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Bessel Collocation Method for Solving Fredholm–Volterra Integro-Fractional Differential Equations of Multi-High Order in the Caputo Sense

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Abstract: The approximate solutions of Fredholm–Volterra integro-differential equations of multi-fractional order within the Caputo sense (F-VIFDEs) under mixed conditions are presented in this article apply a collocation points technique based completely on Bessel polynomials of the first kind. This new approach depends particularly on transforming the linear equation and conditions into the matrix relations (some time symmetry matrix), which results in resolving a linear algebraic equation with unknown generalized Bessel coefficients. Numerical examples are given to show the technique’s validity and application, and comparisons are made with existing results by applying this process in order to express these solutions, most general programs are written in Python V.3.8.8 (2021).

Keywords: Fredholm–Volterra integral Equations; fractional derivative; Bessel polynomials; Caputo derivative; collocation points

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1. Introduction

Fractional calculus (FC) deals with the differentiation and integration of arbitrary order and it is used in the real world to model and analyze big problems. Fluid flow, electrical networks, fractals theory, control theory, electromagnetic theory, probability, statistics, optics, potential theory, biology, chemistry, diffusion, and viscoelasticity are just a few of the many fields where fractional calculus is used [1–4].

In recent years, fractional differential equations and integro-fractional differential equations (IFDEs) have captivated the hobby of many researchers in various fields of science and era due to the reality that realistic modeling of a bodily phenomenon with dependencies not only in the immediate time, but also in the past time history can be accomplished effectively using FC. However, in addition to modeling, the solution approaches and their dependability are crucial in detecting key points when a rapid divergence, convergence, or bifurcation begins. As a result, high-precision solutions are always required. Several strategies for solving fractional order differential equations were presented for this purpose (or integro-differential equations), [1,3,4]. The Adomian decomposition method [5], variational iteration method [6], fractional differential transform method [7], fractional difference method [8], and power series method [9] are the most commonly used ideas.

However, from the beginning of 1994, Laguerre, Legendre, Taylor, Fourier, Hermite, and Bessel polynomials have been employed in works [10–15] to solve linear differential, integral, and integro-differential difference equations and related systems. In addition, the Bessel polynomial of the first kind method has been used to find approximate solutions of differential, fractional differential equations, integro-differential equations of fractional order, LVIDEs, and LF-VIDEs [16–19].

The aim of this paper is to expand and apply the first kind of Bessel polynomial in matrix form, as well as the collocation techniques, to evaluate the approximate solution for the multi-high-order linear Fredholm–Volterra integro-fractional differential equations (FVIFDEs) of the general type:

$$\begin{aligned} {}^c D_x^{\sigma_n} u(x) + \sum_{l=1}^{n-1} p_l(x) {}^c D_x^{\sigma_{n-l}} u(x) + p_n(x) u(x) \\ = g(x) + \sum_{i=0}^{m_1} \lambda_i \int_a^b F_i(x, t) {}^c D_t^{\alpha_i} u(t) dt + \sum_{j=0}^{m_2} \bar{\lambda}_j \int_a^x V_j(x, t) {}^c D_t^{\beta_j} u(t) dt, x \in [a, b] \end{aligned} \quad (1)$$

together with mixed conditions:

$$\sum_{\ell=0}^{\mu-1} \{ \hbar_{k\ell} u^{(\ell)}(a) + \bar{\hbar}_{k\ell} u^{(\ell)}(b) \} = C_k, k = 0, 1, \dots, \mu - 1. \quad (2)$$

where the fractional orders: $\sigma_n > \sigma_{n-1} > \dots > \sigma_1 > \sigma_0 = 0$, $\alpha_{m_1} > \alpha_{m_1-1} > \dots > \alpha_1 > \alpha_0 = 0$, and $\beta_{m_2} > \beta_{m_2-1} > \dots > \beta_1 > \beta_0 = 0$, and $\mu = \max\{\lceil \sigma_n \rceil, \lceil \alpha_{m_1} \rceil, \lceil \beta_{m_2} \rceil\}$. In addition, $u(x)$ is an unknown function, the functions $p_l(x), g(x) \in C([a, b], \mathbb{R})$, for all $l = 1, 2, \dots, n$, and $F_i(x, t), V_j(x, t) \in C(S, \mathbb{R})$, (with $S = \{(x, t): a \leq t \leq x \leq b\}$) are known, with constants $\hbar_{k\ell}, \bar{\hbar}_{k\ell}, \lambda_i, \bar{\lambda}_j$ and $C_k \in \mathbb{R}$ for all $k, \ell = 0, 1, \dots, \mu - 1$, $i = 0, 1, \dots, m_1, j = 0, 1, \dots, m_2$, ($n, m_1, m_2 \in \mathbb{Z}^+$) are given.

2. Preliminary Considerations

2.1. Basic Definitions and Some Lemmas

Many mathematical definitions of fractional integration and differentiation have come to light in recent years. The most frequently used definitions of fractional calculus involves the Riemann–Liouville fractional derivative and Caputo derivative. In terms of applicability, the Caputo concept is more dependable than the Riemann–Liouville definition. In this section, we are interested some basic definitions and lemmas which are used later on in this paper [1,3,4,20,21].

Definition 1. [22] A real valued function u defined on closed bounded interval $[a, b] = I$ be in the space $C_\gamma(I)$, $\gamma \in \mathbb{R}$, if there exist a real number $k > \gamma$, such that $u(x) = (x - a)^k u_0(x)$, where $u_0(x) \in C(I)$, and it is said to be in the space $C_\gamma^n(I)$ iff $u^{(n)}(x) \in C_\gamma(I)$, where $n \in \mathbb{Z}^+ \cup \{0\}$.

Definition 2. [23] The Riemann–Liouville (R-L) fractional integral operator, ${}_a J_x^\alpha$, of order $\alpha > 0$ of a function $u \in C_\gamma(I)$, $\gamma \geq -1$ is defined as:

$$\begin{aligned} {}_a J_x^\alpha u(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} u(t) dt, \quad \alpha \in \mathbb{R}^+ \\ {}_a J_x^0 u(x) &= u(x). \end{aligned}$$

Definition 3. [24] The Riemann–Liouville (R-L) fractional derivative operator, ${}_a^R D_x^\alpha$, of order $\alpha \geq 0$ of a function $u(x)$ and $u \in C_{m-1}^m(I)$, $m = [\alpha]$ is normally defined as:

$${}_a^R D_x^\alpha u(x) = D_x^m {}_a J_x^{m-\alpha} u(x), \quad m - 1 < \alpha \leq m, \quad m \in \mathbb{N}.$$

Definition 4. [23] The Caputo fractional derivative operator, ${}_a^C D_x^\alpha$, of a function $u \in C_{m-1}^m(I)$ and $m = [\alpha]$, (ceiling function), is defined as:

$$\begin{aligned}
{}_a^C D_x^\alpha u(x) &= {}_a J_x^{m-\alpha} [D_x^m u(x)] \\
&= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} \frac{\partial^m u(t)}{\partial t^m} dt, & m-1 < \alpha < m \\ \frac{\partial^m u(x)}{\partial x^m}, & \alpha = m, m \in \mathbb{N}. \end{cases}
\end{aligned}$$

where the parameter α is the order of the derivative and is allowed to be any positive real number. The operators ${}_a J_x^\alpha$ and ${}_a^C D_x^\alpha$ are linear operators. Furthermore, we have

Lemma 1. [4] Let $x > a, a \in \mathbb{R}$ and for $u(x) = (x-a)^\beta$ for some $\beta \neq -m$ is not negative integer, then

$${}_a J_x^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\beta+\alpha}.$$

Lemma 2. [20] The Caputo derivative of order $\alpha \geq 0$ with $n = [\alpha]$ of the power function $u(x) = (x-a)^\beta$ for some $\beta \geq 0$ is formed by:

$${}_a^C D_x^\alpha u(x) = \begin{cases} 0 & \text{if } \beta \in \{0, 1, 2, \dots, n-1\} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha} & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq n \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > n-1 \end{cases}$$

Lemma 3. [20] Let $\alpha \geq 0, m = [\alpha]$. Moreover, assume that $u \in C_{-1}^m(I)$. Then the Caputo fractional derivative ${}_a^C D_x^\alpha u(x)$ is continuous on $I = [a, b]$ and $\lim_{x \rightarrow a} [{}_a^C D_x^\alpha u(x)] = 0$.

2.2. Bessel Polynomial of the First Kind

The r -th degree N -truncated Bessel polynomials of the first kind, [25,26], $J_r(x)$, $r = 0, 1, \dots, N$ are defined by

$$J_r(x) = \sum_{k=0}^{\lfloor \frac{N-r}{2} \rfloor} \frac{(-1)^k}{k! (k+r)!} \left(\frac{x}{2}\right)^{2k+r}, \quad r \in \mathbb{N}, 0 \leq x < \infty.$$

Here, N is a positive integer that is selected in such a way that $N \geq r$. On the other hand, we may express the $J_r(x)$ as follows in the matrix form.

$$\mathbf{J}(x) = \mathbf{X}(x) \mathbf{D}^T \text{ or } \mathbf{J}^T(x) = \mathbf{D} \mathbf{X}^T(x). \quad (3)$$

where $\mathbf{J}(x) = [J_0(x) \ J_1(x) \ \dots \ J_N(x)]$ and $\mathbf{X}(x) = [1 \ x \ x^2 \ \dots \ x^N]$

If N is odd

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0! 0! 2^0} & 0 & \frac{-1}{1! 1! 2^2} & \dots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})! (\frac{N-1}{2})! 2^{N-1}} & 0 \\ 0 & \frac{1}{0! 1! 2^1} & 0 & \dots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})! (\frac{N+1}{2})! 2^N} \\ 0 & 0 & \frac{1}{0! 2! 2^2} & \dots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})! (\frac{N+1}{2})! 2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{0! (N-1)! 2^{N-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{0! N! 2^N} \end{bmatrix}_{(N+1) \times (N+1)}$$

If N is even

$$D = \begin{bmatrix} \frac{1}{0! 0! 2^0} & 0 & \frac{-1}{1! 1! 2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})! (\frac{N}{2})! 2^N} \\ 0 & \frac{1}{0! 1! 2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})! (\frac{N}{2})! 2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0! 2! 2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})! (\frac{N+2}{2})! 2^N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0! (N-1)! 2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0! N! 2^N} \end{bmatrix}_{(N+1) \times (N+1)}$$

3. Fundamental Matrix Relations

Recall Equation (1) and rewrite it as follows:

$$D(\{\sigma_l\}_{l=1}^n, x) = g(x) + I_f(\{\alpha_i\}_{i=0}^{m_1}, x) + I_v(\{\beta_j\}_{j=0}^{m_2}, x). \quad (4)$$

where

$$D(\{\sigma_l\}_{l=1}^n, x) = {}^c D_x^{\sigma_n} u(x) + \sum_{l=1}^{n-1} p_l(x) {}^c D_x^{\sigma_{n-l}} u(x) + p_n(x) u(x)$$

and the integral parts:

$$I_f(\{\alpha_i\}_{i=0}^{m_1}, x) = \sum_{i=0}^{m_1} \lambda_i \int_a^b F_i(x, t) {}^c D_t^{\alpha_i} u(t) dt, \quad I_v(\{\beta_j\}_{j=0}^{m_2}, x) = \sum_{j=0}^{m_2} \bar{\lambda}_j \int_a^x V_j(x, t) {}^c D_t^{\beta_j} u(t) dt.$$

our purpose is to find a close approximation of Equation (1) in the N -truncated Bessel series arrangement

$$u(x) \cong \sum_{r=0}^N a_r J_r(x). \quad (5)$$

So that a_r , for all $r = 0, 1, \dots, N$ are the unknown Bessel coefficients. Before we begin the approximate solution we must convert the solution $u(x)$ and its ${}^c D_x^{\sigma_n} u(x)$, ${}^c D_x^{\sigma_{n-l}} u(x)$, ${}^c D_x^{\alpha_i} u(x)$ and ${}^c D_x^{\beta_j} u(x)$, for all $l = 1, 2, \dots, n-1$, $i = 0, 1, \dots, m_1$, $j = 0, 1, \dots, m_2$ in the parts $D(\{\sigma_l\}_{l=1}^n, x)$, $I_f(\{\alpha_i\}_{i=0}^{m_1}, x)$ and $I_v(\{\beta_j\}_{j=0}^{m_2}, x)$, to matrix form, within the mixed conditions of Equation (2).

3.1. Matrix Relation for the Fractional Derivative Part D

To describe the solution $u(x)$ of Equation (1), which is specified by the N -truncated Bessel series of Equation (5). The function defined in relation (5) in a matrix form

$$[u(x)] = \mathbf{J}(x) \mathbf{A}; \mathbf{A} = [a_0 \ a_1 \ a_2 \ \dots \ a_N]^T \quad (6)$$

or from Equation (3)

$$[u(x)] = \mathbf{X}(x) \mathbf{D}^T \mathbf{A}. \quad (7)$$

The relationship between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(1)}(x)$ is also written as follows:

$$\mathbf{X}^{(1)}(x) = \mathbf{X}(x)\mathbf{B}^T. \quad (8)$$

where

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1) \times (N+1)}$$

We will also get the recurrence relations from Equation (8):

$$\begin{aligned} \mathbf{X}^{(0)}(x) &= \mathbf{X}(x) \\ \mathbf{X}^{(1)}(x) &= \mathbf{X}(x)\mathbf{B}^T \\ \mathbf{X}^{(2)}(x) &= \mathbf{X}^{(1)}(x)\mathbf{B}^T = \mathbf{X}(x)(\mathbf{B}^T)^2 \\ &\vdots \\ \mathbf{X}^{(j)}(x) &= \mathbf{X}^{(j-1)}(x)\mathbf{B}^T = \mathbf{X}(x)(\mathbf{B}^T)^j. \end{aligned} \quad (9)$$

Here, note that $(\mathbf{B}^T)^0 = [\mathbf{I}]_{(N+1) \times (N+1)}$ is an identity matrix of dimension $(N+1)$.

Using mathematical induction, we can prove that Equation (9) is correct. By applying the same concept to Equation (7) and using Equation (9), we attain matrix relation

$$\begin{aligned} u^{(j)}(x) &= \mathbf{X}^{(j)}(x)\mathbf{D}^T\mathbf{A} \\ u^{(j)}(x) &= \mathbf{X}(x)(\mathbf{B}^T)^j\mathbf{D}^T\mathbf{A}, \text{ for each } j = 0, 1, \dots, \mu \text{ and } \mu = \max\{\lceil \sigma_n \rceil, \lceil \alpha_{m_1} \rceil, \lceil \beta_{m_2} \rceil\} \end{aligned} \quad (10)$$

By using Equation (7) with (9) and applying the Caputo definition 4, with Lemma 1 and 2, we can convert the fractional terms ${}_a^c D_x^{\sigma_{n-l}} u(x)$, $\bar{n}(\sigma_{n-l}) - 1 < \sigma_{n-l} \leq \bar{n}(\sigma_{n-l})$, that is $\bar{n}(\sigma_{n-l}) = \lceil \sigma_{n-l} \rceil$, for all $l = 0, 1, \dots, n-1$ to matrix form:

$$\begin{aligned} {}_a^c D_x^{\sigma_{n-l}} u(x) &= {}_a^c D_x^{\sigma_{n-l}} \mathbf{X}(x)\mathbf{D}^T\mathbf{A} \\ &= {}_a J_x^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} D^{\bar{n}(\sigma_{n-l})} \mathbf{X}(x)\mathbf{D}^T\mathbf{A} \\ &= {}_a J_x^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} \mathbf{X}(x)(\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T\mathbf{A} \\ &= x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l}} \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T\mathbf{A}. \end{aligned}$$

Since

$$\begin{aligned} {}_a J_x^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} \mathbf{X}(x) &= {}_a J_x^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} [1 \ x \ x^2 \ \dots \ x^N] \\ &= \left[\frac{\Gamma(1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 1)} x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 0}, \frac{\Gamma(2)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 2)} x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 1}, \right. \\ &\quad \left. \dots, \frac{\Gamma(N+1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + N + 1)} x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + N} \right] \end{aligned}$$

$$= x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l}} [1x x^2 \dots x^N] \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 1)} & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma(N+1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + N + 1)} \end{bmatrix}$$

Putting

$$\mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 1)} & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + 2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\Gamma(N+1)}{\Gamma(\bar{n}(\sigma_{n-l}) - \sigma_{n-l} + N + 1)} \end{bmatrix} \quad (11)$$

Thus, for all $l = 1, 2, \dots, n-1$ in general we obtain

$${}_a^c D_x^{\sigma_{n-l}} u(x) = {}_a^c D_x^{\sigma_{n-l}} \mathbf{X}(x) \mathbf{D}^T \mathbf{A} = x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l}} \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T \mathbf{A}. \quad (12)$$

and

$${}_a^c D_x^{\sigma_n} u(x) = {}_a^c D_x^{\sigma_n} \mathbf{X}(x) \mathbf{D}^T \mathbf{A} = x^{\bar{n}(\sigma_n) - \sigma_n} \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n) (\mathbf{B}^T)^{\bar{n}(\sigma_n)} \mathbf{D}^T \mathbf{A}. \quad (13)$$

Using mathematical induction, we can prove that Equations (12) and (13) are correct. By substituting expressions (7), (12) and (13) into (4), As well we can make this assumption $y(n, x) = x^{\bar{n}(\sigma_n) - \sigma_n}$, $y(n-l, x) = x^{\bar{n}(\sigma_{n-l}) - \sigma_{n-l}}$, for all $l = 1, 2, \dots, n-1$, we have

$$D(\{\sigma_l\}_{l=1}^n, x) = [y(n, x) \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n) (\mathbf{B}^T)^{\bar{n}(\sigma_n)} \mathbf{D}^T + \sum_{l=1}^{n-1} p_l(x) y(n-l, x) \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T + p_n(x) \mathbf{X}(x) \mathbf{D}^T] \mathbf{A}. \quad (14)$$

3.2. Matrix Relation for the Fredholm Integral Part I_f

The N -truncated Taylor series around (0,0), [27] and the N -truncated Bessel series can be used to approximate the Fredholm kernel functions $F_i(x, t)$, $i = 0, 1, \dots, m_1$, respectively

$$F_i(x, t) = \sum_{m=0}^N \sum_{n=0}^N {}_t F_{mn}^i x^m t^n \text{ and } F_i(x, t) = \sum_{m=0}^N \sum_{n=0}^N {}_b F_{mn}^i J_m(x) J_n(t), \quad i = 0, 1, \dots, m_1 \quad (15)$$

where

$$[{}_t F_{mn}^i] = \frac{1}{m! n!} \frac{\partial^{m+n} F_i(0,0)}{\partial x^m \partial t^n}, \quad i = 0, 1, \dots, m_1, \quad m, n = 0, 1, \dots, N.$$

In matrix forms, the Equation (15) may be written as Equations (16) and (17), respectively

$$F_i(x, t) = \mathbf{X}(x) \mathbf{F}_t^i \mathbf{X}^T(t), \quad \mathbf{F}_t^i = [{}_t F_{mn}^i], \quad i = 0, 1, \dots, m_1, \quad m, n = 0, 1, \dots, N. \quad (16)$$

and

$$F_i(x, t) = \mathbf{J}(x) \mathbf{F}_b^i \mathbf{J}^T(t), \quad \mathbf{F}_b^i = [{}_b F_{mn}^i], \quad i = 0, 1, \dots, m_1, \quad m, n = 0, 1, \dots, N. \quad (17)$$

From Equations (16) and (17), it also comes out according to Equation (3), the following relation

$$\begin{aligned} \mathbf{X}(x)\mathbf{F}_t^i\mathbf{X}^T(t) &= \mathbf{J}(x)\mathbf{F}_b^i\mathbf{J}^T(t), \quad i = 0, 1, \dots, m_1 \\ \mathbf{X}(x)\mathbf{F}_t^i\mathbf{X}^T(t) &= \mathbf{X}(x)\mathbf{D}^T\mathbf{F}_b^i\mathbf{D}\mathbf{X}^T(t), \quad i = 0, 1, \dots, m_1 \end{aligned} \quad (18)$$

$$\mathbf{F}_t^i = \mathbf{D}^T\mathbf{F}_b^i\mathbf{D} \quad \text{or} \quad \mathbf{F}_b^i = (\mathbf{D}^T)^{-1}\mathbf{F}_t^i\mathbf{D}^{-1}, \quad i = 0, 1, \dots, m_1$$

In the same way from Equations (12) and (13), convert ${}_a^c D_x^{\alpha_i} u(x)$, and $\bar{n}(\alpha_i) - 1 < \alpha_i \leq \bar{n}(\alpha_i)$, i.e., $\bar{n}(\alpha_i) = [\alpha_i]$ for all $i = 0, 1, \dots, m_1$, by apply the Caputo definition 4 with Lemma 1 to the matrix form, we obtain

$$\begin{aligned} {}_a^c D_x^{\alpha_i} u(x) &= {}_a^c D_x^{\alpha_i} \mathbf{X}(x)\mathbf{D}^T \mathbf{A} \\ &= {}_a J_x^{(\bar{n}(\alpha_i) - \alpha_i)} \mathbf{D}^{\bar{n}(\alpha_i)} \mathbf{X}(x)\mathbf{D}^T \mathbf{A} \\ &= {}_a J_x^{(\bar{n}(\alpha_i) - \alpha_i)} \mathbf{X}(x)(\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} \\ &= x^{\bar{n}(\alpha_i) - \alpha_i} \mathbf{X}(x) \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A}, \quad i = 0, 1, \dots, m_1 \end{aligned} \quad (19)$$

where $\mathbf{C}(\bar{n}(\alpha_i) - \alpha_i)$ is defined at Equation (11). We obtain the matrix relation (20), put the Equation (3) into (17), and then replace the obtained matrix with matrix (19) in the Fredholm integral part I_f in Equation (4)

$$\begin{aligned} I_f(\{\alpha_i\}_{i=0}^{m_1}, x) &= \sum_{i=0}^{m_1} \lambda_i \int_a^b F_i(x, t) {}_a^c D_t^{\alpha_i} u(t) dt \\ &= \sum_{i=0}^{m_1} \lambda_i \int_a^b \mathbf{J}(x)\mathbf{F}_b^i\mathbf{J}^T(t) t^{\bar{n}(\alpha_i) - \alpha_i} \mathbf{X}(t) \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} dt \\ &= \sum_{i=0}^{m_1} \lambda_i \int_a^b \mathbf{X}(x)\mathbf{D}^T\mathbf{F}_b^i\mathbf{D}\mathbf{X}^T(t) t^{\bar{n}(\alpha_i) - \alpha_i} \mathbf{X}(t) \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} dt \\ &= \sum_{i=0}^{m_1} \lambda_i \mathbf{X}(x)\mathbf{D}^T\mathbf{F}_b^i\mathbf{D} \left(\int_a^b \mathbf{X}^T(t) \mathbf{X}(t) t^{\bar{n}(\alpha_i) - \alpha_i} dt \right) \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} \\ &= \sum_{i=0}^{m_1} \lambda_i \mathbf{X}(x)\mathbf{D}^T\mathbf{F}_b^i\mathbf{D} \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A}. \end{aligned} \quad (20)$$

where

$$\begin{aligned} \mathbf{H}_{f,i} &= \int_a^b \mathbf{X}^T(t) \mathbf{X}(t) t^{\bar{n}(\alpha_i) - \alpha_i} dt = [h_{rs}^{f,i}], \quad i = 0, 1, \dots, m_1, \quad r, s = 0, 1, \dots, N. \\ [h_{rs}^{f,i}] &= \frac{b^{\bar{n}(\alpha_i) - \alpha_i + r + s + 1} - a^{\bar{n}(\alpha_i) - \alpha_i + r + s + 1}}{\bar{n}(\alpha_i) - \alpha_i + r + s + 1}, \quad i = 0, 1, \dots, m_1, \quad r, s = 0, 1, \dots, N. \end{aligned}$$

We can get the last matrix form (21) by replacing the matrix relation (18) into expression (20).

$$I_f(\{\alpha_i\}_{i=0}^{m_1}, x) = \sum_{i=0}^{m_1} \lambda_i \mathbf{X}(x)\mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A}. \quad (21)$$

3.3. Matrix Relation for the Volterra Integral Part I_v

The N -truncated Taylor series around $(0,0)$, [27] and the N -truncated Bessel series can be used to approximate the Volterra kernel functions $V_j(x, t)$, $j = 0, 1, \dots, m_2$, respectively

$$V_j(x, t) = \sum_{m=0}^N \sum_{n=0}^N {}_tV_{mn}^j x^m t^n \text{ and } V_j(x, t) = \sum_{m=0}^N \sum_{n=0}^N {}_bV_{mn}^j J_m(x) J_n(t), \quad j = 0, 1, \dots, m_2 \quad (22)$$

where

$$[{}_tV_{mn}^j] = \frac{1}{m! n!} \frac{\partial^{m+n} V_j(0,0)}{\partial x^m \partial t^n}, \quad j = 0, 1, \dots, m_2, \quad m, n = 0, 1, \dots, N.$$

The relations in Equation (22) can be transformed into matrix forms:

$$V_j(x, t) = \mathbf{X}(x) \mathbf{V}_t^j \mathbf{X}^T(t), \quad \mathbf{V}_t^j = [{}_tV_{mn}^j], \quad j = 0, 1, \dots, m_2, \quad m, n = 0, 1, \dots, N \quad (23)$$

and

$$V_j(x, t) = \mathbf{J}(x) \mathbf{V}_b^j \mathbf{J}^T(t), \quad \mathbf{V}_b^j = [{}_bV_{mn}^j], \quad j = 0, 1, \dots, m_2, \quad m, n = 0, 1, \dots, N \quad (24)$$

from Equations (23) and (24), it also comes out according to Equation (3) we obtain the following relation:

$$\begin{aligned} \mathbf{X}(x) \mathbf{V}_t^j \mathbf{X}^T(t) &= \mathbf{J}(x) \mathbf{V}_b^j \mathbf{J}^T(t), \quad j = 0, 1, \dots, m_2 \\ \mathbf{X}(x) \mathbf{V}_t^j \mathbf{X}^T(t) &= \mathbf{X}(x) \mathbf{D}^T \mathbf{V}_b^j \mathbf{D} \mathbf{X}^T(t), \quad j = 0, 1, \dots, m_2 \\ \mathbf{V}_t^j &= \mathbf{D}^T \mathbf{V}_b^j \mathbf{D} \quad \text{or} \quad \mathbf{V}_b^j = (\mathbf{D}^T)^{-1} \mathbf{V}_t^j \mathbf{D}^{-1}, \quad j = 0, 1, \dots, m_2 \end{aligned} \quad (25)$$

Finally, in the same way from Equations (12) and (13), convert ${}_aD_x^{\beta_j} u(x)$, $\bar{n}(\beta_j) - 1 < \beta_j \leq \bar{n}(\beta_j)$, i. e. $\bar{n}(\beta_j) = [\beta_j]$, for all $j = 0, 1, \dots, m_2$, by applying the Caputo definition 4 with Lemma 1, 2 and 3 to matrix form, we obtain

$$\begin{aligned} {}_aD_x^{\beta_j} u(x) &= {}_aD_x^{\beta_j} \mathbf{X}(x) \mathbf{D}^T \mathbf{A} \\ &= {}_aJ_x^{(\bar{n}(\beta_j) - \beta_j)} \mathbf{D}^{\bar{n}(\beta_j)} \mathbf{X}(x) \mathbf{D}^T \mathbf{A} \\ &= {}_aJ_x^{(\bar{n}(\beta_j) - \beta_j)} \mathbf{X}(x) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A} \\ &= x^{\bar{n}(\beta_j) - \beta_j} \mathbf{X}(x) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A}, \quad j = 0, 1, \dots, m_2 \end{aligned} \quad (26)$$

where $\mathbf{C}(\bar{n}(\beta_j) - \beta_j)$ define at Equation (11). We obtain the matrix relation (27), put the Equation (3) into (24) and then replace the obtained matrix with matrix (26) in Fredholm integral part I_v in Equation (4)

$$\begin{aligned}
I_v \left(\{\beta_j\}_{j=0}^{m_2}, x \right) &= \sum_{j=0}^{m_2} \bar{\lambda}_j \int_a^x V_j(x, t) {}^c D_t^{\beta_j} u(t) dt \\
&= \sum_{j=0}^{m_2} \bar{\lambda}_j \int_a^x \mathbf{I}(x) \mathbf{V}_b^j \mathbf{I}^T(t) t^{\bar{n}(\beta_j) - \beta_j} \mathbf{X}(t) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A} dt \\
&= \sum_{j=0}^{m_2} \bar{\lambda}_j \int_a^x \mathbf{X}(x) \mathbf{D}^T \mathbf{V}_b^j \mathbf{D} \mathbf{X}^T(t) t^{\bar{n}(\beta_j) - \beta_j} \mathbf{X}(t) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A} dt \\
&= \sum_{j=0}^{m_2} \bar{\lambda}_j \mathbf{X}(x) \mathbf{D}^T \mathbf{V}_b^j \mathbf{D} \left(\int_a^x \mathbf{X}^T(t) \mathbf{X}(t) t^{\bar{n}(\beta_j) - \beta_j} dt \right) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A} \\
&= \sum_{j=0}^{m_2} \bar{\lambda}_j \mathbf{X}(x) \mathbf{D}^T \mathbf{V}_b^j \mathbf{D} \mathbf{H}_{v,j}(x) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A}.
\end{aligned} \tag{27}$$

where

$$\mathbf{H}_{v,j}(x) = \int_a^x \mathbf{X}^T(t) \mathbf{X}(t) t^{\bar{n}(\beta_j) - \beta_j} dt = [h_{rs}^{v,j}(x)], \quad j = 0, 1, \dots, m_2, \quad r, s = 0, 1, \dots, N.$$

$$[h_{rs}^{v,j}(x)] = \frac{x^{\bar{n}(\beta_j) - \beta_j + r + s + 1} - a^{\bar{n}(\beta_j) - \beta_j + r + s + 1}}{\bar{n}(\beta_j) - \beta_j + r + s + 1}, \quad j = 0, 1, \dots, m_2, \quad r, s = 0, 1, \dots, N.$$

We can get the last matrix form (28) by replacing the matrix relation (25) into expression (27).

$$I_v \left(\{\beta_j\}_{j=0}^{m_2}, x \right) = \sum_{j=0}^{m_2} \bar{\lambda}_j \mathbf{X}(x) \mathbf{V}_b^j \mathbf{H}_{v,j}(x) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A}. \tag{28}$$

3.4. Matrix Relation for the Conditions

For each $k = 0, 1, \dots, \mu - 1$ and $\mu = \max\{\lceil \sigma_n \rceil, \lceil \alpha_{m_1} \rceil, \lceil \beta_{m_2} \rceil\}$, applying the relation (10) to each mixed condition of Equation (2), we obtain the corresponding condition matrix forms as follows.

$$\sum_{\ell=0}^{\mu-1} \{ \bar{h}_{k\ell} \mathbf{X}(a) (\mathbf{B}^T)^\ell \mathbf{D}^T \mathbf{A} + \bar{h}_{k\ell} \mathbf{X}(b) (\mathbf{B}^T)^\ell \mathbf{D}^T \mathbf{A} \} = [C_k], \quad k = 0, 1, \dots, \mu - 1$$

Thus

$$\sum_{\ell=0}^{\mu-1} [\bar{h}_{k\ell} \mathbf{X}(a) + \bar{h}_{k\ell} \mathbf{X}(b)] (\mathbf{B}^T)^\ell \mathbf{D}^T \mathbf{A} = [C_k], \quad k = 0, 1, \dots, \mu - 1. \tag{29}$$

4. Method of Solution

To construct the fundamental matrix equation that corresponds to Equation (1), insert the matrix relations (14), (21), and (28) into Equation (4) to obtain the following matrix equation

$$\begin{aligned}
& \left[y(n, x) \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n) (\mathbf{B}^T)^{\bar{n}(\sigma_n)} \mathbf{D}^T + \sum_{l=1}^{n-1} p_l(x) y(n-l, x) \mathbf{X}(x) \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T \right. \\
& \quad \left. + p_n(x) \mathbf{X}(x) \mathbf{D}^T \right] \mathbf{A} \\
& = g(x) + \sum_{i=0}^{m_1} \lambda_i \mathbf{X}(x) \mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} \\
& \quad + \sum_{j=0}^{m_2} \bar{\lambda}_j \mathbf{X}(x) \mathbf{V}_t^j \mathbf{H}_{v,j}(x) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A}.
\end{aligned} \tag{30}$$

We get the following system of equations by setting the collocation points, [28], described by $x_i = a + \frac{b-a}{N}i$, $i = 0, 1, \dots, N$:

$$\begin{aligned}
& \left[y(n, x_i) \mathbf{X}(x_i) \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n) (\mathbf{B}^T)^{\bar{n}(\sigma_n)} \mathbf{D}^T + \sum_{l=1}^{n-1} p_l(x_i) y(n-l, x_i) \mathbf{X}(x_i) \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T \right. \\
& \quad \left. + p_n(x_i) \mathbf{X}(x_i) \mathbf{D}^T \right] \mathbf{A} \\
& = g(x_i) + \sum_{i=0}^{m_1} \lambda_i \mathbf{X}(x_i) \mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T \mathbf{A} \\
& \quad + \sum_{j=0}^{m_2} \bar{\lambda}_j \mathbf{X}(x_i) \mathbf{V}_t^j \mathbf{H}_{v,j}(x_i) \mathbf{C}(\bar{n}(\beta_j) - \beta_j) (\mathbf{B}^T)^{\bar{n}(\beta_j)} \mathbf{D}^T \mathbf{A}, \quad i = 0, 1, \dots, N.
\end{aligned}$$

or in brief, the most important matrix equation is

$$\begin{aligned}
& \left[\mathbf{y}(n) \mathbf{X} \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n) (\mathbf{B}^T)^{\bar{n}(\sigma_n)} \mathbf{D}^T + \sum_{l=1}^{n-1} \mathbf{p}_l \mathbf{y}(n-l) \mathbf{X} \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}) (\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})} \mathbf{D}^T + \mathbf{p}_n \mathbf{X} \mathbf{D}^T \right. \\
& \quad \left. - \sum_{i=0}^{m_1} \lambda_i \mathbf{X} \mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T - \sum_{j=0}^{m_2} \bar{\lambda}_j \bar{\mathbf{X}} \bar{\mathbf{V}}^j \bar{\mathbf{H}}_j \bar{\mathbf{C}}_j \bar{\mathbf{B}}^j \bar{\mathbf{D}} \right] \mathbf{A} = \mathbf{G}.
\end{aligned} \tag{31}$$

where

$$\mathbf{y}(n-l) = \begin{bmatrix} x_0^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} & 0 & \cdots & 0 \\ 0 & x_1^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N^{(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})} \end{bmatrix}, \mathbf{p}_l = \begin{bmatrix} p_l(x_0) & 0 & \cdots & 0 \\ 0 & p_l(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_l(x_N) \end{bmatrix}$$

for all $l = 0, 1, \dots, n$, and $\mathbf{y}(0) = [\mathbf{I}]_{(N+1) \times (N+1)}$, $\mathbf{p}_0 = [\mathbf{I}]_{(N+1) \times (N+1)}$ are the unit matrix,

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}, \quad \bar{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x_N) \end{bmatrix}$$

also, for each fractional order $\gamma = \{\sigma_n, \sigma_{n-l}, \alpha_i \text{ and } \beta_j, \text{ for all } l = 1, 2, \dots, n, i = 0, 1, \dots, m_1, j = 0, 1, \dots, m_2\}$ we are putting

$$\begin{aligned}
\mathbf{C}(\bar{n}(\gamma) - \gamma) &= \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\bar{n}(\gamma) - \gamma + 1)} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\bar{n}(\gamma) - \gamma + 2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(\bar{n}(\gamma) - \gamma + N + 1)} \end{bmatrix} \\
(\mathbf{B}^T)^{\bar{n}(\gamma)} &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\bar{n}(\gamma)}, \quad \mathbf{F}_t^i = \begin{bmatrix} {}_tF_{00}^i & {}_tF_{01}^i & \cdots & {}_tF_{0N}^i \\ {}_tF_{10}^i & {}_tF_{11}^i & \cdots & {}_tF_{1N}^i \\ \vdots & \vdots & \ddots & \vdots \\ {}_tF_{N0}^i & {}_tF_{N1}^i & \cdots & {}_tF_{NN}^i \end{bmatrix}, \quad \mathbf{V}_t^j = \begin{bmatrix} {}_tV_{00}^j & {}_tV_{01}^j & \cdots & {}_tV_{0N}^j \\ {}_tV_{10}^j & {}_tV_{11}^j & \cdots & {}_tV_{1N}^j \\ \vdots & \vdots & \ddots & \vdots \\ {}_tV_{N0}^j & {}_tV_{N1}^j & \cdots & {}_tV_{NN}^j \end{bmatrix} \\
i &= 0, 1, \dots, m_1, j = 0, 1, \dots, m_2
\end{aligned}$$

$$\begin{aligned}
&\text{where} \\
[{}_tF_{mn}^i] &= \frac{1}{m!n!} \frac{\partial^{m+n} F_i(0,0)}{\partial x^m \partial t^n}, \quad i = 0, 1, \dots, m_1, \quad m, n = 0, 1, \dots, N \\
[{}_tV_{mn}^j] &= \frac{1}{m!n!} \frac{\partial^{m+n} V_j(0,0)}{\partial x^m \partial t^n}, \quad j = 0, 1, \dots, m_2, \quad m, n = 0, 1, \dots, N \\
\mathbf{H}_{f,i} &= \begin{bmatrix} h_{00}^{f,i} & h_{01}^{f,i} & \cdots & h_{0N}^{f,i} \\ h_{10}^{f,i} & h_{11}^{f,i} & \cdots & h_{1N}^{f,i} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N0}^{f,i} & h_{N1}^{f,i} & \cdots & h_{NN}^{f,i} \end{bmatrix}, \quad \mathbf{H}_{v,j}(x_i) = \begin{bmatrix} h_{00}^{v,j}(x_i) & h_{01}^{v,j}(x_i) & \cdots & h_{0N}^{v,j}(x_i) \\ h_{10}^{v,j}(x_i) & h_{11}^{v,j}(x_i) & \cdots & h_{1N}^{v,j}(x_i) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N0}^{v,j}(x_i) & h_{N1}^{v,j}(x_i) & \cdots & h_{NN}^{v,j}(x_i) \end{bmatrix}, \quad i = 0, 1, \dots, N
\end{aligned}$$

$$\begin{aligned}
&\text{where, respectively} \\
[h_{rs}^{f,i}] &= \frac{b^{\bar{n}(\alpha_i) - \alpha_i + r + s + 1} - a^{\bar{n}(\alpha_i) - \alpha_i + r + s + 1}}{\bar{n}(\alpha_i) - \alpha_i + r + s + 1}, \quad i = 0, 1, \dots, m_1, \quad r, s = 0, 1, \dots, N \\
[h_{rs}^{v,j}(x_i)] &= \frac{x_i^{\bar{n}(\beta_j) - \beta_j + r + s + 1} - a^{\bar{n}(\beta_j) - \beta_j + r + s + 1}}{\bar{n}(\beta_j) - \beta_j + r + s + 1}, \quad j = 0, 1, \dots, m_2, \quad r, s = 0, 1, \dots, N \\
i &= 0, 1, \dots, N
\end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{V}}^j &= \begin{bmatrix} \mathbf{V}_t^j & 0 & \cdots & 0 \\ 0 & \mathbf{V}_t^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{V}_t^j \end{bmatrix}, \quad \bar{\mathbf{C}}_j = \begin{bmatrix} \mathbf{C}(\bar{n}(\beta_j) - \beta_j) & 0 & \cdots & 0 \\ 0 & \mathbf{C}(\bar{n}(\beta_j) - \beta_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{C}(\bar{n}(\beta_j) - \beta_j) \end{bmatrix} \\
\bar{\mathbf{H}}_j &= \begin{bmatrix} \mathbf{H}_{v,j}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{H}_{v,j}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{v,j}(x_N) \end{bmatrix}, \quad \bar{\mathbf{B}}^j = \begin{bmatrix} (\mathbf{B}^T)^{\bar{n}(\beta_j)} & 0 & \cdots & 0 \\ 0 & (\mathbf{B}^T)^{\bar{n}(\beta_j)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mathbf{B}^T)^{\bar{n}(\beta_j)} \end{bmatrix} \\
j &= 0, 1, \dots, m_2
\end{aligned}$$

$$\bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T \\ \mathbf{D}^T \\ \vdots \\ \mathbf{D}^T \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \quad \text{and } \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

When the matrices

$$\mathbf{y}(n), \mathbf{y}(n-l), \mathbf{p}_n, \mathbf{p}_l, \mathbf{X}, \mathbf{C}(\bar{n}(\sigma_n) - \sigma_n), \mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l}), \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i), \mathbf{C}(\bar{n}(\beta_j) - \beta_j),$$

$$(\mathbf{B}^T)^{\bar{n}(\sigma_n)},$$

$$(\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})}, (\mathbf{B}^T)^{\bar{n}(\alpha_i)}, (\mathbf{B}^T)^{\bar{n}(\beta_j)}, \mathbf{F}_t^i, \mathbf{V}_t^j, \mathbf{H}_{f,i}, \mathbf{H}_{v,j}(x_i) \text{ and } \mathbf{D}^T$$

For all $l = 1, 2, \dots, n-1$, $i = 0, 1, \dots, m_1$, $j = 0, 1, \dots, m_2$, $i = 0, 1, \dots, N$. In Equation (31) we have explained that their dimensions are similar to those of $(N+1) \times (N+1)$. Moreover, in Equation (31), these matrices $\bar{\mathbf{X}}$, $\bar{\mathbf{V}}^j$, $\bar{\mathbf{H}}_j$, $\bar{\mathbf{C}}_j$, $\bar{\mathbf{B}}^j$ and $\bar{\mathbf{D}}$, for all $j = 0, 1, \dots, m_2$ are written in full, their measured dimensions can be observed by $(N+1) \times (N+1)^2$, $(N+1)^2 \times (N+1)^2$, $(N+1)^2 \times (N+1)^2$, $(N+1)^2 \times (N+1)^2$, $(N+1)^2 \times (N+1)^2$, and $(N+1)^2 \times (N+1)$ respectively.

As a result, the fundamental matrix Equation (31) that corresponds to Equation (1) may be expressed as

$$\mathbf{W}\mathbf{A} = \mathbf{G} \text{ or } [\mathbf{W} : \mathbf{G}]. \quad (32)$$

where

$$\begin{aligned} \mathbf{W} = & \mathbf{y}(n)\mathbf{X}\mathbf{C}(\bar{n}(\sigma_n) - \sigma_n)(\mathbf{B}^T)^{\bar{n}(\sigma_n)}\mathbf{D}^T + \sum_{l=1}^{n-1} \mathbf{p}_l \mathbf{y}(n-l)\mathbf{X}\mathbf{C}(\bar{n}(\sigma_{n-l}) - \sigma_{n-l})(\mathbf{B}^T)^{\bar{n}(\sigma_{n-l})}\mathbf{D}^T \\ & + \mathbf{p}_n \mathbf{X} \mathbf{D}^T - \sum_{i=0}^{m_1} \lambda_i \mathbf{X} \mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i)(\mathbf{B}^T)^{\bar{n}(\alpha_i)}\mathbf{D}^T - \sum_{j=0}^{m_2} \bar{\lambda}_j \bar{\mathbf{X}} \bar{\mathbf{V}}^j \bar{\mathbf{H}}_j \bar{\mathbf{C}}_j \bar{\mathbf{B}}^j \bar{\mathbf{D}}. \end{aligned}$$

Note that, Equation (32) is a set of $(N+1)$ linear algebraic equations with unknown Bessel coefficients $\mathbf{A} = [a_0, a_1, \dots, a_N]$. The matrix form (29) for the conditions, on the other hand, may be represented as

$$\mathbf{U}_k \mathbf{A} = [\mathbf{C}_k] \text{ or } [\mathbf{U}_k : \mathbf{C}_k]; \quad k = 0, 1, \dots, \mu-1, \quad \mu = \max\{\sigma_n, \alpha_{m_1}, \beta_{m_2}\}. \quad (33)$$

$$\mathbf{U}_k = \sum_{\ell=0}^{\mu-1} [\hbar_{k\ell} \mathbf{X}(a) + \bar{\hbar}_{k\ell} \mathbf{X}(b)] (\mathbf{B}^T)^\ell \mathbf{D}^T \mathbf{A}.$$

$$= [u_{k0} \ u_{k1} \ u_{k2} \ \dots \ u_{kN}], \quad k = 0, 1, \dots, \mu-1$$

Hence, we may solve Equation (1) under mixed conditions (2) by substituting the rows of the matrices $\mathbf{W}$ and $\mathbf{G}$ for the rows of the matrices $\mathbf{U}_k$ and $\mathbf{C}_k$, respectively.

$$\bar{\mathbf{W}}\mathbf{A} = \bar{\mathbf{G}}.$$

The new augmented matrix (some time may be symmetry) of the preceding system is as follows if the last μ -rows of the matrix (32) are replaced for simplicity:

$$[\widetilde{W} : \widetilde{G}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & : & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & : & g(x_1) \\ w_{20} & w_{21} & w_{22} & \cdots & w_{2N} & : & g(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ w_{N-m,0} & w_{N-m,1} & w_{N-m,2} & \cdots & w_{N-m,N} & : & g(x_{N-m}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & : & c_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & : & c_1 \\ u_{20} & u_{21} & u_{22} & \cdots & u_{2N} & : & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ u_{\mu-1,0} & u_{\mu-1,1} & u_{\mu-1,2} & \cdots & u_{\mu-1,N} & : & c_{\mu-1} \end{bmatrix} \quad (34)$$

Take note that $\text{rank } \widetilde{W} = \text{rank}[\widetilde{W} : \widetilde{G}] = N + 1$. If it isn't, the suggested technique fails to offer a solution; but in this case, in this situation, the number of collocation points (or, equivalently, the dimension of the matrix W) can be increased to get the specific or general answer. As a result, we may write $A = (\widetilde{W})^{-1} \widetilde{G}$, and therefore the elements $a_0, a_1, \dots, a_N$ of A are uniquely determined.

Moreover, select that N we define needs to be greater than μ , i.e., $N > \mu = \max\{\lceil \sigma_n \rceil, \lceil \alpha_{m_1} \rceil, \lceil \beta_{m_2} \rceil\}$. If it is not, the proposed strategy is thus unable to give a solution, because matrix B^T becomes a zero matrix, we only get zero solution.

5. Numerical Examples

In this work, we choose several examples where the exact solution already exists to demonstrate the accuracy. They were all carried out on a computer using a Python program V3.8.8 (2021). The least square errors (L.S.E) in tables are the values of $\sum_{i=0}^M [u(x_i) - \tilde{u}_N(x_i)]^2$, $M \in \mathbb{N}$ at M -selected collocation points x_i , and the running time is also provided in tabular form.

Example 1. Consider the linear Fredholm–Volterra integro-differential equation of multi-higher fractional order, given by

$${}_0^C D_x^{1.2} u(x) + x {}_0^C D_x^{0.1} u(x) - xu(x) = g(x) + \int_0^1 (e^x u(t) - t {}_0^C D_t^{0.9} u(t)) dt + \int_0^x (-(x-t)u(t) + 2x {}_0^C D_t^{1.8} u(t)) dt$$

where $0 \leq x, t \leq 1$

$$g(x) = \frac{-2}{\Gamma(1.8)} x^{0.8} - \frac{2}{\Gamma(2.9)} x^{2.9} + \frac{1}{\Gamma(1.9)} x^{1.9} + \frac{4}{\Gamma(2.2)} x^{2.2} - \frac{1}{6} e^x + \frac{1.1}{\Gamma(3.1)} - \frac{4.2}{\Gamma(4.1)} - x^2 + \frac{7}{6} x^3 - \frac{1}{12} x^4$$

with the boundary conditions

$$2u^{(1)}(0) + u^{(1)}(1) = 1 \quad \text{and} \quad u^{(1)}(1) = -1$$

which is the exact solution $u(x) = x(1-x)$.

Let us now determine the N -truncated Bessel series approximate solution $u_N(x)$

$$u(x) \cong u_N(x) = \sum_{r=0}^N a_r J_r(x)$$

Here, from the considered, example we have:

$$\sigma_0 = 0, \sigma_1 = 0.1, \sigma_2 = 1.2 \rightarrow \bar{n}(\sigma_0) = \lceil \sigma_0 \rceil = 0, \bar{n}(\sigma_1) = \lceil \sigma_1 \rceil = 1, \bar{n}(\sigma_2) = \lceil \sigma_2 \rceil = 2$$

$$\alpha_0 = 0, \alpha_1 = 0.9 \rightarrow \bar{n}(\alpha_0) = \lceil \alpha_0 \rceil = 0, \bar{n}(\alpha_1) = \lceil \alpha_1 \rceil = 1$$

$$\beta_0 = 0, \beta_1 = 1.8 \rightarrow \bar{n}(\beta_0) = \lceil \beta_0 \rceil = 0, \bar{n}(\beta_1) = \lceil \beta_1 \rceil = 2$$

$$\mu = \max\{\lceil 1.2 \rceil, \lceil 0.9 \rceil, \lceil 1.8 \rceil\} = 2$$

$$p_1(x) = x, p_2(x) = -x, F_0(x, t) = e^x, F_1(x, t) = t, V_0(x, t) = x - t, V_1(x, t) = x,$$

$$\lambda_0 = 1, \lambda_1 = -1, \bar{\lambda}_0 = -1, \bar{\lambda}_1 = 2$$

Hence $\mu = 2$ so take $N = 3$, the collocation point sets are $\{x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1\}$, and the fundamental matrix equation of the given (LF-VFIDEs) is derived from Equation (31), written as

$$\left[\mathbf{y}(2) \mathbf{X} \mathbf{C}(\bar{n}(\sigma_2) - \sigma_2) (\mathbf{B}^T)^{\bar{n}(\sigma_2)} \mathbf{D}^T + \mathbf{p}_1 \mathbf{y}(1) \mathbf{X} \mathbf{C}(\bar{n}(\sigma_1) - \sigma_1) (\mathbf{B}^T)^{\bar{n}(\sigma_1)} \mathbf{D}^T + \mathbf{p}_2 \mathbf{X} \mathbf{D}^T \right. \\ \left. - \sum_{i=0}^1 \lambda_i \mathbf{X} \mathbf{F}_t^i \mathbf{H}_{f,i} \mathbf{C}(\bar{n}(\alpha_i) - \alpha_i) (\mathbf{B}^T)^{\bar{n}(\alpha_i)} \mathbf{D}^T - \sum_{j=0}^1 \bar{\lambda}_j \bar{\mathbf{X}} \bar{\mathbf{V}}^j \bar{\mathbf{H}}_j \bar{\mathbf{C}}_j \bar{\mathbf{B}}^j \bar{\mathbf{D}} \right] \mathbf{A} = \mathbf{G}$$

where

$$\mathbf{X}(x) = [1 \ x \ x^2 \ x^3], \mathbf{X} = \begin{bmatrix} \mathbf{X}(0) \\ \mathbf{X}(1/3) \\ \mathbf{X}(2/3) \\ \mathbf{X}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{p}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{y}(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1/3)^{0.8} & 0 & 0 \\ 0 & 0 & (2/3)^{0.8} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{y}(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1/3)^{0.9} & 0 & 0 \\ 0 & 0 & (2/3)^{0.9} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}(\bar{n}(\sigma_2) - \sigma_2) = \begin{bmatrix} \frac{1}{\Gamma(1.8)} & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2.8)} & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3.8)} & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4.8)} \end{bmatrix}, \mathbf{C}(\bar{n}(\sigma_1) - \sigma_1) = \begin{bmatrix} \frac{1}{\Gamma(1.9)} & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2.9)} & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3.9)} & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4.9)} \end{bmatrix},$$

$$\mathbf{C}(\bar{n}(\alpha_1) - \alpha_1) = \begin{bmatrix} \frac{1}{\Gamma(1.1)} & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2.1)} & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3.1)} & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4.1)} \end{bmatrix}, \mathbf{C}(\bar{n}(\beta_1) - \beta_1) = \begin{bmatrix} \frac{1}{\Gamma(1.2)} & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2.2)} & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3.2)} & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4.2)} \end{bmatrix}$$

$$B^T = (B^T)^{\bar{n}(\sigma_1)} = (B^T)^{\bar{n}(\alpha_1)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (B^T)^2 = (B^T)^{\bar{n}(\sigma_2)} = (B^T)^{\bar{n}(\beta_1)} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$, F_t^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}, F_t^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, V_t^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, V_t^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{f,0} = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, H_{f,1} = \begin{bmatrix} \frac{1}{1.1} & \frac{1}{2.1} & \frac{1}{3.1} & \frac{1}{4.1} \\ \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \\ \frac{2.1}{1} & \frac{3.1}{1} & \frac{4.1}{1} & \frac{5.1}{1} \\ \frac{3.1}{1} & \frac{4.1}{1} & \frac{5.1}{1} & \frac{6.1}{1} \\ \frac{1}{4.1} & \frac{1}{5.1} & \frac{1}{6.1} & \frac{1}{7.1} \end{bmatrix}, D^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 \\ 0 & -\frac{1}{16} & 0 & \frac{1}{48} \end{bmatrix},$$

$$\bar{X} = \begin{bmatrix} X(0) & 0 & 0 & 0 \\ 0 & X(1/3) & 0 & 0 \\ 0 & 0 & X(2/3) & 0 \\ 0 & 0 & 0 & X(1) \end{bmatrix}, \bar{V}^0 = \begin{bmatrix} V_t^0 & 0 & 0 & 0 \\ 0 & V_t^0 & 0 & 0 \\ 0 & 0 & V_t^0 & 0 \\ 0 & 0 & 0 & V_t^0 \end{bmatrix}, \bar{V}^1 = \begin{bmatrix} V_t^1 & 0 & 0 & 0 \\ 0 & V_t^1 & 0 & 0 \\ 0 & 0 & V_t^1 & 0 \\ 0 & 0 & 0 & V_t^1 \end{bmatrix}$$

$$\bar{H}_0 = \begin{bmatrix} H_{v,0}(0) & 0 & 0 & 0 \\ 0 & H_{v,0}(1/3) & 0 & 0 \\ 0 & 0 & H_{v,0}(2/3) & 0 \\ 0 & 0 & 0 & H_{v,0}(1) \end{bmatrix}, \bar{H}_1 = \begin{bmatrix} H_{v,1}(0) & 0 & 0 & 0 \\ 0 & H_{v,1}(1/3) & 0 & 0 \\ 0 & 0 & H_{v,1}(2/3) & 0 \\ 0 & 0 & 0 & H_{v,1}(1) \end{bmatrix},$$

$$H_{v,0}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, H_{v,0}(1/3) = \begin{bmatrix} \frac{1}{3} & \frac{(\frac{1}{3})^2}{2} & \frac{(\frac{1}{3})^3}{3} & \frac{(\frac{1}{3})^4}{4} \\ \frac{(\frac{1}{3})^2}{2} & \frac{(\frac{1}{3})^3}{3} & \frac{(\frac{1}{3})^4}{4} & \frac{(\frac{1}{3})^5}{5} \\ \frac{(\frac{1}{3})^3}{3} & \frac{(\frac{1}{3})^4}{4} & \frac{(\frac{1}{3})^5}{5} & \frac{(\frac{1}{3})^6}{6} \\ \frac{(\frac{1}{3})^4}{4} & \frac{(\frac{1}{3})^5}{5} & \frac{(\frac{1}{3})^6}{6} & \frac{(\frac{1}{3})^7}{7} \end{bmatrix}, H_{v,0}(2/3) = \begin{bmatrix} \frac{2}{3} & \frac{(\frac{2}{3})^2}{2} & \frac{(\frac{2}{3})^3}{3} & \frac{(\frac{2}{3})^4}{4} \\ \frac{(\frac{2}{3})^2}{2} & \frac{(\frac{2}{3})^3}{3} & \frac{(\frac{2}{3})^4}{4} & \frac{(\frac{2}{3})^5}{5} \\ \frac{(\frac{2}{3})^3}{3} & \frac{(\frac{2}{3})^4}{4} & \frac{(\frac{2}{3})^5}{5} & \frac{(\frac{2}{3})^6}{6} \\ \frac{(\frac{2}{3})^4}{4} & \frac{(\frac{2}{3})^5}{5} & \frac{(\frac{2}{3})^6}{6} & \frac{(\frac{2}{3})^7}{7} \end{bmatrix},$$

$$H_{v,0}(1) = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, H_{v,1}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, H_{v,1}(1/3) = \begin{bmatrix} \frac{(\frac{1}{3})^{1.2}}{1.2} & \frac{(\frac{1}{3})^{2.2}}{2.2} & \frac{(\frac{1}{3})^{3.2}}{3.2} & \frac{(\frac{1}{3})^{4.2}}{4.2} \\ \frac{(\frac{1}{3})^{2.2}}{2.2} & \frac{(\frac{1}{3})^{3.2}}{3.2} & \frac{(\frac{1}{3})^{4.2}}{4.2} & \frac{(\frac{1}{3})^{5.2}}{5.2} \\ \frac{(\frac{1}{3})^{3.2}}{3.2} & \frac{(\frac{1}{3})^{4.2}}{4.2} & \frac{(\frac{1}{3})^{5.2}}{5.2} & \frac{(\frac{1}{3})^{6.2}}{6.2} \\ \frac{(\frac{1}{3})^{4.2}}{4.2} & \frac{(\frac{1}{3})^{5.2}}{5.2} & \frac{(\frac{1}{3})^{6.2}}{6.2} & \frac{(\frac{1}{3})^{7.2}}{7.2} \end{bmatrix},$$

$$\begin{aligned}
H_{v,1}(2/3) &= \begin{bmatrix} \frac{(\frac{2}{3})^{1.2}}{1.2} & \frac{(\frac{2}{3})^{2.2}}{2.2} & \frac{(\frac{2}{3})^{3.2}}{3.2} & \frac{(\frac{2}{3})^{4.2}}{4.2} \\ \frac{(\frac{2}{3})^{2.2}}{2.2} & \frac{(\frac{2}{3})^{3.2}}{3.2} & \frac{(\frac{2}{3})^{4.2}}{4.2} & \frac{(\frac{2}{3})^{5.2}}{5.2} \\ \frac{(\frac{2}{3})^{3.2}}{3.2} & \frac{(\frac{2}{3})^{4.2}}{4.2} & \frac{(\frac{2}{3})^{5.2}}{5.2} & \frac{(\frac{2}{3})^{6.2}}{6.2} \\ \frac{(\frac{2}{3})^{4.2}}{4.2} & \frac{(\frac{2}{3})^{5.2}}{5.2} & \frac{(\frac{2}{3})^{6.2}}{6.2} & \frac{(\frac{2}{3})^{7.2}}{7.2} \end{bmatrix}, H_{v,1}(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1.2}{1} & \frac{2.2}{1} & \frac{3.2}{1} & \frac{4.2}{1} \\ \frac{2.2}{1} & \frac{3.2}{1} & \frac{4.2}{1} & \frac{5.2}{1} \\ \frac{3.2}{1} & \frac{4.2}{1} & \frac{5.2}{1} & \frac{6.2}{1} \\ \frac{4.2}{1} & \frac{5.2}{1} & \frac{6.2}{1} & \frac{7.2}{1} \end{bmatrix}, \\
\bar{C}_1 &= \begin{bmatrix} C(\bar{n}(\beta_1) - \beta_1) & 0 & 0 & 0 \\ 0 & C(\bar{n}(\beta_1) - \beta_1) & 0 & 0 \\ 0 & 0 & C(\bar{n}(\beta_1) - \beta_1) & 0 \\ 0 & 0 & 0 & C(\bar{n}(\beta_1) - \beta_1) \end{bmatrix}, \\
\bar{B}^1 &= \begin{bmatrix} (B^T)^{\bar{n}(\beta_1)} & 0 & 0 & 0 \\ 0 & (B^T)^{\bar{n}(\beta_1)} & 0 & 0 \\ 0 & 0 & (B^T)^{\bar{n}(\beta_1)} & 0 \\ 0 & 0 & 0 & (B^T)^{\bar{n}(\beta_1)} \end{bmatrix}, G = \begin{bmatrix} -0.28262788 \\ -0.90165393 \\ -0.47693449 \\ 0.94267344 \end{bmatrix} \text{ and } \bar{D} = \begin{bmatrix} D^T \\ D^T \\ D^T \\ D^T \end{bmatrix}
\end{aligned}$$

putting all above matrices in matrix Equation (32) and calculating it, this fundamental matrix equation's augmented matrix is:

$$[W:G] = \begin{bmatrix} -1.07079233 & -0.02572358 & 0.03539616 & 0.00866477 & : & -0.28262788 \\ -1.85498405 & -0.12829141 & 0.09107227 & 0.01393314 & : & -0.90165393 \\ -2.40595015 & -0.22844205 & 0.01161705 & 0.01241051 & : & -0.47693449 \\ -2.7722549 & -0.23879802 & -0.19720588 & -0.02479584 & : & 0.94267344 \end{bmatrix}$$

For our consider example, the boundary conditions from Equation (33) have the following matrix forms:

$$U_k A = [C_k] \text{ or } [U_k: C_k]; \quad k = 0, 1$$

or clearly

$$[U_0: C_0] = [-0.5 \quad 1.3125 \quad 0.25 \quad 0.0625 \quad : \quad 1]$$

$$[U_1: C_1] = [-0.5 \quad 0.3125 \quad 0.25 \quad 0.0625 \quad : \quad -1]$$

The new augmented matrix depending on conditions is constructed as follows from the system (34):

$$[\tilde{W}: \tilde{G}] = \begin{bmatrix} -1.07079233 & -0.02572358 & 0.03539616 & 0.00866477 & : & -0.28262788 \\ -1.85498405 & -0.12829141 & 0.09107227 & 0.01393314 & : & -0.90165393 \\ -0.5 & 1.3125 & 0.25 & 0.0625 & : & 1 \\ -0.5 & 0.3125 & 0.25 & 0.0625 & : & -1 \end{bmatrix}$$

The Bessel coefficient matrix A is obtained by solving this system.

$$A = [-1.98357176 \times 10^{-06} \quad 2.00000000 \quad -8.00269502 \quad 6.01076421]^T$$

hence, for $N = 3$ the approximate solution of the problem is formed as

$$u_3(x) = 0.0002242544x^3 - 1.0003363816x^2 + 1.0x - 0.0000019836$$

for $N = 6$ and $N = 10$, similarly as steps above and running the general python program which are written for this purpose we obtain the approximate solution of the problem, respectively

$$u_6(x) = -0.0003537963x^6 + 0.0007932088x^5 - 0.0006209581x^4 + 0.000229553x^3 \\ - 1.0000240462x^2 + 1.0x + 0.0000090551$$

and

$$u_{10}(x) = 0.0000000661x^{10} - 0.0000002889x^9 + 0.00000052947x^8 - 0.00000053514x^7 \\ + 0.000000328x^6 - 0.00000012512x^5 + 0.0000000293x^4 \\ - 0.00000000370x^3 - 0.999999999656939x^2 + 1.0x + 0.00000000015$$

In Table 1, Comparison the exact solution $u(x)$ with the approximate solution $u_N(x)$ of example 1 for $N = 3, 6$ and 10 , respectively, in terms of least square error and running time.

Table 1. Compares the exact solution $u(x)$ with the approximate solution $u_N(x)$ of example 1.

x_i	Exact Solution Example 1.	N-Approximate Solution $u_N(x)$		
		$N = 3$	$N = 6$	$N = 10$
0.0	0.00	$-1.9835710 \times 10^{-06}$	$9.05511829 \times 10^{-06}$	$1.53060853 \times 10^{-10}$
0.1	0.09	$8.99948769 \times 10^{-02}$	$9.00089897 \times 10^{-02}$	$9.00000002 \times 10^{-02}$
0.2	0.16	$1.59986355 \times 10^{-01}$	$1.60009167 \times 10^{-01}$	$1.60000000 \times 10^{-01}$
0.3	0.21	$2.09973797 \times 10^{-01}$	$2.10009729 \times 10^{-01}$	$2.10000000 \times 10^{-01}$
0.4	0.24	$2.39958548 \times 10^{-01}$	$2.40010676 \times 10^{-01}$	$2.40000000 \times 10^{-01}$
0.5	0.25	$2.49941953 \times 10^{-01}$	$2.50012188 \times 10^{-01}$	$2.50000000 \times 10^{-01}$
0.6	0.24	$2.39925358 \times 10^{-01}$	$2.40014679 \times 10^{-01}$	$2.40000000 \times 10^{-01}$
0.7	0.21	$2.09910109 \times 10^{-01}$	$2.10018608 \times 10^{-01}$	$2.10000000 \times 10^{-01}$
0.8	0.16	$1.59897550 \times 10^{-01}$	$1.60024025 \times 10^{-01}$	$1.60000000 \times 10^{-01}$
0.9	0.09	$8.98890288 \times 10^{-02}$	$9.00298712 \times 10^{-02}$	$9.00000005 \times 10^{-02}$
1.0	0.00	$-1.141107 \times 10^{-04}$	$3.30162295 \times 10^{-05}$	$5.74892312 \times 10^{-10}$
L.S.E.		$5.5474382 \times 10^{-08}$	$3.72530728 \times 10^{-09}$	$1.13502895 \times 10^{-18}$
Running Time/Sec		0.47589683	0.227055311	0.57032561

Example 2. Let us now consider the LF-VIFDEs on the closed bounded interval $[0,1]$ given by

$${}_0^C D_x^{1.3} u(x) + \frac{x}{2} {}_0^C D_x^{0.8} u(x) + \sqrt{x} {}_0^C D_x^{0.5} u(x) + (x^2 + 1)u(x) \\ = g(x) + \int_0^1 ((\sin(x) - t) {}_0^C D_t^{0.9} u(t) + 2(e^x - t) {}_0^C D_t^{1.1} u(t)) dt + 3 \int_0^x t \sinh(x) {}_0^C D_t^{1.9} u(t) dt$$

where

$$g(x) = \frac{2}{\Gamma(1.7)} x^{0.7} + \frac{1}{\Gamma(2.2)} x^{2.2} + \frac{1}{\Gamma(1.2)} x^{1.2} + \frac{2}{\Gamma(2.5)} x^{1.5} \sqrt{x} + \frac{2}{\Gamma(1.5)} x + (x^2 + 1)(x + 1)^2 \\ - \left(\frac{2}{\Gamma(3.1)} + \frac{2}{\Gamma(2.1)} \right) \sin(x) - \frac{4}{\Gamma(2.9)} e^x - \frac{6.6}{\Gamma(3.1)} x^{2.1} \sinh(x) + \left(\frac{4.2}{\Gamma(4.1)} + \frac{2.2}{\Gamma(3.1)} + \frac{7.6}{\Gamma(3.9)} \right)$$

with the boundary conditions:

$$u(0) + u^{(1)}(1) = 5 \text{ and } u(1) + u^{(1)}(0) = 6$$

The exact solution is $u(x) = (x + 1)^2$.

Let us now calculate the coefficients $a_r, r = \overline{0:N}$ of approximate solution with the aid of the truncated Bessel series:

$$u(x) \cong u_N(x) = \sum_{r=0}^N a_r J_r(x)$$

Here, from the considered example we have:

$$\sigma_1 = 0.5, \sigma_2 = 0.8, \sigma_3 = 1.3 \rightarrow \bar{n}(\sigma_1) = [\sigma_1] = 1, \bar{n}(\sigma_2) = [\sigma_2] = 1, \bar{n}(\sigma_3) = [\sigma_3] = 2$$

$$\alpha_1 = 0.9, \alpha_2 = 1.1 \rightarrow \bar{n}(\alpha_1) = [\alpha_1] = 1, \bar{n}(\alpha_2) = [\alpha_2] = 2$$

$$\beta_1 = 1.9 \rightarrow \bar{n}(\beta_1) = [\beta_1] = 2, \mu = \max\{[1.3], [1.1], [1.9]\} = 2 \text{ and } \lambda_1 = 1, \lambda_2 = 2, \bar{\lambda}_1 = 3$$

$$p_1(x) = \frac{x}{2}, p_2(x) = \sqrt{x}, p_3(x) = (x^2 + 1), F_1(x, t) = \sin(x) - t, F_2(x, t) = (e^x - t), V_1(x, t) = t \sinh(x)$$

Hence $\mu = 2$ so take $N = 3$, the set of collocation points, and the fundamental matrix equation of the given (LF-VFIDEs) is derived from Equation (31), written as

$$\begin{aligned} & [\mathbf{y}(3)\mathbf{X}\mathbf{C}(\bar{n}(\sigma_3) - \sigma_3)(\mathbf{B}^T)^{\bar{n}(\sigma_3)}\mathbf{D}^T + \mathbf{p}_1\mathbf{y}(2)\mathbf{X}\mathbf{C}(\bar{n}(\sigma_2) - \sigma_2)(\mathbf{B}^T)^{\bar{n}(\sigma_2)}\mathbf{D}^T \\ & + \mathbf{p}_2\mathbf{y}(1)\mathbf{X}\mathbf{C}(\bar{n}(\sigma_1) - \sigma_1)(\mathbf{B}^T)^{\bar{n}(\sigma_1)}\mathbf{D}^T + \mathbf{p}_3\mathbf{X}\mathbf{D}^T \\ & - \lambda_1\mathbf{X}\mathbf{F}_t^1\mathbf{H}_{f,1}\mathbf{C}(\bar{n}(\alpha_1) - \alpha_1)(\mathbf{B}^T)^{\bar{n}(\alpha_1)}\mathbf{D}^T \\ & - \lambda_2\mathbf{X}\mathbf{F}_t^2\mathbf{H}_{f,2}\mathbf{C}(\bar{n}(\alpha_2) - \alpha_2)(\mathbf{B}^T)^{\bar{n}(\alpha_2)}\mathbf{D}^T - \bar{\lambda}_1\bar{\mathbf{X}}\bar{\mathbf{V}}^1\bar{\mathbf{H}}_1\bar{\mathbf{C}}_1\bar{\mathbf{B}}^1\bar{\mathbf{D}}] \mathbf{A} = \mathbf{G} \end{aligned}$$

After inputting each of the parameters above by running the general python program, which are written for this purpose for $N = 3$, we obtain the approximate solution of the problem,

$$u_3(x) = 0.0029061023x^3 + 0.99568824199x^2 + 2.0015004466x + 0.9984047624$$

Similarly, the approximate solution of the problem for $N = 6$ and 11, respectively, we obtain

$$\begin{aligned} u_6(x) = & 4.353138854 \times 10^{-7}x^6 - 4.54065912 \times 10^{-5}x^5 + 5.511176789 \times 10^{-5}x^4 \\ & - 1.135105524 \times 10^{-5}x^3 + 1.0000204x^2 + 1.999983583x \\ & + 1.000013642 \end{aligned}$$

and

$$\begin{aligned} u_{11}(x) = & 3.2299188319 \times 10^{-7}x^{11} - 1.5377530537 \times 10^{-6}x^{10} \\ & + 3.1687057159 \times 10^{-6}x^9 - 3.716458436 \times 10^{-6}x^8 \\ & + 2.73904171 \times 10^{-6}x^7 - 1.319310342 \times 10^{-6}x^6 \\ & + 4.181528996 \times 10^{-7}x^5 - 8.549654914 \times 10^{-8}x^4 \\ & + 1.082189316 \times 10^{-8}x^3 + 0.9999999996x^2 + 1.9999999996x \\ & + 1.0000000003. \end{aligned}$$

In Table 2 comparison in terms of least square error and running time the exact solution $u(x)$ with the approximate solution $u_N(x)$ of example 2 for $N = 3, 6$ and 11, respectively.

Table 2. Compares the exact solution $u(x)$ with the approximate solution $u_N(x)$ of example 2.

x_i	Exact Solution	N-Approximate Solutions $u_N(x)$		
	Example 2.	$N = 3$	$N = 6$	$N = 11$
0.0	1.00	0.99840476	1.00001364	1.00
0.1	1.21	1.2085146	1.2100122	1.21
0.2	1.44	1.43855563	1.44001116	1.44
0.3	1.69	1.6885453	1.69001058	1.69
0.4	1.96	1.95850105	1.96001056	1.96
0.5	2.25	2.24844031	2.25001115	2.25
0.6	2.56	2.55838052	2.56001232	2.56
0.7	2.89	2.88833911	2.89001391	2.89
0.8	3.24	3.23833352	3.24001556	3.24
0.9	3.61	3.60838119	3.6100167	3.61
1.0	4.00	3.99849955	4.00001642	4.00
	L.S.E.	$2.66633874 \times 10^{-05}$	$1.94276005 \times 10^{-09}$	$7.33991398 \times 10^{-19}$
	Running Time/Sec	0.449205636	0.2821376323	0.6396300792

Figures 1 and 2 illustrate a comparison between the exact solution and approximate solution of LF-VIFDEs of Examples 1 and 2, respectively. To show the result of the proposed method to an exact solution, we present Tables 1 and 2, respectively. Each of the plots is drawn with our Python program version 3.8.8 (2021).

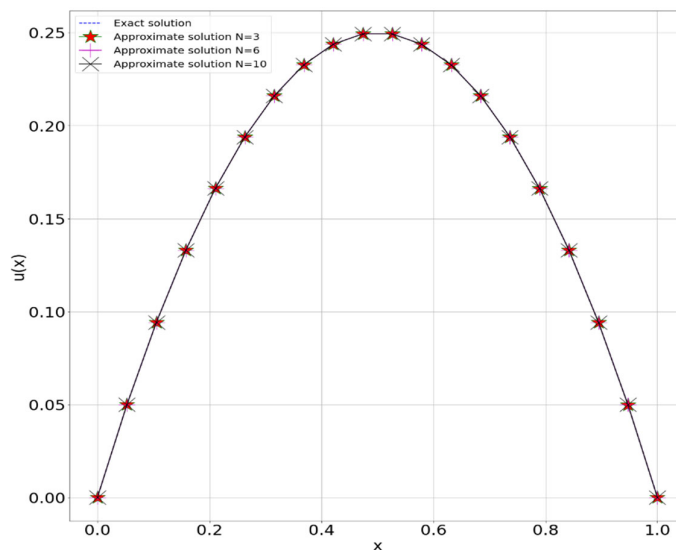


Figure 1. Comparison of the exact and approximate solution of example 1.

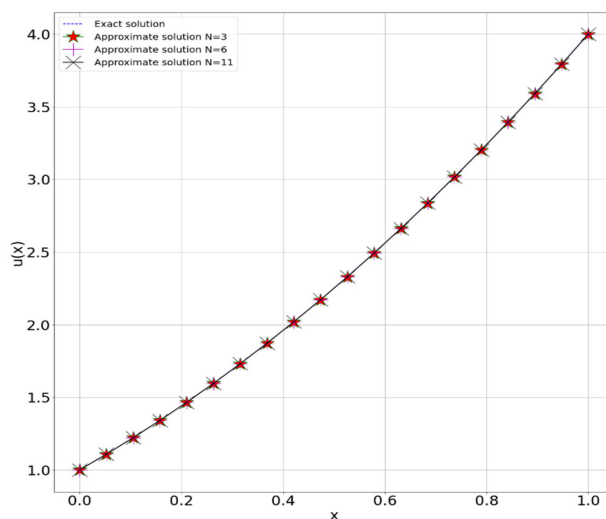


Figure 2. Comparison of the exact and approximate solution of example 2.

Example 3. Let us consider the linear Fredholm–Volterra fractional integro-differential equation on the closed bounded interval $[0,1]$:

$${}_0^C D_x^{0.73} u(x) = g(x) + \int_0^1 \left(t \sin\left(\frac{x}{2}\right) {}_0^C D_t^{0.64} u(t) + 2\left(x - \frac{t}{2}\right) {}_0^C D_t^{1.64} u(t) \right) dt + \int_0^x \left(-e^x {}_0^C D_t^{0.5} u(t) + (-2)te^x {}_0^C D_t^{1.5} u(t) \right) dt$$

where

$$g(x) = \sum_{i=0}^{\infty} \left[\frac{2x}{\Gamma(i+2.36)} - \frac{x^{i+0.27}}{\Gamma(i+1.27)} + \frac{\sin\left(\frac{x}{2}\right)(i+1.36)}{\Gamma(i+3.36)} - \frac{(i+1.36)}{\Gamma(i+3.36)} - \frac{e^x x^{i+1.5}}{\Gamma(i+2.5)} - \frac{2(i+1.5)e^x x^{i+2.5}}{\Gamma(i+3.5)} \right]$$

with the boundary conditions

$$u(0) + u^{(1)}(0) = -1 \quad \text{and} \quad u(1) - u^{(1)}(1) = 1$$

which is the exact solution $u(x) = 1 - e^x$.

Let us now calculate the coefficients $a_r, r = \overline{0:N}$ of approximate solution with the aid of the truncated Bessel series:

$$u(x) \cong u_N(x) = \sum_{r=0}^N a_r J_r(x)$$

Here, from consider example we have:

$$\sigma_1 = 0.73 \rightarrow \bar{n}(\sigma_1) = [\sigma_1] = 1$$

$$\alpha_1 = 0.64 \rightarrow \bar{n}(\alpha_1) = [\alpha_1] = 1, \quad \alpha_2 = 1.64 \rightarrow \bar{n}(\alpha_2) = [\alpha_2] = 2$$

$$\beta_1 = 0.5 \rightarrow \bar{n}(\beta_1) = [\beta_1] = 1, \quad \beta_2 = 1.5 \rightarrow \bar{n}(\beta_2) = [\beta_2] = 2$$

$$\mu = \max \{[\sigma_1], [\alpha_2], [\beta_2]\} = \max\{[0.73], [1.64], [1.5]\} = 2$$

$$p_1(x) = 0, F_1(x, t) = t \sin\left(\frac{x}{2}\right), F_2(x, t) = \left(x - \frac{t}{2}\right), V_1(x, t) = e^x, V_2(x, t) = te^x \quad \text{and} \quad \lambda_1 = 1, \lambda_2 = 1, \bar{\lambda}_1 = -1, \bar{\lambda}_2 = -2.$$

Suppose that, we take $\bar{N}$ terms from the homogeneous part $g(x)$:

$$g(x) = \sum_{i=0}^{\bar{N}} \left[\frac{2x}{\Gamma(i+2.36)} - \frac{x^{i+0.27}}{\Gamma(i+1.27)} + \frac{\sin\left(\frac{x}{2}\right)(i+1.36)}{\Gamma(i+3.36)} - \frac{(i+1.36)}{\Gamma(i+3.36)} - \frac{e^x x^{i+1.5}}{\Gamma(i+2.5)} - \frac{2(i+1.5)e^x x^{i+2.5}}{\Gamma(i+3.5)} \right]$$

Hence $\mu = 2$, the fundamental matrix equation of the given (LF-VFIDEs) is derived from Equation (31), written as

$$\begin{aligned} & [y(1) X C(\bar{n}(\sigma_1) - \sigma_1) (B^T)^{\bar{n}(\sigma_1)} D^T - \lambda_1 X F_t^1 H_{f,1} C(\bar{n}(\alpha_1) - \alpha_1) (B^T)^{\bar{n}(\alpha_1)} D^T \\ & - \lambda_2 X F_t^2 H_{f,2} C(\bar{n}(\alpha_2) - \alpha_2) (B^T)^{\bar{n}(\alpha_2)} D^T - \bar{\lambda}_1 \bar{X} \bar{V}^1 \bar{H}_1 \bar{C}_1 \bar{B}^1 \bar{D} - \bar{\lambda}_2 \bar{X} \bar{V}^2 \bar{H}_2 \bar{C}_2 \bar{B}^2 \bar{D}] A = G \end{aligned}$$

We choose if $\bar{N} = 5$, the approximate solution of the problem for $N = 4, 10, 21$, respectively

$$u_4(x) = -0.0569736258x^4 - 0.1665069441x^3 - 0.49426610634x^2 - 1.0017991281x + 0.001799128086$$

$$\begin{aligned} u_{10}(x) = & -0.0112791176x^{10} + 0.04361473251x^9 - 0.07167515469x^8 + 0.06498524957x^7 \\ & - 0.03711020557x^6 + 0.00378578712x^5 - 0.0441096211x^4 \\ & - 0.1664357675x^3 - 0.4999839118x^2 - 0.999999027x \\ & - 9.72893359856624 \times 10^{-7}, \end{aligned}$$

and

$$\begin{aligned} u_{21}(x) = & 2649.390397x^{21} - 26487.723779x^{20} + 123552.0169x^{19} - 357222.928001x^{18} \\ & + 717359.135397x^{17} - 1062540.52114x^{16} + 1203203.42788x^{15} \\ & - 1065423.12951x^{14} + 748326.26838x^{13} - 420462.50002x^{12} \\ & + 189747.93897x^{11} - 68792.817648x^{10} + 19970.45119x^9 - 4609.812041x^8 \\ & + 836.75503981x^7 - 117.510711061x^6 + 12.4665060412x^5 \\ & - 1.0098682157x^4 - 0.11440971986x^3 - 0.50180747163x^2 \\ & - 0.99996814632x - 3.1859671582 \times 10^{-5} \end{aligned}$$

we choose if $\bar{N} = 10$, the approximate solution of the problem for $N = 4, 10, 21$, respectively

$$u_4(x) = -0.05759757087x^4 - 0.1652553696x^3 - 0.4950425756x^2 - 1.001653973x + 0.001653973$$

$$u_{10}(x) = 3.399204141e - 6x^{10} - 2.0679446987e - 5x^9 + 1.254303583e - 5x^8 \\ - 0.000242474208x^7 - 0.001356395456x^6 - 0.008348879766x^5 \\ - 0.0416618228x^4 - 0.166667629x^3 - 0.4999998846x^2 \\ - 1.0000000056x + 5.656145543 \times 10^{-9},$$

and

$$u_{21}(x) = 0.0366737297x^{21} - 0.36713669x^{20} + 1.714854990x^{19} - 4.9651991545x^{18} \\ + 9.9858420757x^{17} - 14.8143137167x^{16} + 16.8038086x^{15} \\ - 14.90654178x^{14} + 10.49049965x^{13} - 5.9069139812x^{12} \\ + 2.6719828271x^{11} - 0.97127108855x^{10} + 0.282792909x^9 \\ - 0.0655249586x^8 + 0.0117379624x^7 - 0.00307302784x^6 \\ - 0.008153528584x^5 - 0.041680721487x^4 - 0.1666659001x^3 \\ - 0.5000000273x^2 - 0.9999999995x - 4.670435338 \times 10^{-10}$$

Similarly doing it for $\bar{N} = 16$, the approximate solution of the problem for $N = 4, 10, 21$, respectively

$$u_4(x) = -0.057597577x^4 - 0.165255357x^3 - 0.495042583x^2 - 1.001653971x \\ + 0.0016539713$$

$$u_{10}(x) = 3.4214653141e - 6x^{10} - 2.0612514452e - 5x^9 + 1.2077128326e - 5x^8 \\ - 0.000241586656x^7 - 0.0013572657x^6 - 0.0083483752x^5 \\ - 0.0416620032x^4 - 0.1666675894x^3 - 0.499999889x^2 \\ - 1.000000005x + 5.395456491 \times 10^{-9}$$

and

$$u_{21}(x) = -0.000949182285x^{21} + 0.00893091688x^{20} - 0.03880287140x^{19} \\ + 0.103101802x^{18} - 0.186859983473284x^{17} + 0.24354820561x^{16} \\ - 0.2337645213x^{15} + 0.1652048762x^{14} - 0.08287855594x^{13} \\ + 0.0253870484x^{12} - 0.0005131248x^{11} - 0.0043926365x^{10} \\ + 0.00281921684x^9 - 0.0010956089x^8 + 8.623002860e - 5x^7 \\ - 0.00144403537x^6 - 0.00832553568x^5 - 0.0416674549x^4 - 0.16666661x^3 \\ - 0.500000002x^2 - 0.9999999995x - 4.983565034 \times 10^{-11}.$$

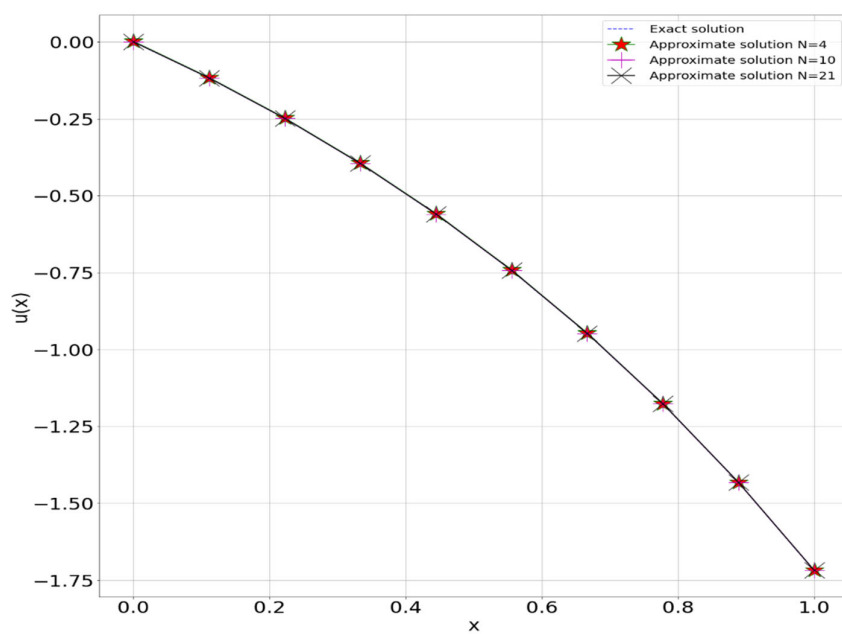
In Table 3 presents a comparison between the exact solution $u(x)$ and approximate solution $u_N(x)$, when we choose $\bar{N} = 5, 10$, and 16 , respectively. For each of them we chose $N = 4, 10$, and 21 , respectively depending on the least square error and running time.

Table 3. Comparison between the exact solution $u(x)$ and approximate solution $u_N(x)$ for example 3.

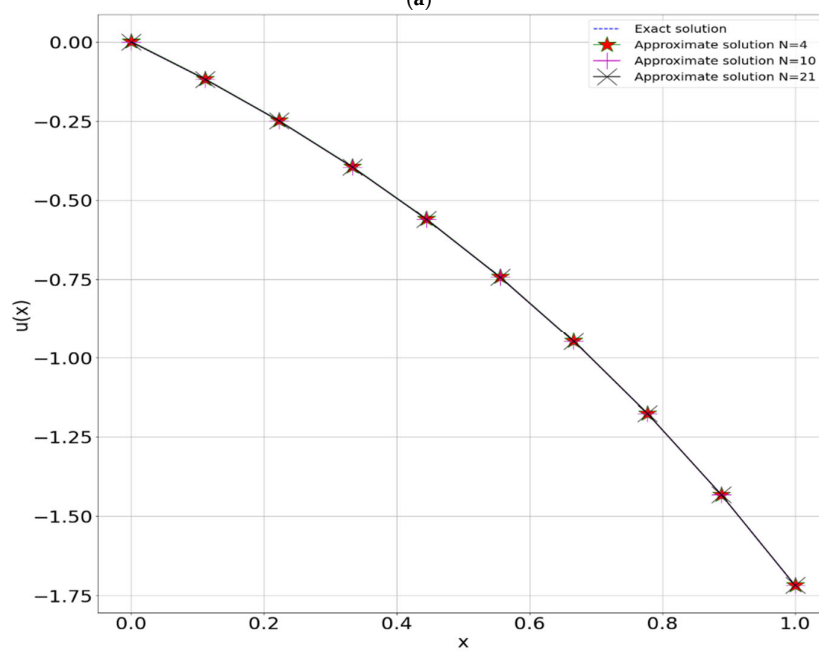
(a)				
x_i	Exact Solution Example 3.	$\bar{N} = 5$		
		N -Approximate Solution $u_N(x)$		
		$N = 4$	$N = 10$	$N = 21$
0.0	0.0	0.00179913	$-9.72893360 \times 10^{-7}$	$-3.1859672 \times 10^{-5}$
0.1	-0.10517092	-0.10349565	-0.105171555	-0.10520278
0.2	-0.22140276	-0.21975456	-0.221402690	-0.22143482
0.3	-0.34985881	-0.34818173	-0.349857839	-0.34989091
0.4	-0.4918247	-0.49011807	-0.491822593	-0.49185648
0.5	-0.64872127	-0.64704118	-0.648717441	-0.64875193
0.6	-0.8221188	-0.82056543	-0.822111722	-0.82214657
0.7	-1.01375271	-1.0124419	-1.01373901	-1.01377394
0.8	-1.22554093	-1.22455843	-1.22551392	-1.22554873

0.9	−1.45960311	−1.45893959	−1.45955258	−1.45958517
1.0	−1.71828183	−1.71774668	−1.71820801	−1.71823455
L.S.E.		$2.3130923 \times 10^{-05}$	$8.99107006 \times 10^{-09}$	$9.87875607 \times 10^{-09}$
Running Time/Sec		0.171678066	0.49988222	2.92118477
(b)				
x_i	Exact Solution Example 3.	$\bar{N} = 10$		
		N -Approximate Solution $u_N(x)$		
		$N = 4$	$N = 10$	$N = 21$
0	0.0	$1.65397131 \times 10^{-03}$	$5.65614554 \times 10^{-09}$	$-4.67043534 \times 10^{-10}$
0.1	−0.10517092	$-1.03632867 \times 10^{-01}$	$-1.05170912 \times 10^{-01}$	$-1.05170919 \times 10^{-01}$
0.2	−0.22140276	$-2.19892725 \times 10^{-01}$	$-2.21402752 \times 10^{-01}$	$-2.21402759 \times 10^{-01}$
0.3	−0.34985881	$-3.48324488 \times 10^{-01}$	$-3.49858802 \times 10^{-01}$	$-3.49858808 \times 10^{-01}$
0.4	−0.4918247	$-4.90265271 \times 10^{-01}$	$-4.91824692 \times 10^{-01}$	$-4.91824698 \times 10^{-01}$
0.5	−0.64872127	$-6.47190428 \times 10^{-01}$	$-6.48721264 \times 10^{-01}$	$-6.48721271 \times 10^{-01}$
0.6	−0.8221188	$-8.20713545 \times 10^{-01}$	$-8.22118794 \times 10^{-01}$	$-8.22118801 \times 10^{-01}$
0.7	−1.01375271	−1.01258644	−1.01375270	−1.01375271
0.8	−1.22554093	−1.22469917	−1.22554092	−1.22554093
0.9	−1.45960311	−1.45908002	−1.45960311	−1.45960311
1	−1.71828183	−1.71789552	−1.71828182	−1.71828183
L.S.E.		$1.89772271 \times 10^{-05}$	$3.87412311 \times 10^{-16}$	$2.32392057 \times 10^{-18}$
Running Time/Sec		0.187295436	0.515326499	2.749377012
(c)				
x_i	Exact Solution Example 3.	$\bar{N} = 16$		
		N -Approximate Solution $u_N(x)$		
		$N = 4$	$N = 10$	$N = 21$
0.0	0.	$1.65397131 \times 10^{-03}$	$5.39545649 \times 10^{-09}$	$-4.98356503 \times 10^{-11}$
0.1	−0.10517092	$-1.03632867 \times 10^{-01}$	$-1.05170913 \times 10^{-01}$	$-1.05170918 \times 10^{-01}$
0.2	−0.22140276	$-2.19892725 \times 10^{-01}$	$-2.21402753 \times 10^{-01}$	$-2.21402758 \times 10^{-01}$
0.3	−0.34985881	$-3.48324488 \times 10^{-01}$	$-3.49858802 \times 10^{-01}$	$-3.49858808 \times 10^{-01}$
0.4	−0.4918247	$-4.90265271 \times 10^{-01}$	$-4.91824692 \times 10^{-01}$	$-4.91824698 \times 10^{-01}$
0.5	−0.64872127	$-6.47190428 \times 10^{-01}$	$-6.48721265 \times 10^{-01}$	$-6.48721271 \times 10^{-01}$
0.6	−0.8221188	$-8.20713545 \times 10^{-01}$	$-8.22118794 \times 10^{-01}$	$-8.22118800 \times 10^{-01}$
0.7	−1.01375271	−1.01258644	−1.01375270	−1.01375271
0.8	−1.22554093	−1.22469917	−1.22554092	−1.22554093
0.9	−1.45960311	−1.45908002	−1.45960311	−1.45960311
1.0	−1.71828183	−1.71789552	−1.71828182	−1.71828183
L.S.E.		$1.89771908 \times 10^{-05}$	$3.4403856 \times 10^{-16}$	$3.45079782 \times 10^{-20}$
Running Time/Sec		0.203105688	0.5199816226	2.90862059

Figure 3a–c illustrates a comparison between the exact solution and approximate solution of (LF-VIFDEs) of equation above, respectively. To show the result of the proposed method to an exact solution, we present Table 3, respectively. Each of the plots is drawn with our Python program version 3.8.8 (2021).



(a)



(b)

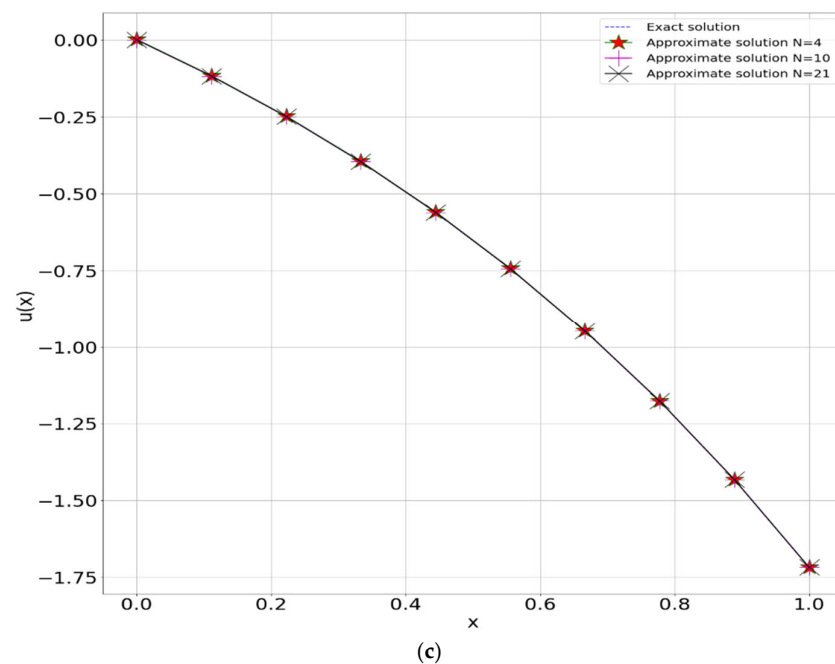


Figure 3. (a) Comparison of the exact and approximate solution, when $\bar{N} = 5$. (b) Comparison of the exact and approximate solution, when $\bar{N} = 10$. (c) Comparison of the exact and approximate solution, when $\bar{N} = 16$.

Example 4. Suppose that the following linear Fredholm–Volterra fractional integro-differential equation given by

$${}_0^C D_x^{0.8} u(x) = g(x) + \int_0^2 \left(\frac{1}{2} (t-2) {}_0^C D_t^{0.3} u(t) + (t + \cos(x)) {}_0^C D_t^{1.3} u(t) \right) dt \\ + \int_0^x \left(\frac{1}{4} \left(\frac{x-t}{2} \right) u(t) + [\tan(x)t] {}_0^C D_t^{2.7} u(t) \right) dt, \quad 0 \leq x, t \leq 2$$

where

$$g(x) = \frac{2}{\Gamma(2.2)} x^{1.2} - \frac{1}{\Gamma(1.2)} x^{0.2} - \frac{(2.7)}{\Gamma(4.7)} 2^{3.7} - \frac{1}{\Gamma(3.7)} ((1.7)2^{3.7} - 2^{3.7} - (1.7)2^{1.7}) \\ - \frac{1}{\Gamma(2.7)} (2^{1.7} + 2^{2.7} \cos(x)) - \frac{1}{16} \left(\frac{1}{6} x^4 - \frac{1}{3} x^3 + x^2 \right)$$

with the boundary conditions

$$u(0) + u(2) = 4, \quad u^{(1)}(0) + u^{(1)}(2) = 2 \quad \text{and} \quad u^{(2)}(0) + u^{(2)}(2) = 4$$

which is the exact solution $u(x) = x^2 - x + 1$.

Now let us find the approximate solution given by the N -truncated Bessel series

$$u(x) \cong u_N(x) = \sum_{r=0}^N a_r J_r(x)$$

Here, from consider example we have:

$$\sigma_1 = 0.8 \rightarrow \bar{n}(\sigma_1) = [\sigma_1] = 1$$

$$\alpha_1 = 0.3, \alpha_2 = 1.3 \rightarrow \bar{n}(\alpha_1) = [\alpha_1] = 1, \bar{n}(\alpha_2) = [\alpha_2] = 2$$

$$\beta_0 = 0, \beta_1 = 2.7 \rightarrow \bar{n}(\beta_0) = [\beta_0] = 0, \bar{n}(\beta_1) = [\beta_1] = 3$$

$$\mu = \max\{[0.8], [1.3], [2.7]\} = 3, \quad \lambda_1 = \frac{1}{2}, \lambda_2 = 1, \quad \bar{\lambda}_0 = \frac{1}{4}, \bar{\lambda}_1 = 1$$

and $p_1(x) = 0, F_1(x, t) = t - 2, F_2(x, t) = t + \cos(x), V_0(x, t) = \left(\frac{x-t}{2}\right), V_1(x, t) = \tan(x) t$,
from Equation (31), the fundamental matrix equation of the given problem is written as

$$\begin{aligned} & [y(1) X C(\bar{n}(\sigma_1) - \sigma_1) (B^T)^{\bar{n}(\sigma_1)} D^T - \lambda_1 X F_t^1 H_{f,1} C(\bar{n}(\alpha_1) - \alpha_1) (B^T)^{\bar{n}(\alpha_1)} D^T \\ & - \lambda_2 X F_t^2 H_{f,2} C(\bar{n}(\alpha_2) - \alpha_2) (B^T)^{\bar{n}(\alpha_2)} D^T - \bar{\lambda}_0 \bar{X} \bar{V}^0 \bar{H}_0 \bar{D} - \bar{\lambda}_1 \bar{X} \bar{V}^1 \bar{H}_1 \bar{C}_1 \bar{B}^1 \bar{D}] A = G \end{aligned}$$

Thus, the approximate solution of the problem for $N = 5, 9, 12$, respectively

$$\begin{aligned} u_5(x) = & 0.00595039974990515x^5 - 0.0295085156834566x^4 + 0.0436273982107945x^3 \\ & + 0.98520400357289x^2 - 0.998052135471447x + 0.99399626495166, \end{aligned}$$

$$\begin{aligned} u_9(x) = & -6.93792068979952 \times 10^{-7}x^9 + 6.12616976553793 \times 10^{-6}x^8 - 2.13390478665744 \times 10^{-5}x^7 \\ & + 3.69079850914571 \times 10^{-5}x^6 - 3.01713615634192 \times 10^{-5}x^5 \\ & + 1.16291867668927 \times 10^{-6}x^4 + 1.66943823538962 \times 10^{-5}x^3 \\ & + 0.999993226980557x^2 - 0.999998942794685x + 0.99999725431996, \end{aligned}$$

and

$$\begin{aligned} u_{12}(x) = & -5.96684668513465 \times 10^{-10}x^{12} + 6.51569719257085 \times 10^{-9}x^{11} \\ & - 3.10471674650442 \times 10^{-8}x^{10} + 8.49087433427835 \times 10^{-8}x^9 \\ & - 1.47156926969634 \times 10^{-7}x^8 + 1.67915537193858 \times 10^{-7}x^7 \\ & - 1.26723286902409 \times 10^{-7}x^6 + 6.23462096628266 \times 10^{-8}x^5 \\ & - 2.05277535525461 \times 10^{-8}x^4 + 5.82167120066757 \times 10^{-9}x^3 \\ & + 0.999999998891922x^2 - 0.999999999885384x + 0.999999999582705. \end{aligned}$$

In Table 4, presents a comparison between the exact solution $u(x)$ and approximate solution $u_N(x)$ for $N = 5, 9$ and 12 , respectively, depending on the least square error and running time.

Table 4. Comparison between the exact solution $u(x)$ and approximate solution $u_N(x)$ for example 4.

x_i	Exact Solution Example 4.	N-Approximate Solution $u_N(x)$		
		$N = 5$	$N = 9$	$N = 12$
0.0	1.00	0.99399626	0.99999725	1.00
0.1	0.91	0.90408383	0.90999731	0.91
0.2	0.84	0.83409771	0.83999732	0.84
0.3	0.79	0.78420236	0.78999737	0.79
0.4	0.76	0.75450572	0.7599975	0.76
0.5	0.75	0.74506629	0.74999774	0.75
0.6	0.76	0.75590034	0.75999808	0.76
0.7	0.79	0.78698902	0.78999852	0.79
0.8	0.84	0.83828549	0.83999904	0.84
0.9	0.91	0.90972207	0.90999961	0.91
1.0	1.00	1.00121742	1.00000023	1.00
L.S.E.		0.0002244	$4.72053254 \times 10^{-11}$	$1.11096853 \times 10^{-18}$
Running Time/Sec		0.568155527	1.649296999	6.3344522

Figure 4 illustrates a comparison between the exact solution and approximate solution of linear (FVIFDEs). To show the result of the proposed method to an exact solution, we present Table 4. Each of the plots is drawn with our Python program version 3.8.8 (2021).

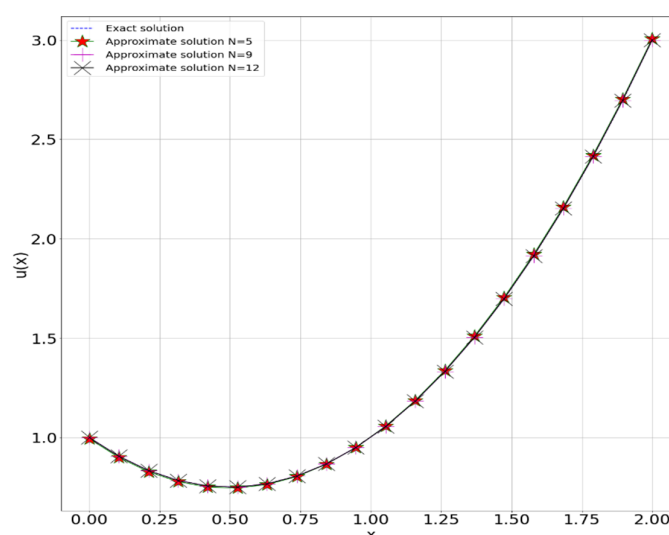


Figure 4. Comparison of the exact and approximate solution of example 4.

6. Conclusions

Multi-fractional order linear integro-differential equations are generally difficult to solve analytically. In many situations, it is necessary to approximate solutions. In this work, we present a new technique for numerically solving the linear Fredholm–Volterra integro-fractional differential equation of multi-fractional order of the Caputo sense using first-order Bessel polynomials. The comparison of the results achieved with the exact solution, the exact solution, and the other methods suggests that the procedure is very effective and convenient. We introduced this with some illustrative examples of the approach and their least square error to minimize the error terms on the specified domain and running time are also given in tabular form. It is obvious that as N rises, the error rate reduces and the answer becomes closer to the exact solution. One significant benefit of the technique is that the Bessel coefficients of the solution may be determined relatively quickly using computer code developed in Python v3.8.8 (2021). As an example, consider the Python v3.8.8 (2021).

Future directions: Using the residual error function, we can enhance the Bessel collocation method for solving the multi-high fractional-order system of Fredholm–Volterra integro-differential equations and their delay. This technique can also be used to make an accurate error estimation.

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