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Space, Matter and Interactions in a Quantum Early Universe Part I: Kac–Moody and Borcherds Algebras

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Abstract: We introduce a quantum model for the universe at its early stages, formulating a mechanism for the expansion of space and matter from a quantum initial condition, with particle interactions and creation driven by algebraic extensions of the Kac–Moody Lie algebra \mathfrak{e}_9 . We investigate Kac–Moody and Borcherds algebras, and we propose a generalization that meets further requirements that we regard as fundamental in quantum gravity.

Keywords: Kac–Moody algebras; Borcherds algebras; early universe cosmology



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1. Introduction

This is the first of two papers—see also [1]—describing an algebraic model of quantum gravity.

The intrinsic difficulty of quantizing gravity, encountered also in the most acclaimed approaches of string theory and loop quantum gravity, has led us to this attempt of thinking outside the box and exploiting only the most fundamental principles of quantum mechanics and general relativity, as we believe that they should apply in the extreme conditions of a hot dense universe in its early stages. We use, therefore, an Occam's razor type of approach by starting with the least possible assumptions and filling in more structure when needed.

Two fundamental and intuitive physical principles are assumed to hold in the regime we study:

FP1 There is no classical observable to be quantized. We start directly from quantum objects and states of the system: interactions, creation and the expansion of spacetime all occur with quantum probability amplitudes; the universe evolves from an initial quantum state. Gravity is identified with the way spacetime is created as particles evolve, whereas the rest of the dynamics, which incorporate electro-weak and strong forces, only depend on the quantum charges of the constituents and the rule for the building blocks of the interactions, which we assume to be algebraic in nature. We depart from the conventional view that quantum gravity should be realized as the *quantization of gravity* with its renormalization, with the four fundamental forces unifying at the Planck scale. We are at the Planck scale: the charges of the constituents are symmetrical, but diversity comes from quantum theory itself, and from the initial conditions.

FP2 There is no spacetime geometry to start with. We can only start by establishing one rule for the interactions and one for the creations of spacetime, bearing in mind that they both occur with probability amplitudes: both interactions and spacetime have an intrinsic quantum nature.

The assumption, supported by physical observations, that at very high temperatures the interactions are tree-like, has led us to consider algebraic models, with the algebra

product playing the role of the building blocks of all interactions. A mechanism for the quantum creation of spacetime suggests the inclusion of momenta within the charges (roots) of the algebra, thus achieving both *charge and energy–momentum conservation*. We reach this goal by considering *infinite dimensional generalized Lie algebras*. A generator in the algebra is related to a particle, with certain charges coming from the algebra roots, but it is also related to a quantum field, since new generators are produced by multiplication in the algebra: a generator expands in spacetime with complex quantum amplitudes, but locally interacts, disappears and contributes to the creation of new generators. This local action can be considered as a vertex, made of generators obeying the rules of an algebra. There is no *vacuum*, since space points exist only where generators are. We see, therefore, that we do have an algebra at the core of the model, but the expansion of spacetime embeds it into a larger picture: that of a vertex-type algebra describing quantum interactions and a quantum-generated spacetime.

We believe that the above considerations are plausible and strongly based on fundamental physics. The evolution of the universe, its quantum interactions and quantum expansion from a chosen initial state of a finite set of generators, can be turned into an algorithm in which all physical quantities are, in principle, calculable, and no infinity occurs. A concrete, calculable realization of the above ideas is what we have achieved with our model, which has no claim other than being physically consistent and mathematically rigorous.

In this first paper, we provide an introduction to the foundations of the model, and we start investigating the mathematical structures that suit our purpose. In the second paper [1], we will deal with a physical model relying on a particular infinite dimensional algebra.

The Lie algebra at the core of our model has the following features and interpretation:

1. It is an infinite dimensional Lie algebra extending \mathfrak{e}_8 that is regarded as the internal quantum number subalgebra, meaning that the \mathfrak{e}_8 roots represent the charges and spin of elementary particles;
2. Its root lattice is Lorentzian;
3. The subspace of the lattice that is complementary to that of \mathfrak{e}_8 is interpreted as momentum space.

Remark 1. *The Lie algebra \mathfrak{e}_8 has been considered by many as a possible algebra for grand unification, as well as for quantum gravity. It has then been considered not suitable after the no-go theorem by Distler and Garibaldi [2]. We will show in Section 2.3 how \mathfrak{e}_8 fulfils the requirements for standard model degrees of freedom and algebras, which seems to contradict the thesis of Distler and Garibaldi. We underline here that it does not, since the hypothesis, denoted TOE1 by the authors of [2]—in particular, the fact that the algebra of the standard model centralizes $\mathfrak{sl}(2, \mathbb{C})$ —not only does not apply, but actually needs not to do so, as will become obvious in the development of Section 2.3.*

Algebraic methods are extensively used and successfully exploited in string theory and conformal field theory in two dimensions, through the concept of vertex operator algebras, [3–5], in order to describe the interactions between different strings, localized at vertices, analogously to the Feynman diagram vertices. Mathematically, the underlying concept of a vertex algebra was introduced by Borchers [6–8], in order to prove the *Monstrous Moonshine* conjecture [9].

Infinite dimensional Kac–Moody algebras have recently entered the loop quantum gravity literature to describe spin network edge modes (generalizing the Gibbons–Hawking boundary term); e.g., [10,11], in which a boundary $SU(2)$ symmetry was found, along with a Virasoro structure that resembles strings with an internal three-dimensional structure. On the other hand, the Kac–Moody algebra \mathfrak{e}_{11} has been investigated in string theory and M-theory by P. West and collaborators (cf. e.g., [12] and Refs. therein). As we will elucidate further below, we consider a Borchers extension of the even larger Kac–Moody algebra

\mathfrak{e}_{12} in order to describe all interactions in a very early universe, and to exploit the Grassmann envelop in order to deal with fermions and implement Pauli's exclusion principle.

The algebras used in this paper may be regarded as *vertex operator algebras* in a broader sense, since they are characterized by interaction operators that look like generators of a Lie algebra, and whose product depends upon parameters related to the spacetime creation and expansion. The Lie algebra acts *locally*, but it is immersed in a wider, vertex-type algebra by means of a mechanism that creates a discrete quantum spacetime.

The Pauli exclusion principle is fulfilled by turning the algebra into a Lie superalgebra using the *Grassmann envelope*.

The resulting model is thus intrinsically relativistic, both because of the way spacetime expands and because the Poincaré group acts locally on the Lie algebra. Furthermore, the conservation of charge and momentum is a consequence of the Lie product, and, in this respect, they are treated at the same level.

1.1. \mathfrak{g}_u , a Lie Algebra for Quantum Gravity

At a very fundamental level, we make the following assumptions on quantum gravity, founded on the current theoretical and experimental knowledge in physics.

- (QG.1) Gravity is a characteristic of spacetime;
- (QG.2) Spacetime is *dynamical* and related to matter. Therefore, we assume that it emerges from the existence of particles and their interactions. There is no way of defining distances and time lapses without interactions, so that the creation and expansion of spacetime is itself a rule followed by particle interactions;
- (QG.3) A suitable mathematical structure at the core of the description of quantum gravity is that of an algebra, which we will henceforth denote by \mathfrak{g}_u , whose generators represent the particles and whose product yields the building blocks of the interactions (let us call them *elementary interactions*). As a consequence, the interactions are endowed with a tree structure, thus opening up the opportunity for a description of scattering amplitudes in terms of what we would call *gravitahedra*, providing a generalization of the associahedra and permutahedra in the current theory of scattering [13–20]. The structure constants of the algebra determine the quantum amplitudes of the elementary interactions; in particular, we assume \mathfrak{g}_u to be a Lie algebra because it enables us to derive the fundamental conservation laws observed in physics directly from the action of the generators as derivations (Jacobi identity). As in the theory of fields, the interactions may only occur locally, point-by-point in the expanding spacetime, which can therefore be viewed as a parameter on which the algebra product depends;
- (QG.4) In agreement with the theory of a big bang, strongly supported by the current observations, we assume the existence of an initial quantum state, mathematically represented by an element of the universal enveloping algebra of \mathfrak{g}_u . Such an element is made of generators that can all interact among themselves, thus yielding the first geometrical interpretation: that of a point where particles may interact;
- (QG.5) A particle has a certain probability amplitude to interact but also not to interact, in which case, it expands, as described in Section 1.2;
- (QG.6) Particles are quantum objects, hence their existence through interactions occurs with certain amplitudes. Therefore, **spacetime acquires a quantum structure**: a point in space and time where particles are present with a certain amplitude and may interact. The amplitude related to the quantum spacetime point is the sum of the amplitudes for particles to be there. Consequently, the fact that gravitation appears as an attractive force has to be explained through amplitudes and their interference;
- (QG.7) The initial set of generators is finite by assumption, being the the initial state represented by an element of the universal enveloping algebra. These generators are all allowed to interact with each other with a certain amplitude and according to the algebra relations, at what we call time 0 of the universal clock.

The outcome of the first finite number of interactions, plus the creation of space, which is a consequence of the momentum part of the root associated with each generator, leads to a second finite set of interactions, and so on. What we call *universal time* is this order parameter of the interactions. The expansions are also countable, hence discrete: the structure of spacetime that emerges is discrete and finite at every instant of the universal clock, as is the universe and the quantum theory describing it. There is no divergence of any sort: quantum field theory in the continuum, with its divergences and related renormalizations, is an approximation that may be useful for calculations long after the big bang;

(QG.8) The finiteness of the expanding universe, and thus the absence of spacetime beyond it, affects the quantum initial state of particles, which are not free to move on the spacetime *stage* but are bound as if they were surrounded by infinitely high barriers. The steady state of such a particle is a **superposition of states with opposite 3-momenta**, representing an object that moves simultaneously in opposite directions, where, by 3-momentum, we denote the spatial component of 4-momentum.

1.2. Expansion

The assignment of opposite 3-momenta is inherent to the quantum behavior of a particle in a box, in which, the square of the momentum, but not the momentum itself, has a definite value in a stationary state. In standard relativistic and non-relativistic quantum mechanics, the ground state is a superposition of *generalized states*, with opposite momenta $e^{i(kx+a)}$ and $e^{-i(kx+a)}$. We maintain the same energy and start enlarging the box on opposite sides along the direction of \vec{p} in steps of $|\vec{p}|/E$ in Planck units, so that a massless particle travels at velocity c , and a massive one travels slower than that. We obtain a wave proportional to $\sin(\frac{\pi}{2}(x/a + n))$ for $a = 1 = |\vec{p}|/E$ and $n = 1, 2, 3, \dots$

The wave function for the first four expansions is shown in Figure 1. We take the discretized picture of the sine function maxima and minima, the dots, with cosmological time $t = n - 1$.

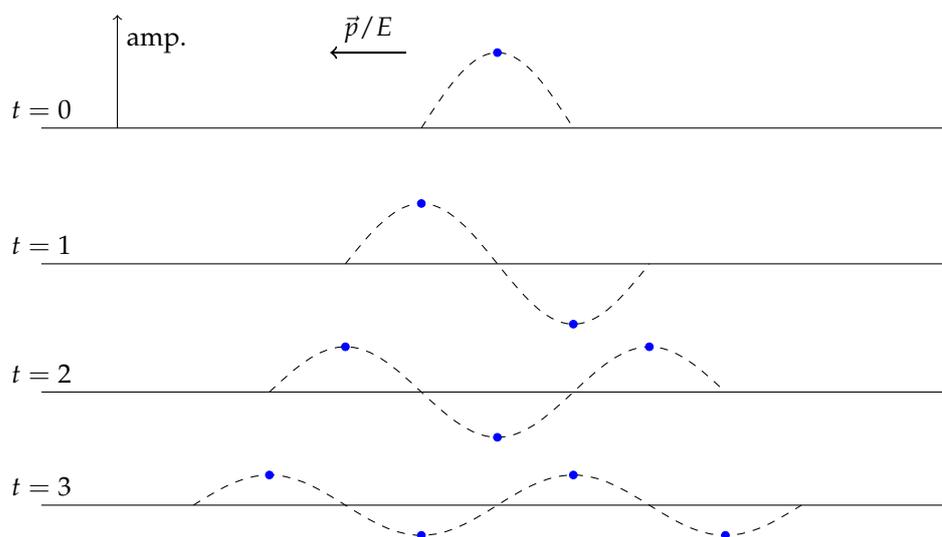


Figure 1. Expansion of a particle (blue dot) along \vec{p} (x -axis) with amplitudes (y -axis).

The amplitude also acquires a time-dependent phase e^{iEt} that makes it complex.

1.3. Fermions and Bosons

The Lie algebras considered in this paper contain \mathfrak{e}_8 , and thus \mathfrak{d}_8 . Under the adjoint action of \mathfrak{d}_8 , the generators of \mathfrak{e}_8 split into spinorial and non-spinorial ones, providing the algebra with a two-graded structure. We give the spinorial generators the physical

meaning of fermions, whereas the non-spinorial generators are given the physical meaning of bosons, in order to automatically comply with the addition of angular momenta.

On the other hand, the Pauli exclusion principle is embodied in the *Grassmann envelope*, which turns the two-graded algebra into a Lie superalgebra. The degrees of freedom of the spin-1/2 fermions originate from the superposition of opposite 3-momenta and the corresponding change in helicity caused by the reflection at the space boundary. The Poincaré group then naturally emerges as a group of transformations of the *local* algebra \mathfrak{g}_u , leaving the charges invariant.

All of these topics will be treated in the companion paper [1].

1.4. Quantum Quasicrystal

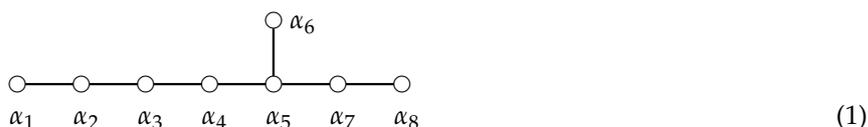
The expansion of the space that we propose has two fundamental features:

1. A space point may exist with a certain *probability amplitude*, this latter being the sum of the amplitudes for some particles—matter or radiation—to be there: no space point can possibly be empty;
2. Space is a quantum object that expands according to algebraic rules.

As a result of these two features, our model of the universe can be conceived as a *quantum quasicrystal* [21–23].

2. \mathfrak{e}_8 , the Charge/Spin Subalgebra

In our treatment, we use the following labels for the Dynkin diagram of \mathfrak{e}_8 :



A way of writing the simple roots of \mathfrak{e}_8 in the orthonormal basis $\{k_1, \dots, k_8\}$ of \mathbb{R}^8 is:

$$\begin{aligned}
 \alpha_1 &= k_1 - k_2; \\
 \alpha_2 &= k_2 - k_3; \\
 \alpha_3 &= k_3 - k_4; \\
 \alpha_4 &= k_4 - k_5; \\
 \alpha_5 &= k_5 - k_6; \\
 \alpha_6 &= k_6 - k_7; \\
 \alpha_7 &= k_6 + k_7; \\
 \alpha_8 &= -\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8).
 \end{aligned}
 \tag{2}$$

The whole root system Φ_8 of \mathfrak{e}_8 (obtained from the simple roots by Weyl reflections) can be written as follows:

$$\begin{aligned}
 \Phi_8 &= \Phi_{\mathcal{O}} \cup \Phi_{\mathcal{S}} \text{ (240 roots);} \\
 \Phi_{\mathcal{O}} &:= \{\pm k_i \pm k_j | 1 \leq i < j \leq 8\}, & 4 \binom{8}{2} &= 112; \\
 \Phi_{\mathcal{S}} &:= \left\{ \frac{1}{2}(\pm k_1 \pm k_2 \pm \dots \pm k_7 \pm k_8), \text{ even \# of +} \right\}, & 2^7 &= 128.
 \end{aligned}
 \tag{3}$$

The first set $\Phi_{\mathcal{O}}$ of 112 roots is the set of roots of $\mathfrak{d}_8 \simeq \mathfrak{so}(\mathbf{16}, \mathbb{C})$. The set $\Phi_{\mathcal{S}}$ is a Weyl spinor of \mathfrak{d}_8 , with respect to the adjoint action (every orthogonal Lie algebra in even dimension $\mathfrak{d}_n \simeq \mathfrak{so}(2n, \mathbb{C})$ has a Weyl spinor representation of dimension 2^{n-1}).

If α is a root, there is a unique way of writing it as $\alpha = \sum \lambda_i \alpha_i$, where the α_i 's are simple (in fact, all λ_i 's are positive for positive roots, and negative for negative roots). The sum $ht(\alpha) := \sum \lambda_i$ is called the *height* of α .

The fact that the roots of $\Phi_{\mathcal{O}}$ are the roots of a subalgebra and those of $\Phi_{\mathcal{S}}$ correspond to a representation of it can be seen by noticing that: $\Phi_{\mathcal{O}} + \Phi_{\mathcal{O}} \subset \Phi_{\mathcal{O}}$, $\Phi_{\mathcal{O}} + \Phi_{\mathcal{S}} \subset \Phi_{\mathcal{S}}$. Moreover, $\Phi_{\mathcal{S}} + \Phi_{\mathcal{S}} \subset \Phi_{\mathcal{O}}$ implies that $\Phi_{\mathcal{O}}$ is embedded into \mathfrak{e}_8 in a symmetric way.

Thus, one can consistently define a non-Cartan generators x_{α} bosonic if $\alpha \in \Phi_{\mathcal{O}}$, and fermionic if $\alpha \in \Phi_{\mathcal{S}}$. We also call *fermionic* or *bosonic* the root α associated to a fermionic (resp. bosonic) non-Cartan generator x_{α} . A Cartan generator h_{α} is always bosonic for any α , since $h_{\alpha} = [x_{-\alpha}, x_{\alpha}]$. The roots of \mathfrak{e}_8 split into 128 fermions (F) and 112 bosons (B).

2.1. Algebraic Structure

The \mathfrak{e}_8 algebra can be defined from its root system Φ_8 [24–26], over the complex field extension \mathbb{C} of the rational integers \mathbb{Z} in the following way:

- (a) We select the set of simple roots Δ_8 of Φ_8 ;
- (b) We select a basis $\{h_1, \dots, h_8\}$ of the eight-dimensional vector space \mathfrak{h} over \mathbb{C} and set $h_{\alpha} = \sum_{i=1}^8 \lambda_i h_i$ for each $\alpha \in \Phi_8$, such that $\alpha = \sum_{i=1}^8 \lambda_i \alpha_i$;
- (c) We associate to each $\alpha \in \Phi_8$ a one-dimensional vector space L_{α} over \mathbb{C} spanned by x_{α} ;
- (d) We define $\mathfrak{e}_8 = \mathfrak{h} \oplus_{\alpha \in \Phi_8} L_{\alpha}$ as a vector space over \mathbb{C} ;
- (e) We give \mathfrak{e}_8 an algebraic structure by defining the following multiplication on the basis $\{h_1, \dots, h_8\} \cup \{x_{\alpha} \mid \alpha \in \Phi_8\}$ by linearity to a bilinear multiplication $\mathfrak{e}_8 \times \mathfrak{e}_8 \rightarrow \mathfrak{e}_8$:

$$\begin{aligned}
 [h_i, h_j] &= 0, \quad 1 \leq i, j \leq 8 \\
 [h_i, x_{\alpha}] &= -[x_{\alpha}, h_i] = (\alpha, \alpha_i) x_{\alpha}, \quad 1 \leq i \leq 8, \quad \alpha \in \Phi_8 \\
 [x_{\alpha}, x_{-\alpha}] &= -h_{\alpha} \\
 [x_{\alpha}, x_{\beta}] &= 0 \text{ for } \alpha, \beta \in \Phi_8 \text{ such that } \alpha + \beta \notin \Phi_8 \text{ and } \alpha \neq -\beta \\
 [x_{\alpha}, x_{\beta}] &= \varepsilon(\alpha, \beta) x_{\alpha+\beta} \text{ for } \alpha, \beta \in \Phi_8 \text{ such that } \alpha + \beta \in \Phi_8
 \end{aligned} \tag{4}$$

where $\varepsilon(\alpha, \beta)$ is the *asymmetry function*, introduced in [27], as in Definition 1; see also [26].

Definition 1. Let $\mathbb{L}_{\mathfrak{e}_8}$ denote the lattice of all linear combinations of the simple roots with integer coefficients

$$\mathbb{L}_{\mathfrak{e}_8} = \left\{ \sum_{i=1}^8 c_i \alpha_i \mid c_i \in \mathbb{Z}, \alpha_i \in \Delta_8 \right\} \tag{5}$$

the asymmetry function $\varepsilon(\alpha, \beta) : \mathbb{L}_{\mathfrak{e}_8} \times \mathbb{L}_{\mathfrak{e}_8} \rightarrow \{-1, 1\}$ is defined by:

$$\varepsilon(\alpha, \beta) = \prod_{i,j=1}^8 \varepsilon(\alpha_i, \alpha_j)^{\ell_i m_j} \quad \text{for } \alpha = \sum_{i=1}^8 \ell_i \alpha_i, \beta = \sum_{j=1}^8 m_j \alpha_j \tag{6}$$

where $\alpha_i, \alpha_j \in \Delta_8$ and

$$\varepsilon(\alpha_i, \alpha_j) = \begin{cases} -1 & \text{if } i = j \\ -1 & \text{if } \alpha_i + \alpha_j \text{ is a root and } \alpha_i < \alpha_j \\ +1 & \text{otherwise} \end{cases} \tag{7}$$

We recall the following standard result on the roots of \mathfrak{e}_8 (normalized to 2), [24,25]:

Proposition 1. For each $\alpha, \beta \in \Phi_8$, the scalar product $(\alpha, \beta) \in \{\pm 2, \pm 1, 0\}$; $\alpha + \beta$ (respectively, $\alpha - \beta$) is a root if and only if $(\alpha, \beta) = -1$ (respectively, $+1$); if both $\alpha + \beta$ and $\alpha - \beta$ are not in $\Phi_8 \cup \{0\}$, then $(\alpha, \beta) = 0$.
For $\alpha, \beta \in \Phi_8$, if $\alpha + \beta$ is a root, then $\alpha - \beta$ is not a root.

The following properties of the asymmetry function follow from its definition [26].

Proposition 2. *The asymmetry function ε satisfies, for $\alpha, \beta, \gamma \in \mathbb{L}_{\mathbf{e}_8}$:*

- (i) $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$
- (ii) $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$
- (iii) $\varepsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}(\alpha, \alpha)} \Rightarrow \varepsilon(\alpha, \alpha) = -1$ if $\alpha \in \Phi_8$
- (iv) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)} \Rightarrow \varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha)$ if $\alpha, \beta, \alpha + \beta \in \Phi_8$
- (v) $\varepsilon(0, \beta) = \varepsilon(\alpha, 0) = 1$
- (vi) $\varepsilon(-\alpha, \beta) = \varepsilon(\alpha, \beta)^{-1} = \varepsilon(\alpha, \beta)$
- (vii) $\varepsilon(\alpha, -\beta) = \varepsilon(\alpha, \beta)^{-1} = \varepsilon(\alpha, \beta)$

Property (iv) shows that the product in (4) is indeed antisymmetric.

2.2. \mathbf{e}_8 Charges and the Magic Star

There are four orthogonal \mathfrak{a}_2 's in \mathbf{e}_8 , where orthogonal means that the planes on which their root systems lie are orthogonal to each other.

We denote one of them \mathfrak{a}_2^c for color, one \mathfrak{a}_2^f for flavor and the other two as $\mathfrak{a}_2^{(1)}$ and $\mathfrak{a}_2^{(2)}$:

$$\mathfrak{a}_2^c : k_i - k_j, i \neq j, i, j \in \{1, 2, 3\}$$

$$\mathfrak{a}_2^f : k_i - k_j, i \neq j, i, j \in \{4, 5, 6\}$$

$$\mathfrak{a}_2^{(1)} : \pm(k_7 + k_8), \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 - k_7 - k_8) \\ \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8)$$

$$\mathfrak{a}_2^{(2)} : \pm(k_7 - k_8), \pm\frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 - k_7 + k_8) \\ \pm\frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 + k_7 - k_8)$$

The generators of \mathfrak{a}_2^c and \mathfrak{a}_2^f are bosonic.

The Magic Star (MS) of \mathbf{e}_8 shown in Figure 2 is obtained by projecting its roots on the plane of \mathfrak{a}_2^c [28]. The pair of integers $\{r, s\}$ are the (Euclidean) scalar products $r := (\alpha, k_1 - k_2)$ and $s := (\alpha, k_1 + k_2 - 2k_3)$ for each root α . The fermions on the tips of the MS are quarks, since they are acted upon by \mathfrak{a}_2^c : they are colored. The fermions within the center of the MS are leptons: they are colorless. A similar MS of \mathbf{e}_6 within \mathbf{e}_8 is obtained by projecting the roots in the center of the MS of \mathbf{e}_8 on the plane of \mathfrak{a}_2^f .

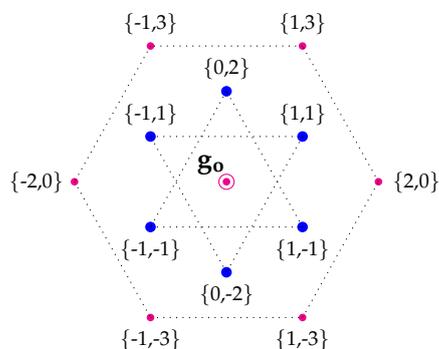


Figure 2. The Magic Star (MS): $g_0 = \mathbf{e}_6$ in the MS of \mathbf{e}_8 , $g_0 = \mathfrak{a}_2^{(1)} \oplus \mathfrak{a}_2^{(2)}$ in the MS of \mathbf{e}_6 ; the triangles represent the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of the \mathfrak{a}_2 with roots in the external hexagon.

Notice that, in each tip of the MS of \mathbf{e}_8 , we obtain 27 roots, 11 of which are bosonic and 16 fermionic; this corresponds to the following decomposition of the irrepr. 27 of \mathbf{e}_6

$$\begin{aligned} \mathbf{e}_6 &\supset \mathbf{d}_5 \oplus \mathbb{C}, \\ 27 &= \mathbf{1}_{-4} \oplus \mathbf{10}_2 \oplus \mathbf{16}_{-1}. \end{aligned} \tag{8}$$

On the other hand, within \mathbf{e}_6 , we have nine roots in each tip of the MS, five of which are bosonic and four fermionic; this corresponds to the following decomposition of the repr. $(\mathbf{3}, \mathbf{3})$ of $\mathfrak{a}_2 \oplus \mathfrak{a}_2$:

$$\begin{aligned} \mathfrak{a}_2 \oplus \mathfrak{a}_2 &\supset \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathbb{C}_I \oplus \mathbb{C}_{II} \supset \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathbb{C}_{I+II}, \\ (\mathbf{3}, \mathbf{3}) &= (\mathbf{1}, \mathbf{1})_{-2,-2} \oplus (\mathbf{2}, \mathbf{2})_{1,1} \oplus (\mathbf{2}, \mathbf{1})_{1,-2} \oplus (\mathbf{1}, \mathbf{2})_{-2,1} \\ &= (\mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{2}, \mathbf{2})_2 \oplus (\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{2})_{-1}. \end{aligned} \tag{9}$$

Tables A1–A4 in Appendix A describe the content of these MS, root by root.

The *magic* of the MS is that each tip $\{r, s\}$ of the star, $\{r, s\} \in \{\{0, \pm 2\}, \{\pm 1, \pm 1\}\}$, both in the case of \mathbf{e}_8 and of \mathbf{e}_6 , can be viewed as a cubic (simple) *Jordan algebra* $J_{\{r,s\}}$, over the octonions and the complex field, respectively, and each pair of opposite tips, with respect to the center of the star, has a natural algebraic structure of a *Jordan pair*. The algebra in the center of the star is the derivation algebra of the Jordan pair; when the Jordan pair is made of a pair of Jordan algebras, its derivations also define the Lie algebra of the structure group of the Jordan algebra itself [28–31].

2.3. The Standard Model

In this section, we relate the \mathbf{e}_8 charges to the degrees of freedom of the Standard Model (SM) of elementary particle physics. It is not our aim to carry through a detailed analysis; in particular, we do not consider symmetry breaking, nor the Higgs mechanism, nor chirality and parity violation by weak interactions in the fermionic sector. We do, however, focus on spin as an internal degree of freedom, and this will be instrumental for the treatment of the Poincaré action on our algebra, which we will investigate in the companion paper [1].

The first important step, after splitting the roots into colored and colorless, as in the previous section, is to find the electromagnetic $\mathbf{u}(\mathbf{1})^{em}$ that gives the right charges to quarks and leptons. The choice may not be unique, although is strongly limited by the requirement to yield the right charges, but the one we make is certainly consistent. We select the $\mathbf{u}(\mathbf{1})^{em}$ generated by

$$h_\gamma = -\mathbf{i} \left(\frac{1}{3}h_{\alpha_1} + \frac{2}{3}h_{\alpha_2} + h_{\alpha_3} \right), \tag{10}$$

giving to x_α , where $\alpha = \sum \lambda_i k_i$, the charge

$$q_{e.m.}(\alpha) := (\alpha, q_\gamma) = -\frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_4, \tag{11}$$

where

$$q_\gamma := -\frac{1}{3}\alpha_1 - \frac{2}{3}\alpha_2 - \alpha_3 = -\frac{1}{3}(k_1 + k_2 + k_3) + k_4. \tag{12}$$

The second column of Tables A1–A4 in Appendix A shows the charges of the \mathbf{e}_8 generators x_α , with α shown in the first column. In particular, Tables A3 and A4 show the charges given to quarks and leptons.

We now select the semi-simple Lie algebra $\mathbf{d}_2 \simeq \mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_1$ with roots $\pm k_5 \pm k_6$ (other choices would be equivalent; thus, this does not imply any loss of generality). We denote by ρ_1, ρ_2 the roots $k_5 - k_6$ and $k_5 + k_6$, respectively, by $\mathfrak{a}_1^{(1)}$, the subalgebra of \mathbf{d}_2

associated to the roots $\pm\rho_1$, and by $\mathfrak{a}_1^{(2)}$, the one associated to the roots $\pm\rho_2$. The non-Cartan generators of \mathfrak{e}_8 fall into $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ irreducible representations of spin (s_1, s_2) :

$$\begin{aligned}
 (0, 0) : & \quad x_\alpha, \alpha = \pm k_i \pm k_j, \quad i, j \notin \{5, 6\}; \\
 (1/2, 0) : & \quad x_\alpha, \alpha = \frac{1}{2}(k \pm (k_5 - k_6)), \quad k := \pm k_1 \pm k_2 \pm k_3 \pm k_4 \pm k_7 \pm k_8; \\
 (0, 1/2) : & \quad x_\alpha, \alpha = \frac{1}{2}(k \pm (k_5 + k_6)); \\
 (1/2, 1/2) : & \quad x_\alpha, \alpha = \pm k_i \pm k_5 \text{ or } \alpha = \pm k_i \pm k_6, \text{ for a fixed } i \notin \{5, 6\}; \\
 (1, 0) : & \quad x_\alpha, \alpha = \pm(k_5 - k_6), \text{ the adjoint of } \mathfrak{a}_1^{(1)} \text{ (adding its Cartan)}; \\
 (0, 1) : & \quad x_\alpha, \alpha = \pm(k_5 + k_6), \text{ the adjoint of } \mathfrak{a}_1^{(2)} \text{ (adding its Cartan)},
 \end{aligned}
 \tag{13}$$

where $(1, 0) \oplus (0, 1)$ corresponds to the six components as a rank 2 antisymmetric tensor in four dimensions, with *selfdual* and *antiselfdual* parts $(1, 0)$ and $(0, 1)$, respectively. Notice that all fermions have a half-integer spin $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, whereas all bosons of type x_α have an integer spin.

In order to define the action of the Poincaré group in [1], we need the covering group of rotations in the internal space. For this purpose, we select the spin (diagonal) subalgebra $\mathfrak{su}(2)^{\text{spin}} \in \mathfrak{a}_1^{(1)} \oplus \mathfrak{a}_1^{(2)}$ as the compact (real) form with generators

$$\begin{aligned}
 R^+ + R^-, \quad \mathbf{i}(R^+ - R^-), \quad \mathbf{i}H_R, \quad \text{where} \\
 R^+ := x_{\rho_1} + x_{\rho_2}, \quad R^- := x_{-\rho_1} + x_{-\rho_2}, \quad H_R := \frac{1}{2}(h_{\rho_1} + h_{\rho_2})
 \end{aligned}
 \tag{14}$$

The $(\frac{1}{2}, \frac{1}{2})$ representation splits into a scalar and a vector under this rotation subalgebra. The spin-1 particle within $(\frac{1}{2}, \frac{1}{2})$ is the linear span of the generators $x_{k_i \pm k_5}$ with z-component of spin $s_z := \frac{1}{2}(\rho_1 + \rho_2, k_i \pm k_5) = (k_5, k_i \pm k_5) = \pm 1$ and $\frac{1}{2}(\varepsilon(\rho_1, k_i - k_5)x_{k_i - k_6} + \varepsilon(\rho_2, k_i - k_5)x_{k_i + k_6})$ with $s_z = 0$; the corresponding scalar is $\frac{1}{2}(\varepsilon(\rho_1, k_i - k_5)x_{k_i - k_6} - \varepsilon(\rho_2, k_i - k_5)x_{k_i + k_6})$, as it can be easily verified.

2.3.1. W^\pm

Let us now consider the W^\pm bosons. There are not many choices for them; indeed, they must be colorless vectors with respect to $\mathfrak{su}(2)^{\text{spin}}$ and have electric charge ± 1 . The W^\pm bosons are therefore the generators associated to $\pm(k_4 - k_5)$ (within \mathfrak{a}_2^f mentioned in Section 2.2) and $\pm(k_4 + k_5)$, and the electric charge ± 1 given by the presence of k_4 ; they change flavor to both quarks and leptons. The above analysis suggests that the extra degree of freedom needed, say, for W^+ , to become massive, from the two degrees of freedom of the massless helicity-1 state, is $\varepsilon(\rho_1, k_4 + k_6)x_{k_4 + k_6} + \varepsilon(\rho_2, k_4 - k_6)x_{k_4 - k_6}$, as a part of the Higgs mechanism, which we will not discuss any further in this paper.

Remark 2. We could have made other equivalent choices for the $\mathfrak{d}_2 \simeq \mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_1$ subalgebra, which has to act non-trivially on W^\pm : once passing to real forms, it cannot possibly commute with the weak interaction $\mathfrak{su}(2)^L$. For what concerns the no-go theorem by Distler-Garibaldi, the hypothesis TOE1 of [2] cannot possibly apply, as outlined in the introduction; see Remark 1. We also emphasize that, contrary to [2], we are dealing with the complex form of \mathfrak{e}_8 because we want complex phases for the particle states.

Using the properties of the asymmetry function and the ordering of the simple roots $\alpha_i < \alpha_{i+1}$, we obtain:

$$\varepsilon(\rho_1, k_4 - k_5) = \varepsilon(\rho_2, k_4 - k_5) = 1,
 \tag{15}$$

hence, the massive W^\pm is described by three components:

$$\begin{aligned}
 W_1^\pm & := x_{\pm k_4 + k_5} & (s_z = 1); \\
 W_0^\pm & := \frac{1}{2}(x_{\pm k_4 - k_6} + x_{\pm k_4 + k_6}) & (s_z = 0); \\
 W_{-1}^\pm & := x_{\pm k_4 - k_5} & (s_z = -1).
 \end{aligned}
 \tag{16}$$

Moreover, using the notation

$$\begin{aligned} R_x &:= \frac{1}{2}(x_{\rho_1} + x_{\rho_2} + x_{-\rho_1} + x_{-\rho_2}); \\ R_y &:= \frac{i}{2}(x_{\rho_1} + x_{\rho_2} - x_{-\rho_1} - x_{-\rho_2}); \\ R_z &:= \frac{i}{2}(h_{\rho_1} + h_{\rho_2}), \end{aligned} \tag{17}$$

we obtain

$$\begin{aligned} [R_x, W_{-1}^{\pm}] &= W_0^{\pm}, & [R_x, W_1^{\pm}] &= W_0^{\pm}, & [R_x, W_0^{\pm}] &= -\frac{1}{2}(W_1^{\pm} + W_{-1}^{\pm}), \\ [R_y, W_{-1}^{\pm}] &= i W_0^{\pm}, & [R_y, W_1^{\pm}] &= -i W_0^{\pm}, & [R_y, W_0^{\pm}] &= -\frac{i}{2}(W_1^{\pm} - W_{-1}^{\pm}), \end{aligned} \tag{18}$$

and

$$[R_z, W_{s_z}^{\pm}] = i s_z W_{s_z}^{\pm}, \quad s_z \in \{1, 0, -1\}. \tag{19}$$

These commutation relations correspond to the action of the rotation matrices:

$$J_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y := \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_z := i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{20}$$

regarding the vectors, $v = v_+e_+ + v_z e_z + v_-e_- \in \mathbb{R}^3$ in the spherical basis $\{e_+, e_z, e_-\}$, which corresponds, for angular momentum 1, to the spherical harmonic basis for the irreducible representations of $SO(3)$. With respect to the same vector $v = v_x e_x + v_y e_y + v_z e_z \in \mathbb{R}^3$ in the standard orthogonal basis $\{e_x, e_y, e_z\}$, we have:

$$\begin{aligned} e_+ &= \frac{1}{\sqrt{2}}(e_x + i e_y), & e_- &= \frac{1}{\sqrt{2}}(e_x - i e_y), \\ v_+ &= \frac{1}{\sqrt{2}}(v_x - i v_y), & v_- &= \frac{1}{\sqrt{2}}(v_x + i v_y). \end{aligned} \tag{21}$$

The transformation between (column) vectors in the two bases is represented by the unitary matrix U :

$$\begin{pmatrix} v_+ \\ v_z \\ v_- \end{pmatrix} = U \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad U := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{22}$$

The correspondence with R 's and W 's is:

$$\begin{aligned} R_x &\leftrightarrow J_x, & R_y &\leftrightarrow J_y, & R_z &\leftrightarrow J_z, \\ W_1^{\pm} &\leftrightarrow v_+, & W_{-1}^{\pm} &\leftrightarrow v_-, & W_0^{\pm} &\leftrightarrow \frac{1}{\sqrt{2}}v_z. \end{aligned} \tag{23}$$

Remark 3. We have an interesting relationship between the weak and rotation generators in the internal space (spin generators) by noticing that

$$\begin{aligned} [W_1^+ + W_{-1}^+, W_0^-] &= [W_1^- + W_{-1}^-, W_0^+] = R_x \\ [W_1^+ - W_{-1}^+, i W_0^-] &= [W_1^- - W_{-1}^-, i W_0^+] = R_y \end{aligned} \tag{24}$$

and, consequently, the relation with $R_z = [R_x, R_y]$.

2.3.2. Z^0

We associate the Z^0 boson with spin $s_z = 0$, denoted by Z_0^0 , to the vector orthogonal to q_γ in the plane of $(k_4 - k_5)$ and q_γ ; hence, it is, up to a scalar, the Cartan generator $Z_0^0 = \frac{1}{4}(h_{\alpha_1} + 2h_{\alpha_2} + 3h_{\alpha_3} + 4h_{\alpha_4})$. It interacts with left-handed neutrinos and right-handed antineutrinos, contrary to the photon; it does not allow for *flavor-changing neutral currents*.

Notice that the generator of *hypercharge* $\mathbf{u}(1)^Y$ of the standard model is, in this, the setting compact Cartan generator $i h_Y$, where $h_Y := -\frac{1}{6}(2h_{\alpha_1} + 4h_{\alpha_2} + 6h_{\alpha_3} + 3h_{\alpha_4})$. The Weinberg angle θ_W is the angle between the axis representing the photon and the axis

representing the hypercharge; therefore, $\theta_W = \pi/2 - \phi$, where ϕ is the angle between q_γ and $k_4 - k_5$, and we obtain $\sin^2\theta_W = 3/8$.

Since $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = k_1 + k_2 + k_3 + k_4 - 4k_5$, $\frac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4, x_{\pm\rho_{1,2}}) = -(k_5, \pm\rho_1) = -(k_5, \pm\rho_2) = \mp 1$; hence, we have the following commutation relations:

$$\begin{aligned} [x_{\rho_1} + x_{\rho_2} + x_{-\rho_1} + x_{-\rho_2}, Z_0^0] &= (x_{\rho_1} + x_{\rho_2} - x_{-\rho_1} - x_{-\rho_2}); \\ [x_{\rho_1} + x_{\rho_2} - x_{-\rho_1} - x_{-\rho_2}, Z_0^0] &= (x_{\rho_1} + x_{\rho_2} + x_{-\rho_1} + x_{-\rho_2}), \end{aligned} \quad (25)$$

that is

$$[R_x, Z_0^0] = -i R_y, \quad [R_y, Z_0^0] = i R_x, \quad [R_z, Z_0^0] = 0. \quad (26)$$

We want Z^0 , as a spin-1 particle, to obey the same commutation relations with the rotation generators as W^\pm . We can define the spin ± 1 components of Z^0 this way by a comparison with the last commutator in each row of (18):

$$\begin{aligned} Z_1^0 &= -[R_x, Z_0^0] + i [R_y, Z_0^0] = -R_x + i R_y = -R_+; \\ Z_{-1}^0 &= -[R_x, Z_0^0] - i [R_y, Z_0^0] = R_x + i R_y = R_-. \end{aligned} \quad (27)$$

(see (14) for the definition of R_\pm).

By looking at Table A3 in Appendix A, we notice that Z_1^0 interacts, for instance, with ν'_e with spin $-\frac{1}{2}$ to give ν'_e with spin $\frac{1}{2}$, and, similarly, for other leptons and for quarks. In particular, there are no flavor-changing neutral currents.

2.3.3. The Tables at a Glance

From the Tables A1–A4 in Appendix A, one can deduce all the standard model charges (in particular, we have denoted with a *prime* possible mixings in Tables A3 and A4). We have:

- (SM.1) The color charges are denoted by the pair $\{r_c, s_c\}$, and one can associate *colors* to them—say, *blue* = $\{1, 1\}$, *green* = $\{-1, 1\}$ and *red* = $\{0, -2\}$ —and, similarly, for the anti-colors;
- (SM.2) The quarks are the fermions in Table A1 with a certain color; they come in three color families, and anti-quarks have anti-colors and opposite electric charges $-\frac{2}{3}, \frac{1}{3}$ with respect to quarks;
- (SM.3) The gluons are the generators of \mathfrak{a}_2^c , change color to the quarks on which they act, as on a $\mathbf{3}$ or $\bar{\mathbf{3}}$ representation, and their electric charge is 0;
- (SM.4) The leptons are in the center of the MS in Table A1 and are the fermions in Table A2;
- (SM.5) The leptons have an integer electric charge in $\{-1, 0, 1\}$;
- (SM.6) There are four flavor families; we have used the notation χ, ν_χ for the fourth lepton family and T, B for the fourth quark family;
- (SM.7) The fourth column of Tables A3 and A4 shows the component of spin along the axis specified by the spin generator $R_z := \frac{i}{2}(h_{\rho_1} + h_{\rho_2})$. Obviously, a rotation by 2π of quarks and leptons changes their sign, regardless of whether it leaves vector bosons invariant.

The consequences of this classification, with respect to the Poincaré action on \mathfrak{g}_u , will be discussed in the companion paper [1].

3. The Kac–Moody Algebras $\mathfrak{e}_9, \mathfrak{e}_{10}, \mathfrak{e}_{12}, \mathfrak{de}_{12}$

Let $\mathfrak{g}(A)$ denote the Kac–Moody algebra associated to the $n \times n$ Cartan matrix A , with Cartan subalgebra \mathfrak{h} . For all algebras in this paper, A is symmetric; its entries are denoted by a_{ij} . We denote the *Chevalley generators* by E_i , associated to the simple root α_i , and by F_i , associated to the root $-\alpha_i$. Let \mathfrak{n}_+ (resp. \mathfrak{n}_-) denote the subalgebra of $\mathfrak{g}(A)$

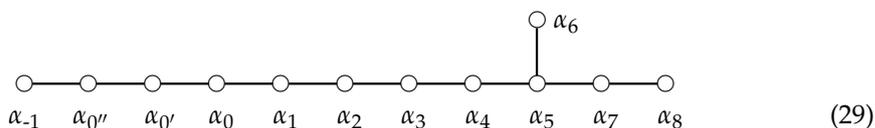
generated by $\{E_1, \dots, E_n\}$ (resp. $\{F_1, \dots, F_n\}$). By Theorem 1.2 (a), (e) in [27], the following triangular decomposition holds:

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \quad (\text{direct sum of vector spaces}). \tag{28}$$

Note that, for a root $\alpha > 0$ (resp. $\alpha < 0$), we have $\alpha \in \mathfrak{h}^*$, the dual of \mathfrak{h} , and the vector space $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ is the linear span of the elements of the form $[\dots [E_{i_1}, E_{i_2}], E_{i_3}] \dots E_{i_s}$ (resp. $[\dots [F_{i_1}, F_{i_2}], F_{i_3}] \dots F_{i_s}$), such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$ (resp. $= -\alpha$). The multiplicity m_α of a root α is defined as $m_\alpha := \dim \mathfrak{g}_\alpha$ ($m_{\alpha_i} = m_{-\alpha_i} = 1$ for each simple root α_i).

Kac–Moody algebras [27,32] can be tackled in terms of simple roots and their (extended) Dynkin diagram, or, equivalently, their Cartan matrix, without any reference to root coordinates. Some physical features or interpretations may, however, be more explicit when roots are expressed on an orthonormal basis rather than on a simple root basis. This is the case of this paper, in which some root coordinates, except for the case of \mathbf{e}_8 , are interpreted as momentum coordinates. We recall that the metric is Euclidean for \mathbf{e}_8 , but Lorentzian in the case of $\mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{12}, \mathbf{de}_{12}$.

Our notation for the simple roots is shown in the Dynkin diagram of \mathbf{e}_{12} :



Analogous diagrams are those of \mathbf{e}_{10} (without $\alpha_{-1}, \alpha_{0''}$), of \mathbf{e}_9 (without $\alpha_{-1}, \alpha_{0''}, \alpha_{0'}$) and of \mathbf{e}_8 (without $\alpha_{-1}, \alpha_{0''}, \alpha_{0'}, \alpha_0$).

3.1. The Simple Roots of \mathbf{e}_9 and \mathbf{e}_{10}

We introduce the following set of simple roots $\alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_8$ of \mathbf{e}_{10} in terms of the basis vectors $k_{-1}, k_0, k_1, \dots, k_8$ spanning the Lorentzian space $\mathbb{R}^{9,1}$, with $(k_{-1}, k_{-1}) = -1$ and $(k_i, k_i) = 1$, for $0 \leq i \leq 8$:

$$\begin{aligned} \alpha_{-1} &= \frac{1}{2}(k_{-1} - 3k_0); \\ \alpha_0 &= \frac{1}{2}(k_{-1} + k_0) - k_1 + k_8; \\ \alpha_1 &= k_1 - k_2; \\ \alpha_2 &= k_2 - k_3; \\ \alpha_3 &= k_3 - k_4; \\ \alpha_4 &= k_4 - k_5; \\ \alpha_5 &= k_5 - k_6; \\ \alpha_6 &= k_6 - k_7; \\ \alpha_7 &= k_6 + k_7; \\ \alpha_8 &= -\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8). \end{aligned} \tag{30}$$

All of these roots have norm 2, with respect to the scalar product (\cdot, \cdot) , and the corresponding Cartan matrix is the Gram matrix of the \mathbf{e}_{10} even unimodular Lorentzian lattice $\text{II}_{9,1}$ made of all the vectors in $\mathbb{R}^{9,1}$, whose components are all in \mathbb{Z} or all in $\mathbb{Z} + \frac{1}{2}$ and have an integer scalar product with $\frac{1}{2}(k_{-1} + k_0 + k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8)$, as can be easily checked.

The affine Kac–Moody algebra \mathbf{e}_9 is obtained by eliminating the root α_{-1} . Notice that

$$\alpha_0 = \alpha_9 + \delta, \quad \begin{cases} \delta := \frac{1}{2}(k_{-1} + k_0), & (\text{hence } (\delta, \delta) = 0); \\ \alpha_9 := -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8), \end{cases} \tag{31}$$

where $\alpha_9 = -k_1 + k_8$ is the lowest height root of \mathbf{e}_8 and δ is a light-like vector.

3.2. \mathfrak{e}_9

The Cartan subalgebra of \mathfrak{e}_9 is the span of 10 generators, containing two new ones with respect to \mathfrak{e}_8 . We write

$$\mathfrak{h} = \text{span}\{K, d, h_i \mid i = 1, \dots, 8\}, \quad (32)$$

where

$$\begin{aligned} K &:= h_\delta, \quad \delta := \frac{1}{2}(k_{-1} + k_0); \\ d &:= h_\rho, \quad \rho := -k_{-1} + k_0, \end{aligned} \quad (33)$$

and K is a central element.

Let Φ_8 be the root system of \mathfrak{e}_8 , \mathfrak{h} be its Cartan subalgebra, h_α (resp. e_α) be a Cartan (resp. non-Cartan) generator associated to the root α , h_i for $1 \leq i \leq 8$ be a Cartan generator associated to the simple root α_i and X and Y be either Cartan or non-Cartan generators of \mathfrak{e}_8 . It is shown by Kac [27] that:

1. The root system Φ_9 of \mathfrak{e}_9 is

$$\Phi_9 := \{\alpha + m\delta \mid \alpha \in \Phi_8, m \in \mathbb{Z}\} \cup \{m\delta \mid m \in \mathbb{Z} \setminus \{0\}\}; \quad (34)$$

2. \mathfrak{e}_8 is determined by the following commutation relations:

$$\begin{aligned} [h, h'] &= 0 && \text{if } h, h' \in \mathfrak{h}; \\ [h, E_\alpha] &= (h^*, \alpha)E_\alpha && \text{if } h \in \mathfrak{h}, \alpha \in \Phi_8; \\ [E_\alpha, E_{-\alpha}] &= -h_\alpha && \text{if } \alpha \in \Phi_8; \\ [E_\alpha, E_\beta] &= 0 && \text{if } \alpha, \beta \in \Phi_8, \alpha + \beta \notin \Phi_8 \cup \{0\}; \\ [E_\alpha, E_\beta] &= \varepsilon(\alpha, \beta)E_{\alpha+\beta} && \text{if } \alpha, \beta, \alpha + \beta \in \Phi_8, \end{aligned} \quad (35)$$

where ε is Kac's asymmetry function, see Section 2.1;

3. The commutation relations in \mathfrak{e}_9 are the same as those for the central extended loop algebra of \mathfrak{e}_8 plus derivations:

$$\begin{aligned} [t^m \otimes X \oplus \lambda K \oplus \mu d, t^n \otimes Y \oplus \lambda_1 K \oplus \nu d] &= \\ (t^{m+n} \otimes [X, Y] - m\nu t^m \otimes X + n\mu t^n \otimes Y) \oplus m\delta_{m,-n}(X|Y) K \end{aligned} \quad (36)$$

with the following correspondence

$$\begin{aligned} t^m \otimes E_\alpha &\leftrightarrow x_{\alpha+m\delta}; \\ t^m \otimes H_\alpha &\leftrightarrow x_{m\delta}^\alpha \text{ if } m \neq 0; \\ t^0 \otimes H_\alpha &\leftrightarrow h_\alpha; \\ K, d &\leftrightarrow h_\delta, h_\rho, \text{ (see Equation (33))} \end{aligned} \quad (37)$$

and with the invariant non-degenerate symmetric bilinear form $(\cdot|\cdot)$ defined by:

$$(X|Y) := \begin{cases} (\alpha, \beta) & \text{if } X = H_\alpha, Y = H_\beta; \\ 0 & \text{if } X = H_\alpha, Y = E_\beta; \\ -\delta_{\alpha,-\beta} & \text{if } X = E_\alpha, Y = E_\beta. \end{cases} \quad (38)$$

For $\alpha, \beta \in \Phi_8$ (roots of \mathfrak{e}_8) and the letter h referring to a Cartan generator, the commutation relations, with no reference to the loop algebra, are:

$$\begin{aligned}
 [h, h'] &= 0 \\
 \begin{bmatrix} h_\alpha, x_{m\delta}^\beta \\ h_\delta, x_{m\delta}^\alpha \\ h_\rho, x_{m\delta}^\alpha \end{bmatrix} &= 0 && m \neq 0 \\
 \begin{bmatrix} h_\beta, x_{\alpha+m\delta} \\ h_\delta, x_{\alpha+m\delta} \\ h_\rho, x_{\alpha+m\delta} \end{bmatrix} &= \begin{matrix} (\beta, \alpha)x_{\alpha+m\delta} \\ 0 \\ mx_{\alpha+m\delta} \end{matrix} && m \neq 0 \\
 \begin{bmatrix} x_{m\delta}^\alpha, x_{n\delta}^\beta \\ x_{m\delta}^\alpha, x_{\beta+n\delta} \end{bmatrix} &= \begin{matrix} m\delta_{m,-n}(\alpha, \beta)h_\delta \\ (\alpha, \beta)x_{\beta+(m+n)\delta} \end{matrix} && \begin{matrix} m, n \neq 0 \\ m \neq 0 \end{matrix}
 \end{aligned} \tag{39}$$

$$[x_{\alpha+m\delta}, x_{\beta+n\delta}] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi_8 \cup \{0\} \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta+(m+n)\delta} & \text{if } \alpha + \beta \in \Phi_8 \\ -x_{(m+n)\delta}^\alpha & \text{if } \alpha + \beta = 0 \text{ and } m + n \neq 0 \\ -h_{\alpha+m\delta} & \text{if } \alpha + \beta = 0 \text{ and } m + n = 0 \end{cases} \tag{40}$$

Remark 4. The commutation relations of \mathfrak{e}_9 are essentially determined by those of \mathfrak{e}_8 , whose main ingredient for explicit calculations is the asymmetry function.

Remark 5. The second correspondence in (37) shows why the so called imaginary roots $m\delta$ are eight-fold degenerate: the space of generators associated to each root $m\delta$ is indeed an eight-dimensional space isomorphic to the span of $\{h_i, i = 1, \dots, 8\}$, namely the Cartan subalgebra of \mathfrak{e}_8 .

3.3. \mathfrak{e}_9 in a 1 + 1 Dimensional Toy Model

The explicit construction presented here, using a realization of the roots in terms of the orthonormal basis $\{k_i, i = -1, 0, 1, \dots, 8\}$ of $\mathbb{R}^{9,1}$, suggests letting the coordinates k_1, \dots, k_8 relate to charge/spin degrees of freedom, and interpreting the coordinates k_{-1}, k_0 as 2-momentum coordinates with a Lorentzian signature.

A crucial step in our model, which describes a universe that expands from an initial quantum state, is to restrict the particles forming that state, and hence their interactions, to lie in the subalgebra \mathfrak{n}_+ of the triangular decomposition Equation (28) of \mathfrak{e}_9 (the reason will be explained in item (TM.3) below). The restriction to \mathfrak{n}_+ has the following consequences:

(TM.1) The only commutation relations within (39) and (40) occurring in \mathfrak{n}_+ are, for $\alpha, \beta \in \Phi_8$:

$$\begin{aligned}
 \begin{bmatrix} x_{m\delta}^\alpha, x_{n\delta}^\beta \\ x_{m\delta}^\alpha, x_{\beta+n\delta} \end{bmatrix} &= \begin{matrix} 0 \\ (\alpha, \beta)x_{\beta+(m+n)\delta} \end{matrix} && \begin{matrix} m, n \neq 0 \\ m > 0, n \geq 0 \end{matrix} \\
 [x_{\alpha+m\delta}, x_{\beta+n\delta}] &= \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi_8 \cup \{0\} \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta+(m+n)\delta} & \text{if } \alpha + \beta \in \Phi_8 \\ -x_{(m+n)\delta}^\alpha & \text{if } \alpha + \beta = 0 \end{cases} \tag{41}
 \end{aligned}$$

We remark that $\alpha + \beta = 0$ in the last commutation relation implies that either α or β is negative, hence $m + n \neq 0$, being $m, n \geq 0$ and, in particular, $m > 0$ in $x_{\alpha+m\delta}$ whenever α is a negative root of \mathfrak{e}_8 ;

(TM.2) The roots involved in the interactions are not only real; for instance, $\alpha_0, -\alpha_9$ are positive roots and $(\alpha_0, -\alpha_9) = -2$. Therefore, by Proposition 5.1 of [27], at least $\alpha_0 + (-\alpha_9) = \delta$ and $\alpha_0 + 2(-\alpha_9)$ are roots; the outgoing generator $x_\delta^{\alpha_9}$ of the interaction between $x_{\alpha_0} = x_{\alpha_9+\delta}$ and $x_{-\alpha_9}$ is similar to the Cartan generator h_{α_9} , except that it carries a momentum $\delta = \frac{1}{2}(k_{-1} + k_0)$. This yields the interesting consideration that *neutral radiation fields, like the photon, are not associated to Cartan elements of infinite dimensional Kac–Moody algebras, but to imaginary roots*, a feature that is not present in Yang–Mills theories. It is also worth noticing that the second

equation in (41) implies that the neutral radiation field keeps memory of the particle–antiparticle pair that produced it, represented in the equation by the root α ;

- (TM.3) The fact that all particles are in n_+ ensures that their energy is always positive, even though they may be related to both positive and negative roots of \mathfrak{e}_8 , as revealed by the fact that α_9 is the negative root of the lowest height. In other words, we obtain both particles and antiparticles, and all of them do have positive energy;
- (TM.4) The momenta given to each particle by the interactions are light-like. Energy momentum is conserved because the outgoing particle in an elementary interaction is associated to a root, which is the sum of the roots of the incoming particles. *All particles are massless*, since momenta add up in the unique spatial direction;
- (TM.5) We give fermionic particles helicity 1/2. So far, everything runs smooth and seems physically plausible. However:
- (TM.6) The initial quantum state of the two-dimensional toy model under consideration is to be a superposition of states with momenta in opposite space directions and opposite helicity;
- (TM.7) Since $\tilde{\delta} := \frac{1}{2}(k_{-1} - k_0)$ is not a root of \mathfrak{e}_9 , we need to introduce the *auxiliary roots*

$$\alpha + m\tilde{\delta} \tag{42}$$

yielding the needed superposition of momenta. We will use the notation:

$$\begin{aligned} \tilde{p} &:= m\tilde{\delta} \text{ for } p = m\delta; \\ \tilde{x}_{\alpha+p} &:= x_{\alpha+p} + \eta_\alpha x_{\alpha+\tilde{p}}, \text{ where } \eta_\alpha^4 = 1; \\ \tilde{x}_p^\alpha &:= x_p^\alpha + \eta_\alpha x_{\tilde{p}}^\alpha, \end{aligned} \tag{43}$$

so that $\{x_{\alpha+p} \rightarrow \eta_\alpha x_{\alpha+\tilde{p}}, x_p^\alpha \rightarrow x_{\tilde{p}}^\alpha\}$ is an isomorphism $\mathfrak{e}_9 \rightarrow \tilde{\mathfrak{e}}_9$ and $x_\alpha \rightarrow \eta_\alpha x_\alpha$ is an \mathfrak{e}_8 involution, up to a sign. The coefficient $\eta_\alpha \in \{\pm 1, \pm i\}$ has been introduced to have the freedom of varying it depending on the spin of the generator related to α .

The commutation relations (41) become, for $\alpha, \beta \in \Phi_8$ and p_1, p_2 linear combination with positive integer coefficients of $\delta, \tilde{\delta}$:

$$\begin{aligned} \begin{bmatrix} x_{p_1}^\alpha, x_{p_2}^\beta \end{bmatrix} &= 0 & p_1, p_2 \neq 0 \\ \begin{bmatrix} x_{p_1}^\alpha, x_{\beta+p_2} \end{bmatrix} &= (\alpha, \beta)x_{\beta+p_1+p_2} & p_1 \neq 0 \\ \begin{bmatrix} x_{\alpha+p_1}, x_{\beta+p_2} \end{bmatrix} &= \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi_8 \cup \{0\} \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta+p_1+p_2} & \text{if } \alpha + \beta \in \Phi_8 \\ -x_{p_1+p_2}^\alpha & \text{if } \alpha + \beta = 0 \end{cases} \end{aligned} \tag{44}$$

$\alpha + \beta = 0$ in the last commutation relation implies $p_1 + p_2 \neq 0$, as remarked in item (TM.1). Moreover, $(p_1 + p_2)^2 \geq 0$, and we still have positive energy associated to all particles. It is no longer true, however, that particles are necessarily massless, as we immediately realize by the fact that $\delta + \tilde{\delta} = k_{-1}$ represents a mass at rest. We also notice that the product is to be antisymmetric; therefore,

$$x_p^{-\alpha} = -x_p^\alpha \text{ and } \begin{bmatrix} x_{p_1}^\alpha, x_{\beta+p_2} \end{bmatrix} = - \begin{bmatrix} x_{\beta+p_2}, x_{p_1}^\alpha \end{bmatrix} \tag{45}$$

Moreover, for consistency

$$x_p^{\alpha+\beta} = x_p^\alpha + x_p^\beta \tag{46}$$

We will prove in the forthcoming paper [1], as a particular case of a more general statement, that the algebra so defined is a Lie algebra;

- (TM.8) We lack two spatial dimensions. This suggests a further extension to \mathbf{e}_{12} or \mathbf{de}_{12} , or to analogous Borcherds (or generalized Kac–Moody) algebras, as we will investigate in the next sections;
- (TM.9) Our toy model still lacks three features, which urges a further extension of the algebra (investigated in the companion paper [1]):
 - (a) *Locality*, i.e., spacetime-related multiplication rules that immerse the algebra into a vertex-type algebra;
 - (b) *Space expansion* within the vertex algebra;
 - (c) *Pauli exclusion principle* that, as we will see, requires an extension to Lie superalgebra.

The above considerations imply the fact that the extension of \mathbf{e}_8 to Kac–Moody, or, even beyond, to generalized Kac–Moody (Borcherds) algebras, is very appealing to particle physics, and not only to two-dimensional conformal field theory [33].

3.4. \mathbf{e}_{12} and \mathbf{de}_{12}

The Dynkin diagram of \mathbf{e}_{12} is shown in (29). We use the same indices for the simple roots $\alpha_{-1}, \alpha_{0''}, \alpha_{0'}, \alpha_0, \alpha_1, \dots, \alpha_8$ and the orthonormal basis vectors $k_{-1}, k_{0''}, k_{0'}, k_0, k_1, \dots, k_8$ of the Lorentzian space $\mathbb{R}^{11,1}$, with $(k_i, k_i) = 1$, for $i = 0'', 0', 0, 1, \dots, 8$, and $(k_{-1}, k_{-1}) = -1$.

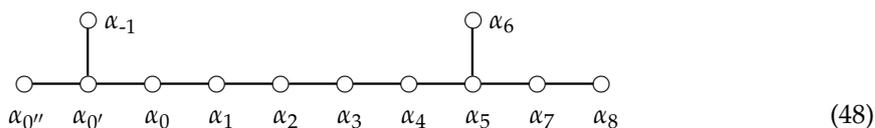
Here, it is worth presenting a different choice for the set of simple roots, with respect to (30), in which the \mathbf{e}_9 simple roots are all fermionic:

$$\begin{aligned}
 \alpha_{-1} &= -\frac{\sqrt{2}}{2}k_{0''} + \frac{\sqrt{6}}{2}k_{0'} + \sqrt{3}\delta; \\
 \alpha_{0''} &= \sqrt{2}k_{0''} + \delta; \\
 \alpha_{0'} &= \sqrt{2}k_{0'} - 2k_0 + 2\delta; \\
 \alpha_0 &= \frac{1}{2}(-k_1 + k_2 + k_3 - k_4 - k_5 - k_6 - k_7 - k_8) + \delta; \\
 \alpha_1 &= \frac{1}{2}(k_1 - k_2 - k_3 - k_4 + k_5 + k_6 + k_7 - k_8); \\
 \alpha_2 &= \frac{1}{2}(k_1 - k_2 + k_3 + k_4 - k_5 - k_6 - k_7 + k_8); \\
 \alpha_3 &= \frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 + k_7 - k_8); \\
 \alpha_4 &= \frac{1}{2}(k_1 + k_2 - k_3 - k_4 + k_5 - k_6 - k_7 + k_8); \\
 \alpha_5 &= \frac{1}{2}(-k_1 - k_2 + k_3 + k_4 + k_5 - k_6 + k_7 - k_8); \\
 \alpha_6 &= \frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 + k_6 + k_7 + k_8); \\
 \alpha_7 &= \frac{1}{2}(-k_1 - k_2 - k_3 - k_4 - k_5 + k_6 - k_7 + k_8); \\
 \alpha_8 &= \frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 - k_7 - k_8),
 \end{aligned}
 \tag{47}$$

where $\delta = \frac{1}{2}(k_0 + k_{-1})$.

The corresponding Cartan matrix is the Gram matrix of the \mathbf{e}_{12} lattice in $\mathbb{R}^{11,1}$, which is not unimodular. We interpret the coordinates $(k_{0''}, k_{0'}, k_0, k_{-1})$ as four-momentum coordinates with a Lorentzian signature.

Similar arguments hold for the Kac–Moody algebra \mathbf{de}_{12} , whose Dynkin diagram is:



This is the extension of \mathbf{e}_8 through the orthogonal Lie algebra \mathbf{d}_4 .

A possible set of simple roots in the orthonormal basis of the Lorentzian space $\mathbb{R}^{11,1}$ is:

$$\begin{aligned}
 \alpha_{-1} &= \sqrt{2}k_{0'} + 3\delta; \\
 \alpha_{0''} &= \sqrt{2}k_{0''} + \delta; \\
 \alpha_{0'} &= \sqrt{2}k_{0'} - 2k_0 + 2\delta; \\
 \alpha_0 &= \frac{1}{2}(-k_1 + k_2 + k_3 - k_4 - k_5 - k_6 - k_7 - k_8) + \delta; \\
 \alpha_1 &= \frac{1}{2}(k_1 - k_2 - k_3 - k_4 + k_5 + k_6 + k_7 - k_8); \\
 \alpha_2 &= \frac{1}{2}(k_1 - k_2 + k_3 + k_4 - k_5 - k_6 - k_7 + k_8); \\
 \alpha_3 &= \frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 + k_7 - k_8); \\
 \alpha_4 &= \frac{1}{2}(k_1 + k_2 - k_3 - k_4 + k_5 - k_6 - k_7 + k_8); \\
 \alpha_5 &= \frac{1}{2}(-k_1 - k_2 + k_3 + k_4 + k_5 - k_6 + k_7 - k_8); \\
 \alpha_6 &= \frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 + k_6 + k_7 + k_8); \\
 \alpha_7 &= \frac{1}{2}(-k_1 - k_2 - k_3 - k_4 - k_5 + k_6 - k_7 + k_8); \\
 \alpha_8 &= \frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 - k_7 - k_8),
 \end{aligned} \tag{49}$$

where $\delta := \frac{1}{2}(k_0 + k_{-1})$. One can realize at a glance that this is the same set of simple roots of \mathbf{e}_{12} , except for the root α_{-1} . We think that the possibility of discriminating between \mathbf{e}_{12} and \mathbf{de}_{12} on physical grounds can only arise when performing explicit computer calculations, which, for the case of \mathbf{de}_{12} , should undergo major simplifications due to the presence of only one irrational number, $\sqrt{2}$.

4. Beyond Kac–Moody

Several difficulties in proceeding with our program arise with Kac–Moody algebras:

- (P.1) As already mentioned above, the presence of irrational numbers in the definition of momentum variables can be very annoying in computer calculations, and it is also quite unnatural in an algebra based on integer numbers, both in the roots and in the structure constants;
- (P.2) The need of roots, as in item (TM.7) for \mathbf{e}_9 , with opposite helicity and opposite signs of the three-momentum components $k_{0''}, k_{0'}, k_0$, complicates the algebra to a large extent, since *they cannot, in pairs, be roots of the Kac–Moody algebras*. In the case of \mathbf{e}_9 , this problem can be overcome by enlarging the explicit commutation relations to consistently include the new roots, but, in the case of \mathbf{e}_{12} or \mathbf{de}_{12} , one needs to further modify the Serre relations, which does not seem to be an easy task to us;
- (P.3) However, the most important issue comes from physics: in \mathbf{e}_{12} and \mathbf{de}_{12} , three simple roots (namely, α_{-1} , $\alpha_{0''}$ and $\alpha_{0'}$) have *tachyon-like* momenta due to their positive norm. The interpretation of such tachyonic momenta, as well as the investigation of their impact on the interactions among charged particles, is beyond the scope of the present paper; computer calculations, starting from an initial state, may reveal the scenario that tachyonic simple roots may yield to.

It is our opinion that these are good motivations for focusing our investigation on generalized Kac–Moody (Borcherds) algebras, where two of the three difficulties listed above disappear.

5. Borcherds \mathfrak{B}_{12}

Borcherds algebras are a generalization of Kac–Moody algebras obtained by releasing the condition on the diagonal elements of the Cartan matrix, which are then allowed to be non-positive, as well as by restricting the Serre relations to the generators associated to positive norm simple roots [34,35].

A generalized Kac–Moody (or Borcherds) algebra \mathfrak{B} is constructed as follows.

Let H be a real vector space with a symmetric bilinear inner product (\cdot, \cdot) , and with elements h_i indexed by a countable set \mathcal{I} , such that $(h_i, h_j) \leq 0$ if $i \neq j$ and $2(h_i, h_j)/(h_i, h_i)$ is an integer if (h_i, h_i) is positive. The matrix A with entries $a_{ij} := (h_i, h_j)$ is called the symmetrized Cartan matrix of \mathfrak{B} .

The generalized Kac–Moody (or Borcherds) algebra \mathfrak{B} associated to A is defined to be the Lie algebra generated by H and elements e_i and f_i , for $i \in \mathcal{I}$, with the following relations:

1. The (injective) image of H in \mathfrak{B} is commutative;
2. If h is in H , then $[h, e_i] = (h, h_i)e_i$ and $[h, f_i] = -(h, h_i)f_i$;
3. $[e_i, f_j] = \delta_{ij}h_i$;
4. If $a_{ii} > 0$ and $i \neq j$, then $\text{ad}(e_i)^n e_j = \text{ad}(f_i)^n f_j = 0$, where $n = 1 - 2a_{ij}/a_{ii}$;
5. If $a_{ij} = 0$, then $[e_i, e_j] = [f_i, f_j] = 0$.

If $a_{ii} > 0$ for all $i \in \mathcal{I}$, then \mathfrak{B} is the Kac–Moody algebra with Cartan matrix A . In general, \mathfrak{B} has almost all of the properties of a Kac–Moody algebra, the only major difference being that \mathfrak{B} is allowed to have *imaginary simple roots*.

The root lattice \mathbb{L} is the free Abelian group generated by elements α_i for $i \in \mathcal{I}$, called *simple roots*, and \mathbb{L} has a real-valued bilinear form defined by $(\alpha_i, \alpha_j) = a_{ij}$. The Lie algebra \mathfrak{B} is then graded by \mathbb{L} with H in degree 0, e_i (resp f_i) in degree α_i (resp. $-\alpha_i$). A root is a non-zero element α of \mathbb{L} such that there are elements of \mathfrak{B} of degree α . A root r is called *real* if $(r, r) > 0$; otherwise, it is called *imaginary*. A root r is positive if it is a sum of simple roots, and negative if $-r$ is positive. Notice that *every root is either positive or negative*, [34].

We build the following symmetrized Cartan matrix for a Borcherds algebra of rank 12, which we denote by \mathfrak{B}_{12} :

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{50}$$

Notice that, for $\delta := \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8$, a four-momentum vector can be written as

$$p := p_0\alpha_{-1} + p_1(\alpha_{0'} - \alpha_{-1}) + p_2(\alpha_{0'} - \alpha_{-1}) + p_3(\delta - \alpha_{-1}). \tag{51}$$

Using the Cartan Matrix (50), we indeed obtain:

$$\begin{aligned} (\alpha_{-1}, \alpha_{-1}) &= -1; \\ (\alpha_{0'} - \alpha_{-1}, \alpha_{0'} - \alpha_{-1}) &= (\alpha_{0'} - \alpha_{-1}, \alpha_{0'} - \alpha_{-1}) = (\delta - \alpha_{-1}, \delta - \alpha_{-1}) = 1; \\ (\alpha_{-1}, \alpha_{0'} - \alpha_{-1}) &= (\alpha_{-1}, \alpha_{0'} - \alpha_{-1}) = (\alpha_{-1}, \delta - \alpha_{-1}) = 0; \\ (\alpha_{0'} - \alpha_{-1}, \alpha_{0'} - \alpha_{-1}) &= (\alpha_{0'} - \alpha_{-1}, \delta - \alpha_{-1}) = (\alpha_{0'} - \alpha_{-1}, \delta - \alpha_{-1}) = 0, \end{aligned} \tag{52}$$

hence the Lorentzian scalar product:

$$(p, p') = -p_0p'_0 + p_1p'_1 + p_2p'_2 + p_3p'_3. \tag{53}$$

Let us restrict to positive roots $r = \sum_{\mathcal{I}} \lambda_i \alpha_i$, $\mathcal{I} = \{-1, 0'', 0', 0, \dots, 8\}$, with $\lambda_i \in \mathbb{N} \cup \{0\}$, and let us denote by \mathfrak{B}^+ the corresponding subalgebra of \mathfrak{B}_{12} . The physical motivation for restricting to \mathfrak{B}^+ is that, given a positive root $r = \sum_{\mathcal{I}} \lambda_i \alpha_i$, its four-momentum is

$$p = (p_0, p_1, p_2, p_3) = (\lambda_{-1} + \lambda_{0''} + \lambda_{0'} + \lambda_0, \lambda_{0''}, \lambda_{0'}, \lambda_0), \tag{54}$$

with $\lambda_{-1}, \lambda_{0''}, \lambda_{0'}, \lambda_0 \geq 0$, implying $m^2 := -p^2 \geq 0$, namely p is either light-like or time-like. In particular:

$$p^2 = -(\lambda_{-1}^2 + 2\lambda_{-1} \sum_{i \neq -1} \lambda_i + \sum_{i \neq j, i, j \neq -1} \lambda_i \lambda_j), \quad i, j \in \{-1, 0'', 0', 0\}$$

$$\begin{cases} = 0 & \text{if } \lambda_{-1} = 0 \text{ and at most one } \lambda_i \neq 0, i \neq -1 \\ = -1 & \text{if } \lambda_{-1} = 1 \text{ and all } \lambda_i = 0, i \neq -1 \\ \leq -2 & \text{otherwise} \end{cases} \tag{55}$$

Remark 6. Notice that the mass of a particle cannot be arbitrarily small, since there is a lower limit $m \geq 1$.

For $r = \sum_{\mathcal{I}} \lambda_i \alpha_i$, $\mathcal{I} = \{-1, 0'', 0', 0, \dots, 8\}$, with $\lambda_i \in \mathbb{N} \cup \{0\}$, we introduce the notation

$$\begin{aligned} r = \alpha + p \quad p &:= \lambda_{-1} \alpha_{-1} + \lambda_{0''} \alpha_{0''} + \lambda_{0'} \alpha_{0'} + \lambda_0 \delta \text{ (see (51) and (54))} \\ \alpha &:= \lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \dots + \lambda_8 \alpha_8 \\ &= (\lambda_1 - 2\lambda_0) \alpha_1 + (\lambda_2 - 3\lambda_0) \alpha_2 + (\lambda_3 - 4\lambda_0) \alpha_3 \\ &\quad + (\lambda_4 - 5\lambda_0) \alpha_4 + (\lambda_5 - 6\lambda_0) \alpha_5 + (\lambda_6 - 3\lambda_0) \alpha_6 \\ &\quad + (\lambda_7 - 4\lambda_0) \alpha_7 + (\lambda_8 - 2\lambda_0) \alpha_8 \end{aligned} \tag{56}$$

Thus, α is in the lattice $\mathbb{L}_{\mathbf{e}_8}$ of \mathbf{e}_8 , and a precise physical meaning is assigned to positive real and imaginary roots when $\alpha \in \mathbb{L}_{\mathbf{e}_8} \setminus \{0\}$:

Proposition 3. A generator in \mathfrak{B}^+ , associated to a positive root $r = \alpha + p$, with $\alpha \in \mathbb{L}_{\mathbf{e}_8} \setminus \{0\}$ and momentum $p \neq 0$, is massive if and only if $\alpha + p$ is an imaginary root; it is massless if and only if r is real, in which case, it is a positive real root of $\mathbf{e}_9 \subset \mathfrak{B}_{12}$.

Proof. The proof consists of the following steps:

1. From Proposition 2.1. of [34], it holds that every positive root $r = \alpha + p$ is conjugate under the Weyl group to a root $r_0 = \alpha' + p'$, such that either r_0 is a simple real root α_i , $i \in \{0, 1, \dots, 8\}$, or it is a positive root in the Weyl chamber (namely $(r_0, \alpha_i) \leq 0$ for all simple roots α_i);
2. Since r and r_0 are conjugate under the Weyl group, then $(r, r) = (r_0, r_0)$;
3. If r_0 is a real simple root, then it is a root of \mathbf{e}_9 and $p'^2 = 0$; r_0 is real and so is r . Since the Weyl group is generated by the reflections $\rho - (\rho, \alpha_i) \alpha_i$, where the α_i simple roots are real, hence $\alpha_i \in \mathbf{e}_9 \subset \mathfrak{B}_{12}$, it coincides with the Weyl group of \mathbf{e}_9 . By applying to r_0 Weyl reflections, we stay within \mathbf{e}_9 , since every Kac–Moody algebra is invariant under the Weyl group; therefore, r is a real root of \mathbf{e}_9 , namely $r = \alpha + m\delta$, $\alpha \in \mathbf{e}_8$ and $m\delta$ is light-like;
4. If r_0 is in the Weyl chamber, then $(r, r) = (r_0, r_0) = \sum \lambda_i (r_0, \alpha_i) \leq 0$, $i \in \mathcal{I}$, since all λ_i are positive, r_0 being a positive root; thus, r is imaginary;
5. Since $(r, r) = \alpha^2 + p^2 \leq 0$ with $\alpha^2 \geq 2$, then $m^2 \geq \alpha^2 \geq 2$, and the particle associated to r is massive.

□

Remark 7. In the massive case, the lower limit of the mass grows with the norm of α : if $\alpha \in \mathbb{L}_{\mathbf{e}_8} \setminus \{0\}$ is not a root of Φ_8 , then the mass is certainly bigger than the lower mass a particle corresponding to a root of Φ_8 may have. We also notice that charged massless particles ($\alpha \neq 0$ in the root $\alpha + p$) are quite peculiar, since their momentum can only be in one direction. The photon is not in this class, since it has $\alpha = 0$, but the (non-virtual) gluons are. A non-virtual photon can be produced in a decay process [1].

Remark 8. We emphasize that two of the three problems listed in Section 4 about Kac–Moody algebras vanish in the Borcherds algebra \mathfrak{B}_{12} . These are obviously (P.1) and (P.3). However, (P.2) still remains [1].

6. Conclusions

In this paper, the first of a series of two papers with the same title, we have described the basic principles of a model of quantum gravity at the early stages of the universe. We have explained how spacetime is generated from an initial state and how it expands and is driven by interactions in a purely algebraic context. We have investigated the mathematical structures that may suit our purpose: they are rank-12 infinite-dimensional algebras, extending \mathbf{e}_8 and including 4-momenta. We have also discussed why a celebrated no-go theorem on \mathbf{e}_8 does not apply in our settings.

The companion paper [1], based on the treatment and considerations of this paper, will focus on a particular rank-12 algebra \mathfrak{g}_u in order to build a model for quantum gravity. In particular, it will turn \mathfrak{g}_u into a Lie superalgebra in order to fulfill the Pauli exclusion principle without producing superpartners. It will also discuss scattering processes and decays.

The algebra \mathfrak{g}_u is based on a simplified version of a Borcherds algebra, where *based on* means that we still have to enlarge the algebra with roots that take into account the coupling of four momenta with those having three momenta of the opposite sign and same energy (see (P.2) of Section 4). Nothing would prevent us from starting from a Borcherds algebra, but we would not have explicit commutation relations, as we do in [1].

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Appendix A

In this appendix, we show the charge content of the \mathbf{e}_8 part of the algebra, root by root. Table A1 show all of the roots of \mathbf{e}_8 , grouped by the points of the Magic Star obtained by projecting \mathbf{e}_8 on the plane of color \mathbf{a}_2^c , see Section 2.2. The leptons, being colorless, lie in the center of this Magic Star, forming the \mathbf{e}_6 roots of Table A2. In addition, the roots of this \mathbf{e}_6 lie on a Magic Star (of a smaller dimension) once projected on the plane of flavor \mathbf{a}_2^f . The families of leptons and quarks are shown in Tables A3 and A4, together with their spin. For the quarks sector, we only consider one color, which we name *blue*; similar tables for *red* and *green* colors are trivially obtained by exchanging indices.

Table A1. The Magic Star of e_8 ; $q_{e.m.}(\alpha) = (\alpha, -\frac{1}{3}(k_1 + k_2 + k_3) + k_4)$; $r_c(\alpha) = (\alpha, k_1 - k_2)$, $s_c(\alpha) = (\alpha, k_1 + k_2 - 2k_3)$.

Roots		$q_{e.m.}$	#	$\{r_c, s_c\}$
$\pm(k_1 - k_2)$		0	2	$\pm\{2, 0\}$
$\pm(k_2 - k_3)$		0	2	$\pm\{-1, 3\}$
$\pm(k_1 - k_3)$		0	2	$\pm\{1, 3\}$
$\pm k_i \pm k_j$	$5 \leq i < j \leq 8$	0	24	{0, 0}
$k_4 \pm k_i$	$5 \leq i \leq 8$	1	8	
$-k_4 \pm k_i$	$5 \leq i \leq 7$	-1	8	
$\frac{1}{2}(\pm(k_1 + k_2 + k_3 + k_4) \pm \dots \pm k_8)$	even # of +	0	16	
$\frac{1}{2}(-k_1 + k_2 + k_3) + k_4 \pm \dots \pm k_8)$	even # of +	1	8	{0, 0}
$\frac{1}{2}(+k_1 + k_2 + k_3) - k_4 \pm \dots \pm k_8)$	even # of +	-1	8	
$-k_2 - k_3, k_1 + k_4$		2/3	2	
$k_1 - k_4$		-4/3	1	
$k_1 \pm k_i$	$i = 5, \dots, 8$	-1/3	8	
$\frac{1}{2}(k_1 - k_2 - k_3 + k_4 \pm \dots \pm k_8)$	even # of +	2/3	8	
$\frac{1}{2}(k_1 - k_2 - k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-1/3	8	{-1, -1}
$+k_2 + k_3, -k_1 - k_4$		-2/3	2	
$-k_1 + k_4$		4/3	1	
$-k_1 \pm k_i$	$i = 5, \dots, 8$	1/3	8	
$\frac{1}{2}(-k_1 + k_2 + k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-2/3	8	{-1, 1}
$\frac{1}{2}(-k_1 + k_2 + k_3 + k_4 \pm \dots \pm k_8)$	even # of +	1/3	8	
$-k_1 - k_3, k_2 + k_4$		2/3	2	
$k_2 - k_4$		-4/3	1	
$k_2 \pm k_i$	$i = 5, \dots, 8$	-1/3	8	
$\frac{1}{2}(-k_1 + k_2 - k_3 + k_4 \pm \dots \pm k_8)$	even # of +	2/3	8	
$\frac{1}{2}(-k_1 + k_2 - k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-1/3	8	{1, -1}
$k_1 + k_3, -k_2 - k_4$		-2/3	2	
$-k_2 + k_4$		4/3	1	
$-k_2 \pm k_i$	$i = 5, \dots, 8$	1/3	8	
$\frac{1}{2}(k_1 - k_2 + k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-2/3	8	{0, -2}
$\frac{1}{2}(k_1 - k_2 + k_3 + k_4 \pm \dots \pm k_8)$	even # of +	1/3	8	
$-k_1 - k_2, k_3 + k_4$		2/3	2	
$k_3 - k_4$		-4/3	1	
$k_3 \pm k_i$	$i = 5, \dots, 8$	-1/3	8	
$\frac{1}{2}(-k_1 - k_2 + k_3 + k_4 \pm \dots \pm k_8)$	even # of +	2/3	8	
$\frac{1}{2}(-k_1 - k_2 + k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-1/3	8	{0, 2}
$k_1 + k_2, -k_3 - k_4$		-2/3	2	
$-k_3 + k_4$		4/3	1	
$-k_3 \pm k_i$	$i = 5, \dots, 8$	1/3	8	
$\frac{1}{2}(k_1 + k_2 - k_3 - k_4 \pm \dots \pm k_8)$	even # of +	-2/3	8	{0, 2}
$\frac{1}{2}(k_1 + k_2 - k_3 + k_4 \pm \dots \pm k_8)$	even # of +	1/3	8	

Table A2. The Magic Star of e_6 ; $q_{e.m.}(\alpha) = (\alpha, -\frac{1}{3}(k_1 + k_2 + k_3) + k_4)$; $r_f(\alpha) = (\alpha, k_4 - k_5)$, $s_f(\alpha) = (\alpha, k_4 + k_5 - 2k_6)$.

Roots	$q_{e.m.}$	# of Roots	$\{r_f, s_f\}$
$\pm(k_4 - k_5)$	± 1	2	$\pm\{2, 0\}$
$\pm(k_5 - k_6)$	0	2	$\pm\{-1, 3\}$
$\pm(k_4 - k_6)$	± 1	2	$\pm\{1, 3\}$
$\pm k_7 \pm k_8$	0	4	$\{0, 0\}$
$\frac{1}{2}(\pm(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \pm k_7 \pm k_8)$ even # of +	0	4	
$\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 + k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	1	2	
$\frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	-1	2	
$-k_5 - k_6$	0	1	$\{1, 1\}$
$k_4 \pm k_i$ $i = 7, 8$	1	4	
$\frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 - k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	1	2	
$k_5 + k_6$	0	1	$\{-1, -1\}$
$-k_4 \pm k_i$ $i = 7, 8$	-1	4	
$\frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	-1	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 - k_4 + k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$-k_4 - k_6$	-1	1	$\{-1, 1\}$
$k_5 \pm k_i$ $i = 7, 8$	0	4	
$\frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	-1	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 - k_4 + k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$k_4 + k_6$	1	1	$\{1, -1\}$
$-k_5 \pm k_i$ $i = 7, 8$	0	4	
$\frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 - k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	1	2	
$-k_4 - k_5$	-1	1	$\{0, -2\}$
$k_6 \pm k_i$ $i = 7, 8$	0	4	
$\frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	-1	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 - k_4 - k_5 + k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$k_4 + k_5$	1	1	$\{0, 2\}$
$-k_6 \pm k_i$ $i = 7, 8$	0	4	
$\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	0	2	
$\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 + k_5 - k_6 \pm k_7 \pm k_8)$ even # of +	1	2	

Table A3. The lepton families with their spin-z.

Roots [$k := k_1 + k_2 + k_3$]	$q_{e.m.}$	Lepton	$\sigma_z := \frac{1}{2}(\alpha, \rho_1 + \rho_2)$	$\{r_f, s_f\}$
$\frac{1}{2}(k + k_4 - k_5 - k_6 + k_7 + k_8)$	0	ν'_τ	-1/2	{1, 1}
$\frac{1}{2}(k + k_4 - k_5 - k_6 - k_7 - k_8)$	0	ν'_χ	-1/2	
$\frac{1}{2}(-k + k_4 - k_5 - k_6 + k_7 - k_8)$	1	e^+	-1/2	
$\frac{1}{2}(-k + k_4 - k_5 - k_6 - k_7 + k_8)$	1	μ^+	-1/2	
$-\frac{1}{2}(k + k_4 - k_5 - k_6 + k_7 + k_8)$	0	$\bar{\nu}'_\tau$	1/2	{-1, -1}
$-\frac{1}{2}(k + k_4 - k_5 - k_6 - k_7 - k_8)$	0	$\bar{\nu}'_\chi$	1/2	
$-\frac{1}{2}(-k + k_4 - k_5 - k_6 + k_7 - k_8)$	-1	e^-	1/2	
$-\frac{1}{2}(-k + k_4 - k_5 - k_6 - k_7 + k_8)$	-1	μ^-	1/2	
$\frac{1}{2}(k - k_4 + k_5 - k_6 + k_7 + k_8)$	-1	τ^-	1/2	{-1, 1}
$\frac{1}{2}(k - k_4 + k_5 - k_6 - k_7 - k_8)$	-1	χ^-	1/2	
$\frac{1}{2}(-k - k_4 + k_5 - k_6 + k_7 - k_8)$	0	$\bar{\nu}'_e$	1/2	
$\frac{1}{2}(-k - k_4 + k_5 - k_6 - k_7 + k_8)$	0	$\bar{\nu}'_\mu$	1/2	
$-\frac{1}{2}(k - k_4 + k_5 - k_6 + k_7 + k_8)$	1	τ^+	-1/2	{1, -1}
$-\frac{1}{2}(k - k_4 + k_5 - k_6 - k_7 - k_8)$	1	χ^+	-1/2	
$-\frac{1}{2}(-k - k_4 + k_5 - k_6 + k_7 - k_8)$	0	ν'_e	-1/2	
$-\frac{1}{2}(-k - k_4 + k_5 - k_6 - k_7 + k_8)$	0	ν'_μ	-1/2	
$\frac{1}{2}(k - k_4 - k_5 + k_6 + k_7 + k_8)$	-1	τ^-	-1/2	{0, -2}
$\frac{1}{2}(k - k_4 - k_5 + k_6 - k_7 - k_8)$	-1	χ^-	-1/2	
$\frac{1}{2}(-k - k_4 - k_5 + k_6 + k_7 - k_8)$	0	$\bar{\nu}'_e$	-1/2	
$\frac{1}{2}(-k - k_4 - k_5 + k_6 - k_7 + k_8)$	0	$\bar{\nu}'_\mu$	-1/2	
$-\frac{1}{2}(k - k_4 - k_5 + k_6 + k_7 + k_8)$	1	τ^+	1/2	{0, 2}
$-\frac{1}{2}(k - k_4 - k_5 + k_6 - k_7 - k_8)$	1	χ^+	1/2	
$-\frac{1}{2}(-k - k_4 - k_5 + k_6 + k_7 - k_8)$	0	ν'_e	1/2	
$-\frac{1}{2}(-k - k_4 - k_5 + k_6 - k_7 + k_8)$	0	ν'_μ	1/2	
$\frac{1}{2}(k - k_4 - k_5 - k_6 + k_7 - k_8)$	-1	μ^-	-1/2	{0, 0}
$\frac{1}{2}(k - k_4 - k_5 - k_6 - k_7 + k_8)$	-1	e^-	-1/2	
$\frac{1}{2}(-k - k_4 - k_5 - k_6 + k_7 + k_8)$	0	$\bar{\nu}'_\chi$	-1/2	
$\frac{1}{2}(-k - k_4 - k_5 - k_6 - k_7 - k_8)$	0	$\bar{\nu}'_\tau$	-1/2	
$-\frac{1}{2}(k - k_4 - k_5 - k_6 + k_7 - k_8)$	1	μ^+	1/2	
$-\frac{1}{2}(k - k_4 - k_5 - k_6 - k_7 + k_8)$	1	e^+	1/2	
$-\frac{1}{2}(-k - k_4 - k_5 - k_6 + k_7 + k_8)$	0	ν'_χ	1/2	
$-\frac{1}{2}(-k - k_4 - k_5 - k_6 - k_7 - k_8)$	0	ν'_τ	1/2	

Table A4. The flavor families of blue $((1,1))$ quarks with their spin-z.

Roots [$k' := k_1 - k_2 - k_3$]	$q_{e.m.}$	Blue Quark	$\sigma_z := \frac{1}{2}(\alpha, \rho_1 + \rho_2)$	$\{r_f, s_f\}$
$\frac{1}{2}(k' + k_4 - k_5 - k_6 + k_7 + k_8)$	1/3	\bar{b}'	-1/2	{1, 1}
$\frac{1}{2}(k' + k_4 - k_5 - k_6 - k_7 - k_8)$	1/3	\bar{B}'	-1/2	
$\frac{1}{2}(-k' + k_4 - k_5 - k_6 + k_7 - k_8)$	2/3	u	-1/2	
$\frac{1}{2}(-k' + k_4 - k_5 - k_6 - k_7 + k_8)$	2/3	c	-1/2	
$-\frac{1}{2}(k' + k_4 - k_5 - k_6 + k_7 + k_8)$	-1/3	b'	1/2	{-1, -1}
$-\frac{1}{2}(k' + k_4 - k_5 - k_6 - k_7 - k_8)$	-1/3	B'	1/2	
$-\frac{1}{2}(-k' + k_4 - k_5 - k_6 + k_7 - k_8)$	-2/3	\bar{u}	1/2	
$-\frac{1}{2}(-k' + k_4 - k_5 - k_6 - k_7 + k_8)$	-2/3	\bar{c}	1/2	
$\frac{1}{2}(k' - k_4 + k_5 - k_6 + k_7 + k_8)$	-2/3	\bar{t}	1/2	{-1, 1}
$\frac{1}{2}(k' - k_4 + k_5 - k_6 - k_7 - k_8)$	-2/3	\bar{T}	1/2	
$\frac{1}{2}(-k' - k_4 + k_5 - k_6 + k_7 - k_8)$	-1/3	d'	1/2	
$\frac{1}{2}(-k' - k_4 + k_5 - k_6 - k_7 + k_8)$	-1/3	s'	1/2	
$-\frac{1}{2}(k' - k_4 + k_5 - k_6 + k_7 + k_8)$	2/3	t	-1/2	{1, -1}
$-\frac{1}{2}(k' - k_4 + k_5 - k_6 - k_7 - k_8)$	2/3	T	-1/2	
$-\frac{1}{2}(-k' - k_4 + k_5 - k_6 + k_7 - k_8)$	1/3	\bar{d}'	-1/2	
$-\frac{1}{2}(-k' - k_4 + k_5 - k_6 - k_7 + k_8)$	1/3	\bar{s}'	-1/2	
$\frac{1}{2}(k' - k_4 - k_5 + k_6 + k_7 + k_8)$	-2/3	\bar{t}	-1/2	{0, -2}
$\frac{1}{2}(k' - k_4 - k_5 + k_6 - k_7 - k_8)$	-2/3	\bar{T}	-1/2	
$\frac{1}{2}(-k' - k_4 - k_5 + k_6 + k_7 - k_8)$	-1/3	d'	-1/2	
$\frac{1}{2}(-k' - k_4 - k_5 + k_6 - k_7 + k_8)$	-1/3	s'	-1/2	
$-\frac{1}{2}(k' - k_4 - k_5 + k_6 + k_7 + k_8)$	2/3	t	1/2	{0, 2}
$-\frac{1}{2}(k' - k_4 - k_5 + k_6 - k_7 - k_8)$	2/3	T	1/2	
$-\frac{1}{2}(-k' - k_4 - k_5 + k_6 + k_7 - k_8)$	1/3	\bar{d}'	1/2	
$-\frac{1}{2}(-k' - k_4 - k_5 + k_6 - k_7 + k_8)$	1/3	\bar{s}'	1/2	
$\frac{1}{2}(k' - k_4 - k_5 - k_6 + k_7 - k_8)$	-2/3	\bar{c}	-1/2	{0, 0}
$\frac{1}{2}(k' - k_4 - k_5 - k_6 - k_7 + k_8)$	-2/3	\bar{u}	-1/2	
$\frac{1}{2}(-k' - k_4 - k_5 - k_6 + k_7 + k_8)$	-1/3	B'	-1/2	
$\frac{1}{2}(-k' - k_4 - k_5 - k_6 - k_7 - k_8)$	-1/3	b'	-1/2	
$-\frac{1}{2}(k' - k_4 - k_5 - k_6 + k_7 - k_8)$	2/3	c	1/2	
$-\frac{1}{2}(k' - k_4 - k_5 - k_6 - k_7 + k_8)$	2/3	u	1/2	
$-\frac{1}{2}(-k' - k_4 - k_5 - k_6 + k_7 + k_8)$	1/3	\bar{B}'	1/2	
$-\frac{1}{2}(-k' - k_4 - k_5 - k_6 - k_7 - k_8)$	1/3	\bar{b}'	1/2	

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