; Birgeneau, R.J. Charge ordering in superconducting copper oxides. *J. Phys. Condens. Matter* **2020**, *32*, 374005. [\[CrossRef\]](#) [\[PubMed\]](#)
- Comin, R.; Damascelli, A. Resonant X-ray Scattering Studies of Charge Order in Cuprates. *Annu. Rev. Condens. Matter Phys.* **2016**, *7*, 369–405. [\[CrossRef\]](#)
- Uchida, S. Ubiquitous Charge Order Correlations in High-Temperature Superconducting Cuprates. *J. Phys. Soc. Jpn.* **2021**, *90*, 111001. [\[CrossRef\]](#)
- Fujita, M.; Goka, H.; Yamada, K.; Tranquada, J.M.; Regnault, L.P. Stripe order, depinning, and fluctuations in  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$  and  $\text{La}_{1.875}\text{Ba}_{0.075}\text{Sr}_{0.050}\text{CuO}_4$ . *Phys. Rev. B* **2004**, *70*, 104517. [\[CrossRef\]](#)
- Hücker, M.; van Zimmermann, M.; Gu, G.D.; Xu, Z.J.; Wen, J.S.; Xu, G.; Kang, H.J.; Zheludev, A.; Tranquada, J.M. Stripe order in superconducting  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  ( $0.095 \leq x \leq 0.155$ ). *Phys. Rev. B* **2011**, *83*, 104506. [\[CrossRef\]](#)
- Fujita, M.; Hiraka, H.; Matsuda, M.; Matsuura, M.; Tranquada, J.; Wakimoto, S.; Xu, G.; Yamada, K. Progress in Neutron Scattering Studies of Spin Excitations in High- $T_c$  Cuprates. *J. Phys. Soc. Jpn.* **2012**, *81*, 011007. [\[CrossRef\]](#)
- Wen, J.J.; Huang, H.; Lee, S.J.; Jang, H.; Knight, J.; Lee, Y.S.; Fujita, M.; Suzuki, K.M.; Asano, S.; Kivelson, S.A.; et al. Observation of two types of charge-density-wave orders in superconducting  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ . *Nat. Commun.* **2019**, *10*, 3269. [\[CrossRef\]](#) [\[PubMed\]](#)
- Miao, H.; Fabbri, G.; Koch, R.J.; Mazzone, D.G.; Nelson, C.S.; Acevedo-Estevés, R.; Gu, G.D.; Li, Y.; Yilmaz, T.; Kaznatcheev, K.; et al. Charge density waves in cuprate superconductors beyond the critical doping. *NPJ Quantum Mater.* **2021**, *6*, 31. [\[CrossRef\]](#)
- Tranquada, J.M.; Sternlieb, B.J.; Axe, J.D.; Nakamura, Y.; Uchida, S. Evidence for stripe correlations of spins and holes in copper oxide superconductors. *Nature* **1995**, *375*, 561–563. [\[CrossRef\]](#)
- Ma, Q.; Rule, K.C.; Cronkright, Z.W.; Dragomir, M.; Mitchell, G.; Smith, E.M.; Chi, S.; Kolesnikov, A.I.; Stone, M.B.; Gaulin, B.D. Parallel spin stripes and their coexistence with superconducting ground states at optimal and high doping in  $\text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4$ . *Phys. Rev. Res.* **2021**, *3*, 023151. [\[CrossRef\]](#)
- Kivelson, S.; Emery, V. Topological doping of correlated insulators. *Synth. Met.* **1996**, *80*, 151–158. [\[CrossRef\]](#)
- Hasan, M.Z.; Kane, C.L. Colloquium: Topological Insulators. *Rev. Mod. Phys.* **2010**, *82*, 3045–3067. [\[CrossRef\]](#)
- Qi, X.L.; Zhang, S.C. Topological insulators and superconductors. *Rev. Mod. Phys.* **2011**, *83*, 1057–1110. [\[CrossRef\]](#)
- Tranquada, J.M. Spins, stripes, and superconductivity in hole-doped cuprates. *AIP Conf. Proc.* **2013**, *1550*, 114–187. [\[CrossRef\]](#)
- Fujita, M.; Yamada, K.; Hiraka, H.; Gehring, P.M.; Lee, S.H.; Wakimoto, S.; Shirane, G. Static magnetic correlations near the insulating-superconducting phase boundary in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ . *Phys. Rev. B* **2002**, *65*, 064505. [\[CrossRef\]](#)
- Pryadko, L.P.; Kivelson, S.A.; Emery, V.J.; Bazaliy, Y.B.; Demler, E.A. Topological doping and the stability of stripe phases. *Phys. Rev. B* **1999**, *60*, 7541–7557. [\[CrossRef\]](#)
- Zaanen, J.; Gunnarsson, O. Charged magnetic domain lines and the magnetism of high- $T_c$  oxides. *Phys. Rev. B* **1989**, *40*, 7391. [\[CrossRef\]](#)
- Löw, U.; Emery, V.J.; Fabricius, K.; Kivelson, S.A. Study of an Ising model with competing long- and short-range interactions. *Phys. Rev. Lett.* **1994**, *72*, 1918–1921. [\[CrossRef\]](#)
- White, S.R.; Scalapino, D.J. Density Matrix Renormalization Group Study of the Striped Phase in the 2D  $t - J$  Model. *Phys. Rev. Lett.* **1998**, *80*, 1272–1275. [\[CrossRef\]](#)
- Zheng, B.X.; Chung, C.M.; Corboz, P.; Ehlers, G.; Qin, M.P.; Noack, R.M.; Shi, H.; White, S.R.; Zhang, S.; Chan, G.K.L. Stripe order in the underdoped region of the two-dimensional Hubbard model. *Science* **2017**, *358*, 1155–1160. [\[CrossRef\]](#) [\[PubMed\]](#)
- Huang, E.W.; Mendl, C.B.; Jiang, H.C.; Moritz, B.; Devereaux, T.P. Stripe order from the perspective of the Hubbard model. *NPJ Quantum Mater.* **2018**, *3*, 22. [\[CrossRef\]](#)
- Tranquada, J.M. Cuprate superconductors as viewed through a striped lens. *Adv. Phys.* **2020**, *69*, 437–509. [\[CrossRef\]](#)
- Dagotto, E.; Rice, T.M. Surprises on the Way from One- to Two-Dimensional Quantum Magnets: The Ladder Materials. *Science* **1996**, *271*, 618–623. [\[CrossRef\]](#)
- Tsvetlik, A.M. Field theory for a fermionic ladder with generic intrachain interactions. *Phys. Rev. B* **2011**, *83*, 104405. [\[CrossRef\]](#)
- Emery, V.J.; Kivelson, S.A.; Zachar, O. Spin-gap proximity effect mechanism of high-temperature superconductivity. *Phys. Rev. B* **1997**, *56*, 6120–6147. [\[CrossRef\]](#)
- Berg, E.; Fradkin, E.; Kivelson, S.A.; Tranquada, J.M. Striped superconductors: how spin, charge and superconducting orders intertwine in the cuprates. *New J. Phys.* **2009**, *11*, 115004. [\[CrossRef\]](#)
- Agterberg, D.F.; Davis, J.S.; Edkins, S.D.; Fradkin, E.; Van Harlingen, D.J.; Kivelson, S.A.; Lee, P.A.; Radzihovsky, L.; Tranquada, J.M.; Wang, Y. The Physics of Pair-Density Waves: Cuprate Superconductors and Beyond. *Annu. Rev. Condens. Matter Phys.* **2020**, *11*, 231–270. [\[CrossRef\]](#)
- Himeda, A.; Kato, T.; Ogata, M. Stripe States with Spatially Oscillating  $d$ -Wave Superconductivity in the Two-Dimensional  $t - t' - J$  Model. *Phys. Rev. Lett.* **2002**, *88*, 117001. [\[CrossRef\]](#)
- Berg, E.; Fradkin, E.; Kim, E.A.; Kivelson, S.A.; Oganessian, V.; Tranquada, J.M.; Zhang, S.C. Dynamical Layer Decoupling in a Stripe-Ordered High- $T_c$  Superconductor. *Phys. Rev. Lett.* **2007**, *99*, 127003. [\[CrossRef\]](#) [\[PubMed\]](#)
- Li, Q.; Hücker, M.; Gu, G.D.; Tsvetlik, A.M.; Tranquada, J.M. Two-Dimensional Superconducting Fluctuations in Stripe-Ordered  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$ . *Phys. Rev. Lett.* **2007**, *99*, 067001. [\[CrossRef\]](#) [\[PubMed\]](#)
- Tajima, S.; Noda, T.; Eisaki, H.; Uchida, S.  $c$ -Axis Optical Response in the Static Stripe Ordered Phase of the Cuprates. *Phys. Rev. Lett.* **2001**, *86*, 500–503. [\[CrossRef\]](#) [\[PubMed\]](#)

32. Lozano, P.M.; Ren, T.; Gu, G.D.; Tsvetlik, A.M.; Tranquada, J.M.; Li, Q. Phase-sensitive evidence of pair-density-wave order in a cuprate. *arXiv* **2021**, arXiv:2110.05513.
33. Li, Y.; Terzic, J.; Baity, P.G.; Popović, D.; Gu, G.D.; Li, Q.; Tsvetlik, A.M.; Tranquada, J.M. Tuning from failed superconductor to failed insulator with magnetic field. *Sci. Adv.* **2019**, *5*, eaav7686. [\[CrossRef\]](#) [\[PubMed\]](#)
34. Li, Y.; Zhong, R.; Stone, M.B.; Kolesnikov, A.I.; Gu, G.D.; Zaliznyak, I.A.; Tranquada, J.M. Low-energy antiferromagnetic spin fluctuations limit the coherent superconducting gap in cuprates. *Phys. Rev. B* **2018**, *98*, 224508. [\[CrossRef\]](#)
35. Edkins, S.D.; Kostin, A.; Fujita, K.; Mackenzie, A.P.; Eisaki, H.; Uchida, S.; Sachdev, S.; Lawler, M.J.; Kim, E.A.; Séamus Davis, J.C.; et al. Magnetic field-induced pair density wave state in the cuprate vortex halo. *Science* **2019**, *364*, 976–980. [\[CrossRef\]](#) [\[PubMed\]](#)
36. Du, Z.; Li, H.; Joo, S.H.; Donoway, E.P.; Lee, J.; Davis, J.C.S.; Gu, G.; Johnson, P.D.; Fujita, K. Imaging the energy gap modulations of the cuprate pair-density-wave state. *Nature* **2020**, *580*, 65–70. [\[CrossRef\]](#)
37. Chen, C.T.; Sette, F.; Ma, Y.; Hybertsen, M.S.; Stechel, E.B.; Foulkes, W.M.C.; Schuller, M.; Cheong, S.W.; Cooper, A.S.; Rupp, L.W.; et al. Electronic states in  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_{4+\delta}$  probed by soft-X-ray absorption. *Phys. Rev. Lett.* **1991**, *66*, 104–107. [\[CrossRef\]](#)
38. Emery, V.J.; Reiter, G. Mechanism for high-temperature superconductivity. *Phys. Rev. B* **1988**, *38*, 4547–4556. [\[CrossRef\]](#)
39. Matsuda, M.; Fujita, M.; Yamada, K.; Birgeneau, R.J.; Endoh, Y.; Shirane, G. Electronic phase separation in lightly doped  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ . *Phys. Rev. B* **2002**, *65*, 134515. [\[CrossRef\]](#)
40. Vajk, O.P.; Mang, P.K.; Greven, M.; Gehring, P.M.; Lynn, J.W. Quantum Impurities in the Two-Dimensional Spin One-Half Heisenberg Antiferromagnet. *Science* **2002**, *295*, 1691–1695. [\[CrossRef\]](#)
41. Tranquada, J.M.; Buttrey, D.J.; Sachan, V.; Lorenzo, J.E. Simultaneous Ordering of Holes and Spins in  $\text{La}_2\text{NiO}_{4.125}$ . *Phys. Rev. Lett.* **1994**, *73*, 1003–1006. [\[CrossRef\]](#) [\[PubMed\]](#)
42. Yoshizawa, H.; Kakeshita, T.; Kajimoto, R.; Tanabe, T.; Katsufuji, T.; Tokura, Y. Stripe order at low temperatures in  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  with  $0.289 \lesssim x \lesssim 0.5$ . *Phys. Rev. B* **2000**, *61*, R854–R857. [\[CrossRef\]](#)
43. Ulbrich, H.; Braden, M. Neutron scattering studies on stripe phases in non-cuprate materials. *Phys. C Supercond.* **2012**, *481*, 31–45. [\[CrossRef\]](#)
44. Lee, S.H.; Cheong, S.W. Melting of Quasi-Two-Dimensional Charge Stripes in  $\text{La}_{5/3}\text{Sr}_{1/3}\text{NiO}_4$ . *Phys. Rev. Lett.* **1997**, *79*, 2514–2517. [\[CrossRef\]](#)
45. Boothroyd, A.T.; Prabhakaran, D.; Freeman, P.G.; Lister, S.J.S.; Enderle, M.; Hiess, A.; Kulda, J. Spin dynamics in stripe-ordered  $\text{La}_{5/3}\text{Sr}_{1/3}\text{NiO}_4$ . *Phys. Rev. B* **2003**, *67*, 100407(R). [\[CrossRef\]](#)
46. Woo, H.; Boothroyd, A.T.; Nakajima, K.; Perring, T.G.; Frost, C.D.; Freeman, P.G.; Prabhakaran, D.; Yamada, K.; Tranquada, J.M. Mapping spin-wave dispersions in stripe-ordered  $\text{La}_{2-x}\text{Sr}_x\text{NiO}_4$  ( $x = 0.275, 0.333$ ). *Phys. Rev. B* **2005**, *72*, 064437. [\[CrossRef\]](#)
47. Boothroyd, A.T.; Freeman, P.G.; Prabhakaran, D.; Hiess, A.; Enderle, M.; Kulda, J.; Altorfer, F. Spin Correlations among the Charge Carriers in an Ordered Stripe Phase. *Phys. Rev. Lett.* **2003**, *91*, 257201. [\[CrossRef\]](#)
48. Merritt, A.M.; Reznik, D.; Garlea, V.O.; Gu, G.D.; Tranquada, J.M. Nature and impact of stripe freezing in  $\text{La}_{1.67}\text{Sr}_{0.33}\text{NiO}_4$ . *Phys. Rev. B* **2019**, *100*, 195122. [\[CrossRef\]](#)
49. Xu, G.; Tranquada, J.M.; Perring, T.G.; Gu, G.D.; Fujita, M.; Yamada, K. High-energy magnetic excitations from dynamic stripes in  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$ . *Phys. Rev. B* **2007**, *76*, 014508. [\[CrossRef\]](#)
50. Tranquada, J.M.; Woo, H.; Perring, T.G.; Goka, H.; Gu, G.D.; Xu, G.; Fujita, M.; Yamada, K. Quantum magnetic excitations from stripes in copper oxide superconductors. *Nature* **2004**, *429*, 534–538. [\[CrossRef\]](#) [\[PubMed\]](#)
51. Sato, K.; Ikeuchi, K.; Kajimoto, R.; Wakimoto, S.; Arai, M.; Fujita, M. Coexistence of Two Components in Magnetic Excitations of  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$  ( $x = 0.10$  and  $0.16$ ). *J. Phys. Soc. Jpn.* **2020**, *89*, 114703. [\[CrossRef\]](#)
52. Barnes, T.; Dagotto, E.; Riera, J.; Swanson, E.S. Excitation spectrum of Heisenberg spin ladders. *Phys. Rev. B* **1993**, *47*, 3196–3203. [\[CrossRef\]](#)
53. Poilblanc, D.; Scalapino, D.J.; Capponi, S. Superconducting Gap for a Two-Leg  $t-J$  Ladder. *Phys. Rev. Lett.* **2003**, *91*, 137203. [\[CrossRef\]](#) [\[PubMed\]](#)
54. Anderson, P.W. The Resonating Valence Bond State in  $\text{La}_2\text{CuO}_4$  and Superconductivity. *Science* **1987**, *235*, 1196–1198. [\[CrossRef\]](#)
55. Kivelson, S.A.; Rokhsar, D.S.; Sethna, J.P. Topology of the resonating valence-bond state: Solitons and high- $T_c$  superconductivity. *Phys. Rev. B* **1987**, *35*, 8865(R). [\[CrossRef\]](#) [\[PubMed\]](#)
56. Reznik, D.; Pintschovius, L.; Ito, M.; Iikubo, S.; Sato, M.; Goka, H.; Fujita, M.; Yamada, K.; Gu, G.D.; Tranquada, J.M. Electron-phonon coupling reflecting dynamic charge inhomogeneity in copper-oxide superconductors. *Nature* **2006**, *440*, 1170. [\[CrossRef\]](#)
57. Miao, H.; Ishikawa, D.; Heid, R.; Le Tacon, M.; Fabbri, G.; Meyers, D.; Gu, G.D.; Baron, A.Q.R.; Dean, M.P.M. Incommensurate Phonon Anomaly and the Nature of Charge Density Waves in Cuprates. *Phys. Rev. X* **2018**, *8*, 011008. [\[CrossRef\]](#)
58. Peng, Y.Y.; Husain, A.A.; Mitran, M.; Sun, S.X.L.; Johnson, T.A.; Zakrzewski, A.V.; MacDougall, G.J.; Barbour, A.; Jarrige, I.; Bisogni, V.; et al. Enhanced Electron-Phonon Coupling for Charge-Density-Wave Formation in  $\text{La}_{1.8-x}\text{Eu}_{0.2}\text{Sr}_x\text{CuO}_{4+\delta}$ . *Phys. Rev. Lett.* **2020**, *125*, 097002. [\[CrossRef\]](#) [\[PubMed\]](#)
59. Wang, Q.; von Arx, K.; Horio, M.; Mukkattukavil, D.J.; Küspert, J.; Sassa, Y.; Schmitt, T.; Nag, A.; Pyon, S.; Takayama, T.; et al. Charge order lock-in by electron-phonon coupling in  $\text{La}_{1.675}\text{Eu}_{0.2}\text{Sr}_{0.125}\text{CuO}_4$ . *Sci. Adv.* **2021**, *7*, eabg7394. [\[CrossRef\]](#)
60. Kivelson, S.A.; Fradkin, E.; Emery, V.J. Electronic liquid-crystal phases of a doped Mott insulator. *Nature* **1998**, *393*, 550–553. [\[CrossRef\]](#)
61. Dolfi, M.; Bauer, B.; Keller, S.; Troyer, M. Pair correlations in doped Hubbard ladders. *Phys. Rev. B* **2015**, *92*, 195139. [\[CrossRef\]](#)

62. Song, J.P.; Mazumdar, S.; Clay, R.T. Absence of Luther-Emery superconducting phase in the three-band model for cuprate ladders. *Phys. Rev. B* **2021**, *104*, 104504. [\[CrossRef\]](#)
63. Tranquada, J.M.; Gu, G.D.; Hücker, M.; Jie, Q.; Kang, H.J.; Klingeler, R.; Li, Q.; Tristan, N.; Wen, J.S.; Xu, G.Y.; et al. Evidence for unusual superconducting correlations coexisting with stripe order in  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$ . *Phys. Rev. B* **2008**, *78*, 174529. [\[CrossRef\]](#)
64. Ding, J.F.; Xiang, X.Q.; Zhang, Y.Q.; Liu, H.; Li, X.G. Two-dimensional superconductivity in stripe-ordered  $\text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4$  single crystals. *Phys. Rev. B* **2008**, *77*, 214524. [\[CrossRef\]](#)
65. Büchner, B.; Breuer, M.; Freimuth, A.; Kampf, A.P. Critical Buckling for the Disappearance of Superconductivity in Rare-Earth-Doped  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ . *Phys. Rev. Lett.* **1994**, *73*, 1841–1844. [\[CrossRef\]](#) [\[PubMed\]](#)
66. Axe, J.D.; Crawford, M.K. Structural instabilities in lanthanum cuprate superconductors. *J. Low Temp. Phys.* **1994**, *95*, 271–284. [\[CrossRef\]](#)
67. Fulde, P.; Ferrell, R.A. Superconductivity in a Strong Spin-Exchange Field. *Phys. Rev.* **1964**, *135*, A550–A563. [\[CrossRef\]](#)
68. Larkin, A.I.; Ovchinnikov, Y.N. Nonuniform State of Superconductors. *Zh. Eksp. Teor. Fiz.* **1964**, *47*, 1136.
69. Koutoulakis, G.; Kühne, H.; Schlueter, J.A.; Wosnitzer, J.; Brown, S.E. Microscopic Study of the Fulde-Ferrell-Larkin-Ovchinnikov State in an All-Organic Superconductor. *Phys. Rev. Lett.* **2016**, *116*, 067003. [\[CrossRef\]](#)
70. Sugiura, S.; Isono, T.; Terashima, T.; Yasuzuka, S.; Schlueter, J.A.; Uji, S. Fulde-Ferrell-Larkin-Ovchinnikov and vortex phases in a layered organic superconductor. *NPJ Quantum Mater.* **2019**, *4*, 7. [\[CrossRef\]](#)
71. Castellani, C.; Di Castro, C.; Grilli, M. Singular Quasiparticle Scattering in the Proximity of Charge Instabilities. *Phys. Rev. Lett.* **1995**, *75*, 4650–4653. [\[CrossRef\]](#) [\[PubMed\]](#)
72. Grilli, M.; Seibold, G.; Di Ciolo, A.; Lorenzana, J. Fermi surface dichotomy in systems with fluctuating order. *Phys. Rev. B* **2009**, *79*, 125111. [\[CrossRef\]](#)
73. Caprara, S.; Di Castro, C.; Seibold, G.; Grilli, M. Dynamical charge density waves rule the phase diagram of cuprates. *Phys. Rev. B* **2017**, *95*, 224511. [\[CrossRef\]](#)
74. Wang, Y.; Abanov, A.; Altshuler, B.L.; Yuzbashyan, E.A.; Chubukov, A.V. Superconductivity near a Quantum-Critical Point: The Special Role of the First Matsubara Frequency. *Phys. Rev. Lett.* **2016**, *117*, 157001. [\[CrossRef\]](#)
75. Fradkin, E.; Kivelson, S.A.; Tranquada, J.M. Colloquium: Theory of intertwined orders in high temperature superconductors. *Rev. Mod. Phys.* **2015**, *87*, 457–482. [\[CrossRef\]](#)
76. Carlson, E.W.; Emery, V.J.; Kivelson, S.A.; Orgad, D. Concepts in High Temperature Superconductivity. In *The Physics of Superconductors Vol II: Superconductivity in Nanostructures, High- $T_c$  and Novel Superconductors, Organic Superconductors*; Bennemann, K.H.; Ketterson, J.B., Eds.; Springer: Berlin, Germany, 2004.
77. Tsai, W.F.; Yao, H.; Läuchli, A.; Kivelson, S.A. Optimal inhomogeneity for superconductivity: Finite-size studies. *Phys. Rev. B* **2008**, *77*, 214502. [\[CrossRef\]](#)
78. Jiang, H.C.; Kivelson, S.A. Stripe order enhanced superconductivity in the Hubbard model. *arXiv* **2021**, arXiv:2105.07048.
79. Yang, K. Detection of Striped Superconductors Using Magnetic Field Modulated Josephson Effect. *J. Supercond. Nov. Magn.* **2013**, *26*, 2741–2742. [\[CrossRef\]](#)
80. Shi, Z.; Baity, P.G.; Terzic, J.; Pokharel, B.K.; Sasagawa, T.; Popović, D. Magnetic field reveals vanishing Hall response in the normal state of stripe-ordered cuprates. *Nat. Commun.* **2021**, *12*, 3724. [\[CrossRef\]](#)
81. Ren, T.; Tselik, A.M. How magnetic field can transform a superconductor into a Bose metal. *New J. Phys.* **2020**, *22*, 103021. [\[CrossRef\]](#)
82. Zhou, P.; Chen, L.; Liu, Y.; Sochnikov, I.; Bollinger, A.T.; Han, M.G.; Zhu, Y.; He, X.; Božović, I.; Natelson, D. Electron pairing in the pseudogap state revealed by shot noise in copper oxide junctions. *Nature* **2019**, *572*, 493–496. [\[CrossRef\]](#) [\[PubMed\]](#)
83. Enoki, M.; Fujita, M.; Nishizaki, T.; Iikubo, S.; Singh, D.K.; Chang, S.; Tranquada, J.M.; Yamada, K. Spin-Stripe Density Varies Linearly With the Hole Content in Single-Layer  $\text{Bi}_{2+x}\text{Sr}_{2-x}\text{CuO}_{6+y}$  Cuprate Superconductors. *Phys. Rev. Lett.* **2013**, *110*, 017004. [\[CrossRef\]](#)
84. Valla, T.; Federov, A.V.; Lee, J.; Davis, J.C.; Gu, G.D. The Ground State of the Pseudogap in Cuprate Superconductors. *Science* **2006**, *314*, 1914. [\[CrossRef\]](#)
85. He, R.H.; Tanaka, K.; Mo, S.K.; Sasagawa, T.; Fujita, M.; Adachi, T.; Mannella, N.; Yamada, K.; Koike, Y.; Hussain, Z.; et al. Energy gaps in the failed high- $T_c$  superconductor  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$ . *Nat. Phys.* **2009**, *5*, 119–123. [\[CrossRef\]](#)
86. Homes, C.C.; Hücker, M.; Li, Q.; Xu, Z.J.; Wen, J.S.; Gu, G.D.; Tranquada, J.M. Determination of the optical properties of  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  for several dopings, including the anomalous  $x = \frac{1}{8}$  phase. *Phys. Rev. B* **2012**, *85*, 134510. [\[CrossRef\]](#)
87. Damascelli, A.; Hussain, Z.; Shen, Z.X. Angle-resolved photoemission studies of the cuprate superconductors. *Rev. Mod. Phys.* **2003**, *75*, 473–541. [\[CrossRef\]](#)
88. Basov, D.N.; Timusk, T. Electrodynamics of high- $T_c$  superconductors. *Rev. Mod. Phys.* **2005**, *77*, 721–779. [\[CrossRef\]](#)
89. Chakravarty, S.; Halperin, B.I.; Nelson, D.R. Low-temperature behavior of two-dimensional quantum antiferromagnets. *Phys. Rev. Lett.* **1988**, *60*, 1057–1060. [\[CrossRef\]](#) [\[PubMed\]](#)
90. White, S.R.; Scalapino, D.J. Doping asymmetry and striping in a three-orbital  $\text{CuO}_2$  Hubbard model. *Phys. Rev. B* **2015**, *92*, 205112. [\[CrossRef\]](#)
91. Lee, W.S.; Vishik, I.M.; Tanaka, K.; Lu, D.H.; Sasagawa, T.; Nagaosa, N.; Devereaux, T.P.; Hussain, Z.; Shen, Z.X. Abrupt onset of a second energy gap at the superconducting transition of underdoped  $\text{Bi}2212$ . *Nature* **2007**, *450*, 81–84. [\[CrossRef\]](#) [\[PubMed\]](#)



92. Yoshida, M.; Ishii, K.; Naka, M.; Ishihara, S.; Jarrige, I.; Ikeuchi, K.; Murakami, Y.; Kudo, K.; Koike, Y.; Nagata, T.; et al. Observation of momentum-resolved charge fluctuations proximate to the charge-order phase using resonant inelastic X-ray scattering. *Sci. Rep.* **2016**, *6*, 23611. [\[CrossRef\]](#) [\[PubMed\]](#)
93. Drozdov, I.K.; Pletikosić, I.; Kim, C.K.; Fujita, K.; Gu, G.D.; Davis, J.C.S.; Johnson, P.D.; Božović, I.; Valla, T. Phase diagram of  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$  revisited. *Nat. Commun.* **2018**, *9*, 5210. [\[CrossRef\]](#)
94. Munnikes, N.; Muschler, B.; Venturini, F.; Tassini, L.; Prestel, W.; Ono, S.; Ando, Y.; Peets, D.C.; Hardy, W.N.; Liang, R.; et al. Pair breaking versus symmetry breaking: Origin of the Raman modes in superconducting cuprates. *Phys. Rev. B* **2011**, *84*, 144523. [\[CrossRef\]](#)
95. Sacuto, A.; Gallais, Y.; Cazayous, M.; Méasson, M.A.; Gu, G.D.; Colson, D. New insights into the phase diagram of the copper oxide superconductors from electronic Raman scattering. *Rep. Prog. Phys.* **2013**, *76*, 022502. [\[CrossRef\]](#)
96. Guguchia, Z.; Das, D.; Wang, C.N.; Adachi, T.; Kitajima, N.; Elender, M.; Brückner, F.; Ghosh, S.; Grinenko, V.; Shiroka, T.; et al. Using Uniaxial Stress to Probe the Relationship between Competing Superconducting States in a Cuprate with Spin-stripe Order. *Phys. Rev. Lett.* **2020**, *125*, 097005. [\[CrossRef\]](#) [\[PubMed\]](#)
97. Leroux, M.; Mishra, V.; Ruff, J.P.C.; Claus, H.; Smylie, M.P.; Opagiste, C.; Rodière, P.; Kayani, A.; Gu, G.D.; Tranquada, J.M.; et al. Disorder raises the critical temperature of a cuprate superconductor. *Proc. Natl. Acad. Sci. USA* **2019**, *116*, 10691–10697. [\[CrossRef\]](#)
98. Lee, P.A. Amperean Pairing and the Pseudogap Phase of Cuprate Superconductors. *Phys. Rev. X* **2014**, *4*, 031017. [\[CrossRef\]](#)
99. Miao, H.; Lorenzana, J.; Seibold, G.; Peng, Y.Y.; Amorese, A.; Yakhov-Harris, F.; Kummer, K.; Brookes, N.B.; Konik, R.M.; Thampy, V.; et al. High-temperature charge density wave correlations in  $\text{La}_{1.875}\text{Ba}_{0.125}\text{CuO}_4$  without spin-charge locking. *Proc. Natl. Acad. Sci. USA* **2017**, *114*, 12430–12435. [\[CrossRef\]](#) [\[PubMed\]](#)
100. Hücker, M.; Christensen, N.B.; Holmes, A.T.; Blackburn, E.; Forgan, E.M.; Liang, R.; Bonn, D.A.; Hardy, W.N.; Gutowski, O.; Zimmermann, M.V.; et al. Competing charge, spin, and superconducting orders in underdoped  $\text{YBa}_2\text{Cu}_3\text{O}_y$ . *Phys. Rev. B* **2014**, *90*, 054514. [\[CrossRef\]](#)
101. Birgeneau, R.J.; Stock, C.; Tranquada, J.M.; Yamada, K. Magnetic Neutron Scattering in Hole-Doped Cuprate Superconductors. *J. Phys. Soc. Jpn.* **2006**, *75*, 111003. [\[CrossRef\]](#)
102. Blanco-Canosa, S.; Frano, A.; Schierle, E.; Porras, J.; Loew, T.; Minola, M.; Bluschke, M.; Weschke, E.; Keimer, B.; Le Tacon, M. Resonant X-ray scattering study of charge-density wave correlations in  $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ . *Phys. Rev. B* **2014**, *90*, 054513. [\[CrossRef\]](#)
103. Lee, S.; Huang, E.W.; Johnson, T.A.; Guo, X.; Husain, A.A.; Mitrano, M.; Lu, K.; Zakrzewski, A.V.; de la Peña, G.; Peng, Y.; et al. Generic character of charge and spin density waves in superconducting cuprates. *arXiv* **2021**, arXiv:2110.13991.
104. Hoffman, J.E.; Hudson, E.W.; Lang, K.M.; Madhavan, V.; Eisaki, H.; Uchida, S.; Davis, J.C. A Four Unit Cell Periodic Pattern of Quasi-Particle States Surrounding Vortex Cores in  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ . *Science* **2002**, *295*, 466–469. [\[CrossRef\]](#) [\[PubMed\]](#)
105. Xu, G.; Gu, G.D.; Hücker, M.; Fauque, B.; Perring, T.G.; Regnault, L.P.; Tranquada, J.M. Testing the itinerancy of spin dynamics in superconducting  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ . *Nat. Phys.* **2009**, *5*, 642–646. [\[CrossRef\]](#)
106. Nachumi, B.; Keren, A.; Kojima, K.; Larkin, M.; Luke, G.M.; Merrin, J.; Tchernyshöv, O.; Uemura, Y.J.; Ichikawa, N.; Goto, M.; et al. Muon Spin Relaxation Studies of Zn-Substitution Effects in High-  $T_c$  Cuprate Superconductors. *Phys. Rev. Lett.* **1996**, *77*, 5421–5424. [\[CrossRef\]](#) [\[PubMed\]](#)
107. Tranquada, J.M.; Dean, M.P.M.; Li, Q. Superconductivity from Charge Order in Cuprates. *J. Phys. Soc. Jpn.* **2021**, *90*, 111002. [\[CrossRef\]](#)
108. Wen, J.; Jie, Q.; Li, Q.; Hücker, M.V.; Zimmermann, M.; Han, S.J.; Xu, Z.; Singh, D.K.; Konik, R.M.; Zhang, L.; et al. Uniaxial linear resistivity of superconducting  $\text{La}_{1.905}\text{Ba}_{0.095}\text{CuO}_4$  induced by an external magnetic field. *Phys. Rev. B* **2012**, *85*, 134513. [\[CrossRef\]](#)
109. Stegen, Z.; Han, S.J.; Wu, J.; Pramanik, A.K.; Hücker, M.; Gu, G.; Li, Q.; Park, J.H.; Boebinger, G.S.; Tranquada, J.M. Evolution of superconducting correlations within magnetic-field-decoupled  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$  ( $x = 0.095$ ). *Phys. Rev. B* **2013**, *87*, 064509. [\[CrossRef\]](#)
110. Lozano, P.M.; Gu, G.D.; Tranquada, J.M.; Li, Q. Experimental evidence that zinc impurities pin pair-density-wave order in  $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$ . *Phys. Rev. B* **2021**, *103*, L020502. [\[CrossRef\]](#)
111. Huang, H.; Lee, S.J.; Ikeda, Y.; Taniguchi, T.; Takahama, M.; Kao, C.C.; Fujita, M.; Lee, J.S. Two-Dimensional Superconducting Fluctuations Associated with Charge-Density-Wave Stripes in  $\text{La}_{1.87}\text{Sr}_{0.13}\text{Cu}_{0.99}\text{Fe}_{0.01}\text{O}_4$ . *Phys. Rev. Lett.* **2021**, *126*, 167001. [\[CrossRef\]](#)
112. Hirota, K. Neutron scattering studies of Zn-doped  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ . *Physica C* **2001**, *357–360*, 61–68. [\[CrossRef\]](#)
113. Guguchia, Z.; Roessli, B.; Khasanov, R.; Amato, A.; Pomjakushina, E.; Conder, K.; Uemura, Y.J.; Tranquada, J.M.; Keller, H.; Shengelaya, A. Complementary Response of Static Spin-Stripe Order and Superconductivity to Nonmagnetic Impurities in Cuprates. *Phys. Rev. Lett.* **2017**, *119*, 087002. [\[CrossRef\]](#)
114. Ganin, A.Y.; Takabayashi, Y.; Jeglič, P.; Arčon, D.; Potočnik, A.; Baker, P.J.; Ohishi, Y.; McDonald, M.T.; Tzirakis, M.D.; McLennan, A.; et al. Polymorphism control of superconductivity and magnetism in  $\text{Cs}_3\text{C}_{60}$  close to the Mott transition. *Nature* **2010**, *466*, 221–225. [\[CrossRef\]](#)
115. Chakravarty, S.; Kivelson, S.A. Electronic mechanism of superconductivity in the cuprates,  $\text{C}_{60}$ , and polyacenes. *Phys. Rev. B* **2001**, *64*, 064511. [\[CrossRef\]](#)
116. Ardavan, A.; Brown, S.; Kagoshima, S.; Kanoda, K.; Kuroki, K.; Mori, H.; Ogata, M.; Uji, S.; Wosnitza, J. Recent Topics of Organic Superconductors. *J. Phys. Soc. Jpn.* **2012**, *81*, 011004. [\[CrossRef\]](#)

- 
117. Kang, N.; Salameh, B.; Auban-Senzier, P.; Jérôme, D.; Pasquier, C.R.; Brazovskii, S. Domain walls at the spin-density-wave endpoint of the organic superconductor (TMTSF)<sub>2</sub>PF<sub>6</sub> under pressure. *Phys. Rev. B* **2010**, *81*, 100509. [[CrossRef](#)]
  118. Kawasugi, Y.; Seki, K.; Tajima, S.; Pu, J.; Takenobu, T.; Yunoki, S.; Yamamoto, H.M.; Kato, R. Two-dimensional ground-state mapping of a Mott-Hubbard system in a flexible field-effect device. *Sci. Adv.* **2019**, *5*, eaav7282. [[CrossRef](#)]
  119. Motoyama, E.M.; Yu, G.; Vishik, I.M.; Vajk, O.P.; Mang, P.K.; Greven, M. Spin correlations in the electron-doped high-transition-temperature superconductor Nd<sub>2-x</sub>Ce<sub>x</sub>CuO<sub>4±δ</sub>. *Nature* **2007**, *445*, 186–189. [[CrossRef](#)] [[PubMed](#)]
  120. Yamada, K.; Kurahashi, K.; Uefuji, T.; Fujita, M.; Park, S.; Lee, S.H.; Endoh, Y. Commensurate Spin Dynamics in the Superconducting State of an Electron-Doped Cuprate Superconductor. *Phys. Rev. Lett.* **2003**, *90*, 137004. [[CrossRef](#)]
  121. Zhao, Y.; Gaulin, B.D.; Castellan, J.P.; Ruff, J.P.C.; Dunsiger, S.R.; Gu, G.D.; Dabkowska, H.A. High-resolution x-ray scattering studies of structural phase transitions in underdoped La<sub>2-x</sub>Ba<sub>x</sub>CuO<sub>4</sub>. *Phys. Rev. B* **2007**, *76*, 184121. [[CrossRef](#)]
  122. Yu, G.; Li, Y.; Motoyama, E.M.; Zhao, X.; Barišić, N.; Cho, Y.; Bourges, P.; Hradil, K.; Mole, R.A.; Greven, M. Magnetic resonance in the model high-temperature superconductor HgBa<sub>2</sub>CuO<sub>4+δ</sub>. *Phys. Rev. B* **2010**, *81*, 064518. [[CrossRef](#)]