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A Generalized Class of Functions Defined by the q-Difference Operator

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Abstract: The goal of the present investigation is to introduce a new class of analytic functions ($K_{t,q}$), defined in the open unit disk, by means of the q-difference operator, which may have symmetric or assymetric properties, and to establish the relationship between the new defined class and appropriate subordination. We derived relationships of this class and obtained sufficient conditions for an analytic function to be $K_{t,q}$. Finally, in the concluding section, we have taken the decision to restate the clearly-proved fact that any attempt to create the rather simple (p, q)-variations of the results, which we have provided in this paper, will be a rather inconsequential and trivial work, simply because the added parameter p is obviously redundant.

Keywords: analytic functions; close-to-convex functions; Schwartz's lemma; differential subordination; q-derivative

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1. Introduction

The exceptional importance and necessity of introducing new subclasses of analytic functions and investigating their properties, such as convexity, close-to-convexity, distortion properties, and coefficient estimates, are due to their applications in a wide variety of areas, including mathematics, physics, mechanics, electrotechnics, and many others. In geometric function theory, different classes of analytic functions have already been considered and examined through several views; nowadays, new prospects are destined to encourage interest among the mathematicians in this area. A bright interest in geometric properties, such as univalency, starlikeness, convexity, and uniform convexity of several special functions, such as Bessel, hyper-Bessel, Struve, Wright, Lommel, q-Bessel functions, and Mittag-Leffler functions has been noticed in recent years. Toklu et al. studied, in [1], the radii of lemniscate starlikeness, lemniscate convexity, Janowski starlikeness, and Janowski convexity of a certain normalized hyper-Bessel function. In [2], Aktas et al. obtained several original results by extending, in a natural way, the features of classical Bessel functions to hyper-Bessel functions.

The field of q-analysis, in the recent past, has captivated the significant consideration of mathematicians. An exhaustive research, applying q-analysis in function theory, can be accessed in [3]. As a result of this expansion in operator theory, several researchers were inspired, as has been seen in many articles. The q-calculus offers precious instruments that have been comprehensively utilized for the goal of investigating several classes of analytic functions. Several geometric features, such as coefficient estimates, distortion bounds, radii of starlikeness, convexity, and close to convexity, have been considered for such proposed classes of functions. Ismail et al. [4] were the first to use the q-derivative operator D_q to investigate a certain q-analogue of the class of starlike functions in open unit disk. Purohit and Raina [5] have introduced the generalization q-Taylor's formula in fractional q-calculus. Mohammed and Darus [6] designed geometric properties and approximation of q-operators in some subclasses of analytic functions in compact disks.

By utilizing the concept of the conic domain, Kanas and Raducanu [7] applied the fractional q-calculus operators in examinations of certain classes of functions. Ramachandran et al. [8] already used the fractional q-calculus operators in the study of certain bounds for q-starlike and q-convex functions, with respect to symmetric points. Srivastava et al. [9] established several general results, concerning the partial sums of meromorphically starlike functions, defined by means of a certain class of q-derivative operators. Ibrahim et al. [10] established a new q-differential operator in the open unit disk that characterizes the analytic geometric representation of the solution of the widely known Beltrami differential equation in a complex domain. In [11], Nezir et al. have introduced certain subclasses of analytic and univalent functions in the open unit disk defined by q-derivative. and they studied several conditions for an analytic and univalent function belonging to these classes. Through the use of the well-known idea of neighborhoods of analytic functions, Deniz et al. [12] have researched the (j, δ)-neighborhoods of different subclasses of convex and starlike convex functions, defined by the q-Ruscheweyh derivative operator. All these, and many other results, motivate significant further developments on q-calculus and fractional q-calculus in geometric function theory of complex analysis.

The results contained in this study were inspired by the outstanding results previously obtained for certain classes of functions analytic in the open disk, making use of differential operators. In our current investigation, we present a new class of analytic functions, defined in the open unit disk by means of a new q-difference operator, and we obtain some interesting results in this generalized class. The relations with this new class, and the appropriate subordination, is discussed. Among the results investigated for the new introduced class, we derive the relationship of this class and obtain sufficient conditions for an analytic function to belong to this class. The Bieberbach-de Branges theorem for the class $K_{t,q}$ is also given.

Let $\mathcal{A}$ be the class of analytic functions (f), defined on the open unit disk $U = \{z : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. In other words, functions (f) in $\mathcal{A}$ have the power set representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in U\,. \tag{1}$$

Let Ω be the family of functions, which are regular in U and satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for every $z \in U$. Denote, by $\mathcal{P}$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ regular in U and such that $p(z)$ is $\mathcal{P}$ if, and only if, $p(z) \prec \frac{1+z}{1-z} \iff p(z) = \frac{1+\phi(z)}{1-\phi(z)}$, for some function $\phi(z) \in \Omega$ and every $z \in U$ [13].

Principle of subordination (see [14]): If f and g are two analytic functions in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, analytic in the open unit disk U, the convolution product (or Hadamard product) of $f(z)$ and $g(z)$, written as $(f * g)(z)$, is defined by:

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n\,.$$

Let C, K, and S^* signify the common subclass of $\mathcal{A}$, whose members are convex, close-to-convex, and starlike in the open unit disk U. We also denote, by $S^*(\alpha)$, the class of starlike functions of order α, $0 \leq \alpha < 1$. The subclasses of K, C, and S^* were investigated by several researchers (see [15–22]).

Gao and Zhou [16] introduced the class K_s of analytic functions, which is a subclass of the class C. We assert that a function $f \in \mathcal{A}$ is in the class K_s, if there exists a starlike function, $g \in S^*\left(\frac{1}{2}\right)$, for which:

$$\mathrm{Re}\left(\frac{\mathrm{z}^2\mathrm{f}'(\mathrm{z})}{\mathrm{g}(\mathrm{z})\mathrm{g}(-\mathrm{z})}\right) > 0,\ \mathrm{z} \in \mathrm{U}.$$

In [18], Goyal et al. defined and studied a subclass of analytic functions related to starlike functions. If $f \in \mathcal{A}$, we say that $f \in K_s(A, B; u, v)$, if there exists a function $g \in S^*\left(\frac{1}{2}\right)$, such that:

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \frac{1+Az}{1+Bz},\ z \in U,$$

$(\ u, v \in \mathbb{C}^*, |u| \leq 1, |v| \leq 1, 1 \leq B < A \leq 1)$.

Recently, Prajapat [20] introduced and studied a new subclass of analytic functions $\chi_t(\gamma)$. A function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)$, $(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$ if there exists a function $g \in S^*\left(\frac{1}{2}\right)$, for which:

$$\mathrm{Re}\left(\frac{\mathrm{t}\mathrm{z}^2\mathrm{f}'(\mathrm{z})}{\mathrm{g}(\mathrm{z})\mathrm{g}(\mathrm{t}\mathrm{z})}\right) > \text{fl},\ \mathrm{z} \in \mathrm{U}.$$

In the current article, inspired by the work of Goyal et al. [18] and Prajapat [20], we introduce a new class of analytic functions ($K_{t,q}$), by means of a q-analogue of a difference operator acting on analytic functions in the unit disk U, and we obtain several results in this generalized class.

We recall some concepts and notations of q-calculus that will be used throughout this paper.

The theory of q-analogues or q-extensions of classical formulas and functions, based on the observation that:

$$\lim_{q \to 1^-} \frac{1-q^n}{1-q} = n,\ q \in (0,1),\ n \in \mathbb{N}, \tag{2}$$

therefore, the number $\frac{1-q^n}{1-q}$ is sometimes called the basic number ($[n]_q$). The q-factorial ($[n]_q!$) is defined by:

$$[n]_q! = \begin{cases} [n]_q \cdot [n-1]_q \cdots [1]_q, & \text{for } n = 1, 2, \ldots; \\ 1 & \text{, for } n = 0. \end{cases} \tag{3}$$

As $q \to 1^-$, $[n]_q \to n$, and this is the bookmark of a q-analogue: the limit as $q \to 1^-$ recovers the classical object.

In 1908, Jackson introduced the Euler–Jackson q-difference operator.

The q-difference operator $(D_q f)(z)$, acting on functions $f(z) \in \mathcal{A}$, is defined by:

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z},\ z \neq 0, \tag{4}$$

and $(D_q f)(0) = f'(0)$, where $q \in (0,1)$. One can see that $(D_q f)(z) \to f'(z)$ as $q \to 1^-$. The q-difference operator plays a notable place in the theory of hypergeometric series and quantum phisics (see [23,24]).

Therefore, for a function $f(z) = z^n$ the q-derivative is given by

$$(D_q f)(z) = D_q(z^n) = \frac{1-q^n}{1-q} \cdot z^{n-1} = [n]_q z^{n-1}, \tag{5}$$

then $\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} [n]_q z^{n-1} = n z^{n-1} = f'(z)$, where $f'(z)$ is the ordinary derivative.

The difference operator helps us to generalize the class of starlike, convex, and close-to-convex functions analytically. The q-analogues, to the functions classes S^* and $\mathcal{K}$, are given as follows.

A function $f \in \mathcal{A}$ is said to belong to the class S_q^* of q-starlike functions if it satisfies:

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; z \in U. \tag{6}$$

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_q$ of q-close-to-convex functions if there exists a starlike function $g \in S^*$, such that:

$$\left| \frac{z(D_q f)(z)}{g(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; z \in U. \tag{7}$$

We say that $f \in \mathcal{K}_q$, with respect to g.

In [22,25], the authors described and investigated some important properties of functions f in the class $\mathcal{K}_q$. In [26], Y. Polatoglu studied essential description, growth, and distortion theorem for the class S_q^*.

In following, we present a generalization of the class introduced in [20], by using the q-difference operator.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $K_{t,q}$, $(|t| \leq 1, t \neq 0, q \in (0,1))$, if there exists a function $g \in S^*\left(\frac{1}{2}\right)$, for which:*

$$\left| \frac{t z^2 (D_q f)(z)}{g(z) g(tz)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; z \in U. \tag{8}$$

We say that $f \in K_{t,q}$, with respect to g.

We give an example of a function belonging to this class.

Example 1. *The function $f_0(z) = z + \frac{z^2}{1+q}$ belongs to the class $K_{t,q}$, with respect to $g_0(z) = z, |t| \leq 1, t \neq 0, q \in (0,1), z \in U$. Indeed, f_0 is analytic in U, with $f_0(0) = 0 = f_0'(0) - 1$ and $g_0 \in S^*\left(\frac{1}{2}\right)$. We have:*

$$\left| \frac{t z^2 (D_q f)(z)}{g(z) g(tz)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; z \in U \iff |z(1-q) - q| \leq 1, \; z \in U\,.$$

This means that $f_0 \in K_{t,q}$, with respect to $g_0(z) = z$.

2. Main Results

We will prove the first result.

Theorem 1. *Let $g \in S^*\left(\frac{1}{2}\right)$, defined by:*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \; z \in U, \tag{9}$$

and

$$G(z) = \frac{g(z) g(tz)}{tz} = z + \sum_{n=2}^{\infty} c_n z^n, \; z \in U, \tag{10}$$

where $c_n = \sum_{j=1}^{n} b_j b_{n-j+1} t^{j-1}$, *with* $b_1 = 1$, $|t| \le 1$, $t \ne 0$. *Then,* $G(z) \in S^*$.

Proof. Since $g \in S^*\left(\frac{1}{2}\right)$, and from the definition of starlike functions, we get $\mathrm{Re}\left(\frac{\mathrm{zg}'(\mathrm{z})}{\mathrm{g}(\mathrm{z})}\right) > \frac{1}{2}$, $z \in U$. We note that for $z \in U$, we have $|tz| \le |z| \le 1$. So, we obtain $\mathrm{Re}\left(\frac{\mathrm{tzg}'(\mathrm{tz})}{\mathrm{g}(\mathrm{tz})}\right) > \frac{1}{2}$.

Therefore, $\mathrm{Re}\left(\frac{\mathrm{zG}'(\mathrm{z})}{\mathrm{G}(\mathrm{z})}\right) = \mathrm{Re}\left(\frac{\mathrm{zg}'(\mathrm{z})}{\mathrm{g}(\mathrm{z})} + \frac{\mathrm{tzg}'(\mathrm{tz})}{\mathrm{g}(\mathrm{tz})} - 1\right) > \frac{1}{2} + \frac{1}{2} - 1 = 0$.

This proves the conclusion of the theorem. □

Remark 1. *From theorem 1, (8) is equivalent to* $\left|\frac{z(D_q f)(z)}{G(z)} - \frac{1}{1-q}\right| \le \frac{1}{1-q}$, *thus it is obvious that* $K_{t,q} \subset \mathcal{K}_q$.

Theorem 2. *Let* $f \in \mathcal{A}$, *given by (1). We have* $f \in K_{t,q}$, *with respect to the function* $g \in S^*\left(\frac{1}{2}\right)$, *if, and only if:*

$$\frac{tz^2(D_q f)(z)}{g(z)g(tz)} \prec \frac{1+z}{1-qz},\ z \in U. \tag{11}$$

Proof. Let $f(z)$ be a member of $K_{t,q}$. We have:

$$\left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - \frac{1}{1-q}\right| \le \frac{1}{1-q},\ q \in (0,1), |t| \le 1, t \ne 0, z \in U,$$

which is equivalent to:

$$\left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - M\right| \le M,\ \ M = \frac{1}{1-q},\ M > 1, |t| \le 1, t \ne 0, z \in U.$$

So, the function:

$$\varphi(z) = \frac{1}{M}\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - 1,$$

has modulus at most 1 in the unit disk U. Therefore,

$$\phi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} = \frac{\frac{1}{M}\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - \left(\frac{1}{M} - 1\right)}{1 - \left(\frac{1}{M} - 1\right)\left(\frac{1}{M}\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - 1\right)}, \tag{12}$$

satisfies the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$. By Schwarz Lemma, we get:

$$\phi(z) \le z. \tag{13}$$

From (12) and (13), we obtain:

$$\frac{tz^2(D_q f)(z)}{g(z)g(tz)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)}. \tag{14}$$

The equality (14) shows that:

$$\frac{tz^2(D_q f)(z)}{g(z)g(tz)} \prec \frac{1+z}{1-qz},\ z \in U.$$

Conversely, let $\frac{tz^2(D_q f)(z)}{g(z)g(tz)} \prec \frac{1+z}{1-qz}$. We have:

$$\frac{tz^2(D_q f)(z)}{g(z)g(tz)} = \frac{1+\phi(z)}{1-\left(1-\frac{1}{M}\right)\phi(z)}, \ M = \frac{1}{1-q}, \ M > 1, |t| \leq 1, t \neq 0, z \in U.$$

So,

$$\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - M = M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}.$$

On the other hand, the function $\left(\frac{\frac{1-M}{M}+\phi(z)}{1+\frac{1-M}{M}\phi(z)}\right)$, with $\phi(0) = 0$ and $|\phi(z)| < 1$ maps the unit circle onto itself, so that:

$$\begin{aligned}
\left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - M\right| &= \left|M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}\right| < M \Rightarrow \\
\left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - \frac{1}{1-q}\right| &\leq \frac{1}{1-q}, \ q \in (0,1), |t| \leq 1, t \neq 0, z \in U,
\end{aligned}$$

and the proof is now complete. □

Remark 2. *If we use the notation (10), the relation (11) is equivalent to:*

$$\frac{z(D_q f)(z)}{G(z)} \prec \frac{1+z}{1-qz}, \ q \in (0,1), \ z \in U. \tag{15}$$

Corollary 1. *Let $f \in K_{t,q}$, with respect to the function $g \in S^*\left(\frac{1}{2}\right)$. Then:*

$$\frac{1-r}{1+qr} \leq \left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)}\right| \leq \frac{1+r}{1-qr}, \ q \in (0,1), |t| \leq 1, t \neq 0, z \in U. \tag{16}$$

Proof. The linear transformation $\omega(z) = \frac{1+z}{1-qz}$ maps $|z| = r$ onto the circle with the centre $C\left(\frac{1+qr^2}{1-q^2r^2}, 0\right)$ and radius $\rho(r) = \frac{(1+q)r}{1-q^2r^2}$. By using the subordination principle, we obtain:

$$\left|\frac{tz^2(D_q f)(z)}{g(z)g(tz)} - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{(1+q)r}{1-q^2r^2}, \ q \in (0,1), |t| \leq 1, t \neq 0, z \in U,$$

which implies the desired result. □

Theorem 3. *Let $f \in K_{t,q}$ with respect to the function g, f given by (1), $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $g \in S^*\left(\frac{1}{2}\right)$. Then:*

$$\sum_{n=1}^{k}\left|a_n\frac{1-q^n}{1-q} - c_n\right|^2 \leq \sum_{n=1}^{k-1}\left|a_n\frac{q-q^{n+1}}{1-q} + c_n\right|^2, \tag{17}$$

where $a_1 = 1, c_n = \sum_{j=1}^{n} b_j b_{n-j+1} t^{n-j}$, with $b_1 = 1$, $|t| \leq 1$, $t \neq 0$, $q \in (0,1)$, $z \in U$.

Proof. Using the above theorem and Remark 2, we have:

$$\begin{aligned}
\frac{z(D_q f)(z)}{G(z)} &= \frac{1+\phi(z)}{1-q\phi(z)} \Leftrightarrow z(D_q f)(z) - zq\phi(z)(D_q f)(z) = \\
G(z) + G(z)\phi(z) &\Leftrightarrow z(D_q f)(z) - G(z) = \phi(z)(zq(D_q f)(z) + G(z)).
\end{aligned}$$

From the definition of $(D_q f)(z)$, we get:

$$\sum_{n=1}^{\infty}\left(a_n\frac{1-q^n}{1-q}-c_n\right)z^n=\phi(z)\sum_{n=1}^{\infty}\left(a_n\frac{q-q^{n+1}}{1-q}+c_n\right)z^n,$$

where $a_1=1, c_n=\sum\limits_{j=1}^{n}b_jb_{n-j+1}t^{n-j}$, with $b_1=1$, $|t|\le 1$, $t\neq 0$, $q\in(0,1)$, $z\in U$. Thus,

$$\sum_{n=1}^{k}\left(a_n\frac{1-q^n}{1-q}-c_n\right)z^n+\sum_{n=k+1}^{\infty}d_nz^n=\phi(z)\sum_{n=1}^{k-1}\left(a_n\frac{q-q^{n+1}}{1-q}+c_n\right)z^n,$$

where the sum $\sum\limits_{n=k+1}^{\infty}d_nz^n$ is convergent in U. Let $z=re^{i\theta}$. Since $|\phi(z)|<1$, we deduce that:

$$\sum_{n=1}^{k}\left|a_n\frac{1-q^n}{1-q}-c_n\right|^2 r^{2k}\le\sum_{n=1}^{k-1}\left|a_n\frac{q-q^{n+1}}{1-q}+c_n\right|^2 r^{2k}. \tag{18}$$

Passing to the limit in (18) as $r\to 1$, we obtain the inequality (17), which completes our proof. □

This proof is based on a method introduced by Clunie (see [17]).

If we consider $g(z)=z$, in theorem 3, we get the following result.

Corollary 2. *Let $f\in K_{t,q}$, with respect to the function g, f given by (1) and $g(z)=z$, $g\in S^*\left(\frac{1}{2}\right)$. Then:*

$$\sum_{n=2}^{k}|a_n|^2\left(\frac{1-q^n}{1-q}\right)^2\le(1+q)^2+\sum_{n=2}^{k-1}|a_n|^2\left(\frac{q-q^{n+1}}{1-q}\right)^2, \tag{19}$$

where $|t|\le 1$, $t\neq 0$, $q\in(0,1)$, $z\in U$.

Next, we provide a sufficient condition for functions to belong to the class $K_{t,q}$.

Theorem 4. *Let $f\in\mathcal{A}$, given by (1), $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$, $g\in S^*\left(\frac{1}{2}\right)$, and $c_n=\sum\limits_{j=1}^{n}b_jb_{n-j+1}tz^{j-1}$, with $b_1=1$, $|t|\le 1$, $t\neq 0$. If*

$$\sum_{n=2}^{\infty}\left(\left|\frac{c_n}{1-q}-a_n[n]_q\right|+\frac{|c_n|}{1-q}\right)\le 1,\ q\in(0,1), z\in U, \tag{20}$$

then $f\in K_{t,q}$, with respect to g.

Proof. If $f\in K_{t,q}$, the relation (8) is equivalent to:

$$\frac{q+\sum\limits_{n=2}^{\infty}|c_n-a_n(1-q^n)|}{1-\sum\limits_{n=2}^{\infty}|c_n|}\le 1,\ q\in(0,1), |t|\le 1, t\neq 0, z\in U. \tag{21}$$

From (21) we obtain:

$$\frac{\left|qz+\sum\limits_{n=2}^{\infty}(c_n+a_nq^n-a_n)z^n\right|}{\left|z+\sum\limits_{n=2}^{\infty}c_nz^n\right|}\le 1,\ q\in(0,1), |t|\le 1, t\neq 0, z\in U,$$

or, equivalently:

$$\left|\frac{z(D_q f)(z)}{G(z)} - \frac{1}{1-q}\right| \le \frac{1}{1-q},\ q \in (0,1), |t| \le 1, t \ne 0, z \in U. \tag{22}$$

Therefore, if the function f satisfies the inequality (20), then $f \in K_{t,q}$, with respect to g. The proof of our theorem is now complete. □

Remark 3. *In the particular case, when $f_0(z) = z + \frac{z^2}{1+q}$ and $g_0(z) = z$, $|t| \le 1$, $t \ne 0$, $q \in (0,1)$, $z \in U$, f_0 belongs to the class $K_{t,q}$, with respect to g_0 (see Example 1). However,*

$$\sum_{n=2}^{\infty}\left(\left|\frac{c_n}{1-q} - a_n[n]_q\right| + \frac{|c_n|}{1-q}\right) = \frac{1}{1-q}[2]_q = 1 + q \nleq 1, \quad q \in (0,1).$$

This show that (20) is only a sufficies condition.

Next, we will prove the Bieberbach-de Branges theorem for the generalized class $K_{t,q}$. We need the following result:

Lemma 1. *A function $f \in K_{t,q}$, with respect to g if, and only if, there exists a function $g \in S^*\left(\frac{1}{2}\right)$, such that:*

$$\frac{|g(z)g(tz) + tz(f(qz) - f(z))|}{|g(z)g(tz)|} \le 1,\ |t| \le 1, t \ne 0, q \in (0,1), z \in U. \tag{23}$$

Proof. The proof can be obtained from: $\frac{tz^2(D_q f)(z)}{g(z)g(tz)} = \frac{tz(f(qz)-f(z))}{g(z)g(tz)(q-1)}$. □

If we use the notation (10), the inequality (23) is equivalent to:

$$\frac{|G(z) + f(qz) - f(z)|}{|G(z)|} \le 1,\ q \in (0,1), z \in U. \tag{24}$$

We now proceed to state and prove the Bieberbach-de Branges theorem for the generalized class $K_{t,q}$. The Bieberbach-de Branges conjecture for close-to-convex functions is proved by Reade (see [21]). It states that if $f \in K$, then $|a_n| \le n$, for all $n \ge 2$. The Bieberbach-de Branges theorem for the class of q-close-to-convex functions is proved in [22].

Theorem 5. *Let $f \in K_{t,q}$, with respect to g, f given by (1), $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $g \in S^*\left(\frac{1}{2}\right)$, then:*

$$|a_n| \le \frac{1-q}{1-q^n}\left[n + (1+q)\frac{n(n-1)}{2}\right],\ \text{for all } n \ge 2. \tag{25}$$

Proof. Since $f \in K_{t,q}$, by (24), there exists a function $w : U \to \bar{U}, w(z) = q + \sum_{n=2}^{\infty} w_n z^n$, such that:

$$G(z) + f(qz) - f(z) = w(z)G(z). \tag{26}$$

Evidently, $w(0) = q$. By assuming $a_1 = c_1 = 1$, we have:

$$\sum_{n=1}^{\infty}(c_n + a_n q^n - a_n)z^n = \sum_{n=1}^{\infty} q c_n z^n + \sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} w_{n-j}c_j\right)z^n,\ q \in (0,1).$$

Equating the coefficients of z^n, for $n \geq 2$, we obtain:

$$a_n(q^n - 1) = c_n(q - 1) + \sum_{j=1}^{n-1} w_{n-j}c_j,\ q \in (0,1).$$

It is easy to verify that $|w_n| \leq 1 - |w_0^2| = 1 - q^2$, for all $n \geq 1$, $q \in (0,1)$. Since $G(z) \in S^*$, then $|c_n| \leq n$, for all $n \geq 2$. Therefore, we get the bound:

$$|a_n| \leq \frac{1-q}{1-q^n}\left[n + (1+q)\sum_{j=1}^{n-1} j\right],\ q \in (0,1), \text{ for all } n \geq 2,$$

which shows that the desired assertion (25) holds. □

Corollary 3. *Let $f \in K_{t,q}$, with respect to $g = \frac{z}{1-z} \in S^*\left(\frac{1}{2}\right)$, f given by (1), $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$, then, for all $n \geq 2$, we have:*

$$|a_n| \leq \frac{1}{[n]_q}\frac{1}{1-t}\left[(1+q)(n+1) + \frac{(1-t^n)(t+q)}{t-1}\right],\ q \in (0,1). \tag{27}$$

Proof. By rewriting the function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$, we get:

$$G(z) = \frac{g(z)g(tz)}{tz} = \sum_{n=1}^{\infty} \frac{t^n - 1}{t-1} z^n,\ |t| \leq 1, t \neq 0, z \in U.$$

From (1), we obtain:

$$\sum_{n=1}^{\infty} \frac{t^n-1}{t-1}z^n + (q-1)z + \sum_{n=2}^{\infty}(q^n-1)a_n z^n = \left(q + \sum_{n=2}^{\infty} w_n z^n\right)\sum_{n=1}^{\infty} . \frac{t^n-1}{t-1}z^n.$$

This is equivalent to:

$$\sum_{n=2}^{\infty}(q^n-1)a_n z^n = (1-q)z + (q-1)\sum_{n=1}^{\infty} . \frac{t^n-1}{t-1}z^n + \sum_{n=1}^{\infty} . \frac{t^n-1}{t-1}z^n \sum_{n=2}^{\infty} w_n z^n.$$

Equating the coefficients of z^n, on both sides of (1), for $n \geq 2$, we have:

$$(q^n-1)a_n = (q-1).\frac{t^n-1}{t-1} + w_2\frac{t^n-1}{t-1} + w_3\frac{t^{n-1}-1}{t-1} + ... + w_{n-1}.$$

Since $|w_n| \leq 1 - |w_0^2| = 1 - q^2$, for all $n \geq 1$, $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$, we get:

$$(q^n-1)|a_n| \leq (q-1).\frac{t^n-1}{t-1} + \left(q^2-1\right)\left(\frac{t^n-1}{t-1} + \frac{t^{n-1}-1}{t-1} + \ldots + 1\right),$$

which implies:

$$|a_n| \leq \frac{1-q}{1-q^n}\left[\frac{1-t^n}{1-t} + (1+q)\sum_{k=1}^{n-1}\frac{1-t^{n-1}}{1-t}\right], \text{ for } n \geq 2.$$

Or, equivalently:

$$|a_n| \leq \frac{1}{[n]_q}\frac{1}{1-t}\left[(1+q)(n+1) + \frac{(1-t^n)(t+q)}{t-1}\right], \text{ for } n \geq 2.$$

This completes the proof. □

If we consider $g(z) = z$, in theorem 5, we obtain:

Corollary 4. *Let $f \in K_{t,q}$, with respect to $g(z) = z \in S^*\left(\frac{1}{2}\right)$, f given by (1), $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$, then for all $n \geq 2$, we have:*

$$|a_n| \leq \frac{1+q}{[n]_q},\ q \in (0,1), \text{ for all } n \geq 2. \tag{28}$$

Remark 4. *Our usages in the current investigation potentially own local or non-local symmetric or asymmetric properties. Our purpose for further investigation is to study the local symmetry in the new introduced class of analytic functions and also to introduce and study an extention, symmetric under the interchange of q and q^{-1}.*

3. Discussion

By making use the previously introduced q-difference operator, we formulated a new class of analytic functions ($K_{t,q}$) in the unit disk (U), and we obtained some results in this generalized class. The relationship between the new defined class, and the appropriate subordination were investigated; additionally, we derived relationships of this class and obtained sufficient conditions for an analytic function to be $K_{t,q}$. The Bieberbach-de Branges theorem, for the class $K_{t,q}$, has also been given.

In the past few decades, applications of q-calculus have been considered and explored comprehensively. Stimulated by these applications, numerous mathematicians have created the theory of quantum calculus based on two parameter (p,q)-integer (with $0 < q < p \leqq 1$). In the newly published survey-cum-expository review article by Srivastava it was clearly highlighting the triviality and inconsequential nature of the (p,q)-variation of the traditional q-calculus, the extra parameter (p), being obviously superfluous (see, for additional information, [27], p. 340). This remark by Srivastava [27] will certainly utilize any attempt to create the quite simple (p,q)-variations of the outcomes, which we have provided in this article. These observations have already been addressed in many papers, as seen in [28–30].

We hope that this work offers a foundation for further study in investigating several other classes of analytic functions, by using the previously introduced q- difference operators and their varied geometric properties, such as their associated coefficient estimates, sufficiency criteria, radii of starlikeness, convexity, and close to convexity, extreme points, and distortion bounds, which can be considered for such classes of analytic functions.

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