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New Self-Adaptive Inertial-like Proximal Point Methods for the Split Common Null Point Problem

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Abstract: Symmetry plays an important role in solving practical problems of applied science, especially in algorithm innovation. In this paper, we propose what we call the self-adaptive inertial-like proximal point algorithms for solving the split common null point problem, which use a new inertial structure to avoid the traditional convergence condition in general inertial methods and avoid computing the norm of the difference between x_n and x_{n-1} before choosing the inertial parameter. In addition, the selection of the step-sizes in the inertial-like proximal point algorithms does not need prior knowledge of operator norms. Numerical experiments are presented to illustrate the performance of the algorithms. The proposed algorithms provide enlightenment for the further development of applied science in order to dig deep into symmetry under the background of technological innovation.

Keywords: split common null point problem; inertial-like proximal point algorithm; resolvent operator; strong convergence

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1. Introduction

We are concerned with the following split common null point problem (*SCNPP*):

$$\text{find } x^* \in H_1 \text{ to solve } 0 \in A(x^*) \tag{1}$$

and

$$y^* = Tx^* \in H_2 \text{ to solve } 0 \in B(y^*), \tag{2}$$

where H_1 and H_2 are Hilbert spaces, $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$ are set-valued mappings, and $T : H_1 \to H_2$ is a nonzero bounded linear operator.

The SCNPP (1) and (2), which covers the convex feasibility problem (CFP) (Censor and Elfving [1]), variational inequalities (VIs) (Moudafi [2]), and many constrained optimization problems as special cases, has attracted important attention both theoretically and practically (see Byrne [3], Moudafi and Thukur [4]).

The main idea to solve SCNPP comes from symmetry, that is, invariance. Therefore, fixed point theory plays a key role here. We recall the resolvent operator $J_r^A = (I + rA)^{-1}, r > 0$, which plays an essential role in the approximation theory for zero points of maximal monotone operators as well as in solving (1) and (2) and has the following key facts.

Fact 1: The resolvent is not only always single-valued but also firmly monotone:

$$\langle J_r^A x - J_r^A y, x - y\rangle \geq \|J_r^A x - J_r^A y\|^2. \tag{3}$$

Fact 2: Using the resolvent operator, the problem (1) and (2) can be written as a fixed point problem:

$$x^* = J_\lambda^A(I - \gamma T^*(I - J_\lambda^B)T)x^*, \lambda > 0, \gamma > 0.$$

Fact 2 transforms the problem (1) and (2) into a fixed-point problem, and the research of the latter reflects the invariance in transformation, which is the essence of symmetry. Based on **Fact 2**, Byrne, et al. [5] proposed the following forward–backward algorithm:

$$x_{n+1} = J_\lambda^A(x_n - \gamma T^*(I - J_\lambda^B)Tx_n) \tag{4}$$

and obtained weak convergence, where T^* is the adjoint of T, the stepsize $\gamma \in (0, \frac{2}{L})$ with $L = \|T^*T\|$.

At the same time, the inertial method originating from the heavy ball with friction system has attracted increasing attention thanks to its convergence properties in the field of continuous optimization. Therefore, many scholars have combined the forward–backward method (4) with the inertial algorithm to study the SCNPP and proposed some iterates. For related works, one can consult Alvarez and Attouch [6], Alvarez [7], Attouch et al. [8–10], Akgül [11], Hasan et al. [12], Khdhr et al. [13], Ochs et al. [14,15], Dang et al. [16], Soleymani and Akgül [17], Suantai et al. [18,19], Dong et al. [20], Sitthithakerngkiet et al. [21], Kazmi and Rizvi [22], Promluang and Kumman [23], Eslamian et al. [24], and references therein.

Although these algorithms improved the numerical solution of the split common null point problem, there exist two common drawbacks: one is that the step size depends on the norm of the linear operator T, which means there is a high computation cost, because the norm of the linear operator must be estimated before selecting the step size; another drawback is that the following condition is required:

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty, \tag{5}$$

which means that one not only needs to calculate the norm of the difference between x_n and x_{n-1} in advance at each step but also check if α_n satisfies (5).

So it is natural to ask the following questions:

Question 1.1 Can we construct the iterate for SCNPP whose step size does not depend on the norm of the linear operator T?

Question 1.2 Can condition (5) be removed from the inertial method and still ensure the convergence of the sequence? Namely, can we construct a new inertial algorithm to solve SCNPP (1) and (2) without prior computation of the norm of the difference between x_n and x_{n-1}?

The purpose of this paper is to present a new self-adaptive inertial-like technique to give an affirmative answer to the above questions. Importantly, the innovative algorithms provide an idea of how to use symmetry to solve real-world problems in applied science.

2. Preliminaries

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be the inner product and the induced norm in a Hilbert space H, respectively. For a sequence $\{x_n\}$ in H, denote $x_n \to x$ and $x_n \rightharpoonup x$ by the strong and weak convergence to x of $\{x_n\}$, respectively. Moreover, the symbol $\omega_w(x_n)$ represents the ω-weak limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The identity below is useful:

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ & - \alpha\beta\|x - y\|^2 - \beta\gamma\|y - z\|^2 - \gamma\alpha\|x - z\|^2 \end{aligned} \tag{6}$$

for all $x, y, z \in \mathbb{R}$ and $\alpha + \beta + \gamma = 1$.

Definition 1. *A multivalued mapping $A : H \to 2^H$ with domain $D(A) = \{x \in H, Ax \neq \emptyset\}$ is monotone if*

$$\langle x - y, x^* - y^* \rangle \geq 0, \tag{7}$$

for all $x, y \in D(A)$, $x^* \in A(x)$, and $y^* \in A(y)$. A monotone operator A is referred to be maximal if its graph is not properly contained in the graph of any other monotone operator.

Definition 2. *Let H be a real Hilbert space and let $h : H \to H$ be a mapping.*

(i) h is called Lipschitz with constant $\kappa > 0$ if $\|h(x) - h(y)\| \leq \kappa\|x - y\|$ for all $x, y \in H$.
(ii) h is called nonexpansive if $\|h(x) - h(y)\| \leq \|x - y\|$ for all $x, y \in H$.

From **Fact 1**, we can conclude that J_r^A is a nonexpansive operator if A is a maximal monotone mapping. Moreover, due to the work of Aoyama et al. [25], we have the following property:

$$\langle J_r^A x - y, x - J_r^A x \rangle \geq 0, y \in A^{-1}(0), \tag{8}$$

where $A^{-1}(0) = \{z \in H : 0 \in Az\}$.

Definition 3. *Let C be a nonempty closed convex subset of H. We use P_C to denote the projection from H onto C; namely,*

$$P_C x = \arg\min\{\|x - y\| : y \in C\}, \quad x \in H.$$

The following significant characterization of the projection P_C should be recalled: given $x \in H$ and $y \in C$,

$$P_C x = z \quad \Longleftrightarrow \quad \langle x - z, y - z \rangle \leq 0, \quad y \in C. \tag{9}$$

Lemma 1. *(Xu [26], Maingé [27]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n a_n)a_n + \gamma_n \delta_n + c_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \sup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$;
(3) $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2. *(see e.g., Opial [28]) Let H be a real Hilbert space and $\{x_n\}$ be a bounded sequence in H. Assume there exists a nonempty subset $S \subset H$ satisfying the properties:*

(i) $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in S$,
(ii) $\omega_w(x_n) \subset S$.

Then, there exists $\bar{x} \in S$ such that $\{x_n\}$ converges weakly to $\bar{x}$.

Lemma 3. *(Maingé [29]) Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at the infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\sigma(n)\}_{n \geq n_0}$ defined by*

$$\sigma(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, $\{\sigma(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty}\sigma(n) = \infty$ *and, for all* $n \geq n_0$,

$$\max\{\Gamma_{\sigma(n)}, \Gamma_n\} \leq \Gamma_{\sigma(n)+1}.$$

3. Main Results

3.1. Variant of Discretization

Inspired by the discretization of the second order dynamic system $\frac{d^2x}{dt^2} + \lambda(t)\frac{dx}{dt} + Ax = 0$, we consider the following iterative sequence

$$x_{n+1} - x_{n-1} - \theta_n(x_n - x_{n-1}) + \gamma_n A(x_{n+1}) \ni 0, \tag{10}$$

where x_0, x_1 are two arbitrary initial points, and γ_n is a real nonnegative number. This recursion can be rewritten as

$$x_{n+1} = J^A_{\gamma_n}(x_{n-1} + \theta_n(x_n - x_{n-1})),$$

which proves that the sequence $\{x_n\}$ satisfying (10) always exists for any choice of the sequences $\{\gamma_n\}$ and $\{\theta_n\}$, provided that $\gamma_n > 0$.

To distinguish from Alvarez and Attouch's *Inertial-Prox* algorithm ([6]), we call it inertial-like proximal point algorithm. Combining the inertial-like proximal point algorithm and the forward–backward method, we propose the following self adaptive inertial-like proximal algorithms.

3.2. Some Assumptions

Assumption 1. *Throughout the rest of this paper, we assume that* H_1 *and* H_2 *are Hilbert spaces. We study the split common null point problem (SCNPP) as (1) and (2), where* $A : H_1 \to 2^{H_1}$ *and* $B : H_2 \to 2^{H_2}$ *are set-valued maximal monotone mappings, respectively, and* $T : H_1 \to H_2$ *is a bounded linear operator,* T^* *means the adjoint of* T.

Assumption 2. *The functions are defined as:*

$$f(x) = \frac{1}{2}\|(I - J^A_r)x\|^2, \quad F(x) = (I - J^A_r)x, \quad r > 0;$$

and

$$g(x) = \frac{1}{2}\|(I - J^B_\mu)Tx\|^2, \quad G(x) = T^*(I - J^B_\mu)Tx, \quad \mu > 0.$$

Assumption 3. *Denote by* Ω *the solution set of the SCNPP (1) and (2); namely,*

$$\Omega = \{x^* \in H_1 : 0 \in A(x^*) \text{ and } 0 \in B(Tx^*)\},$$

and we always assume $\Omega \neq \emptyset$.

3.3. Inertial-like Proximal Point Algorithms

Remark 1. *It is not hard to find that if* $\|F(y_n)\|^2 + \|G(y_n)\|^2 = 0$ *for some* $n \geq 0$*, then* x_n *is a solution of the SCNPP (1) and (2), and the iteration process is terminated in finite iterations. In addition, if* $\theta_n \equiv 1$ *and step size* τ_n *depends on the norm of linear operator* T*, Algorithm 1 recovers Byrne et al. [5].*

Remark 2. *In the subsequent convergence analysis, we will always assume that the two algorithms generate an infinite sequence, namely, the algorithms are not terminated in finite iterations. In addition, in the simulation experiments, we will give a stop criterion to end the iteration for practice.*

Algorithm 1 Self adaptive inertial-like algorithm

Initialization: Choose a sequence $\{\theta_n\} \subset [0,1]$ satisfying one of the three cases: **(I.)** $\theta_n \in (0,1)$ such that $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$; **(II.)** $\theta_n \equiv 0$; or **(III.)** $\theta_n \equiv 1$. Select arbitrary initial points x_0, x_1.

Iterative Step: After constructing the nth-iterate x_n, compute

$$y_n = x_{n-1} + \theta_n(x_n - x_{n-1}), \tag{11}$$

and define the $(n+1)$th iterate by

$$x_{n+1} = J_r^A(I - \tau_n T^*(I - J_\mu^B)T)y_n, \tag{12}$$

where τ_n is defined as

$$\tau_n = \begin{cases} \frac{g(y_n)}{\|F(y_n)\|^2 + \|G(y_n)\|^2}, & \text{if } \|F(y_n)\|^2 + \|G(y_n)\|^2 \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

3.4. Convergence Analysis of Algorithms

Theorem 1. *If the assumptions (**A1**)–(**A3**) are satisfied, then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point $z \in \Omega$.*

Proof. To this end, the following three situations will be discussed: (I). $\theta_n \in (0,1)$, $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$; (II). $\theta_n \equiv 0$; and (III). $\theta_n \equiv 1$.

(**I**). First, we consider the case of $\theta_n \in (0,1)$, $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$.

Without loss of generality, we take $z \in \Omega$, and then we have $z = J_r^A z$, $Tz = J_\mu^B Tz$ and $J_r^A(I - \tau_n T^*(I - J_\mu^B)T)z = z$ from **Fact 2**. It turns out from (11) and (7) that

$$\begin{aligned} \|y_n - z\|^2 &= \|x_{n-1} + \theta_n(x_n - x_{n-1}) - z\|^2 \\ &= \theta_n\|x_n - z\|^2 + (1-\theta_n)\|x_{n-1} - z\|^2 - \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{13}$$

Since J_r^A is nonexpansive, it follows from (12) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|J_r^A(I - \tau_n T^*(I - J_\mu^B)T)y_n - z\|^2 \\ &\leq \|(I - \tau_n T^*(I - J_\mu^B)T)y_n - z\|^2 \\ &= \|y_n - z\|^2 - 2\tau_n\langle y_n - z, T^*(I - J_\mu^B)Ty_n\rangle + \tau_n^2\|T^*(I - J_\mu^B)Ty_n\|^2 \\ &= \|y_n - z\|^2 - 2\tau_n\langle Ty_n - Tz, (I - J_\mu^B)Ty_n\rangle + \tau_n^2\|G(y_n)\|^2. \end{aligned} \tag{14}$$

It follows from the property (8) of resolvent operator that

$$\langle J_\mu^B Ty_n - Tz, (I - J_\mu^B)Ty_n\rangle \geq 0, Tz \in B^{-1}(0),$$

and then from the definition of $g(x)$, we have that

$$\begin{aligned} & \langle Ty_n - Tz, (I - J_\mu^B)Ty_n\rangle \\ = & \langle Ty_n - J_\mu^B Ty_n, (I - J_\mu^B)Ty_n\rangle + \langle J_\mu^B Ty_n - Tz, (I - J_\mu^B)Ty_n\rangle \\ = & \|J_\mu^B Ty_n - Ty_n\|^2 + \langle J_\mu^B Ty_n - Tz, (I - J_\mu^B)Ty_n\rangle \\ \geq & 2g(y_n). \end{aligned}$$

Notice the definition of τ_n, we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\le \|y_n-z\|^2-4\tau_n g(y_n)+\tau_n^2\|G(y_n)\|^2 \\
&= \theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2-\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2 \\
&\quad -4\tau_n g(y_n)+\tau_n^2\|G(y_n)\|^2 \\
&\le \theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2-\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2 \\
&\quad -\frac{3g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2},
\end{aligned}
\tag{15}
$$

which means that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\le \theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2 \\
&\le \max\{\|x_n-z\|^2,\|x_{n-1}-z\|^2\},
\end{aligned}
$$

and, hence, the sequence $\{\|x_n-z\|\}$ is bounded, and so in turn is $\{y_n\}$.

It may be assumed that the sequence $\{\|x_n-z\|\}$ is not decreasing at the infinity in the sense that there exists a subsequence $\{\sigma(n)\}$ of positive integers such that there exists a nondecreasing sequence $\sigma(n)$ for $n \ge N_1$ (for some N_1 large enough) such that $\sigma(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\sigma(n)}-z\| \le \|x_{\sigma(n)+1}-z\|,$$

for each $n \ge 0$.

Notice that (15) holds for each $\sigma(n)$, so from (15) with n replaced by $\sigma(n)$, we have

$$
\begin{aligned}
\|x_{\sigma(n)+1}-z\|^2 &\le \theta_{\sigma(n)}\|x_{\sigma(n)}-z\|^2+(1-\theta_{\sigma(n)})\|x_{\sigma(n)-1}-z\|^2 \\
&\quad -\theta_{\sigma(n)}(1-\theta_{\sigma(n)})\|x_{\sigma(n)}-x_{\sigma(n)-1}\|^2-\frac{3g^2(y_{\sigma(n)})}{\|F(y_{\sigma(n)})\|^2+\|G(y_{\sigma(n)})\|^2} \\
&\le \theta_{\sigma(n)}\|x_{\sigma(n)}-z\|^2+(1-\theta_{\sigma(n)})\|x_{\sigma(n)-1}-z\|^2,
\end{aligned}
$$

which means that

$$\|x_{\sigma(n)+1}-z\|^2-\|x_{\sigma(n)}-z\|^2 \le (1-\theta_{\sigma(n)})(\|x_{\sigma(n)-1}-z\|^2-\|x_{\sigma(n)}-z\|^2),$$

observe the relation $\|x_{\sigma(n)}-z\| \le \|x_{\sigma(n)+1}-z\|$ for each $n \ge 0$, the above inequality concludes a contradiction.

Therefore, there exists an integer $N_0 \ge 0$ such that $\|x_{n+1}-z\| \le \|x_n-z\|$ for all $n \ge N_0$. Then, we have the limit of the sequence $\{\|x_n-z\|^2\}$, denoted by $l = \lim_{n\to}\|x_n-z\|^2$, and so

$$\lim_{n\to\infty}(\|x_n-z\|^2-\|x_{n+1}-z\|^2)=0.$$

In addition, we have

$$\sum_{n=0}^{\infty}(\|x_{n+1}-z\|^2-\|x_n-z\|^2)=\lim_{n\to\infty}(\|x_{n+1}-z\|^2-\|x_0-z\|^2)<\infty.$$

It turns out from (15) that

$$
\begin{aligned}
&\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2+\frac{3g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2} \\
\le\ & \theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2-\|x_{n+1}-z\|^2 \\
=\ & \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2)
\end{aligned}
$$

and so

$$\lim_{n\to} \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 = 0, \quad \frac{g^2(y_n)}{\|F(y_n)\|^2 + \|G(y_n)\|^2} \to 0,$$

as $n \to \infty$, furthermore, we can conclude that $g(y_n) \to 0$ since F and G are Lipschitz continuous (see Censor et al. [30]), and so $\tau_n \to 0$. Therefore, we have

$$\|(I - J_\mu^B)Ty_n\|^2 \to 0. \tag{16}$$

Now, it remains to show that

$$\omega_w(x_n) \subset \Omega.$$

Since the sequence $\{x_n\}$ is bounded, let $\bar{x} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly converging to $\bar{x}$. This suffices to verify that $\bar{x} \in A^{-1}(0)$ and $T\bar{x} \in B^{-1}(0)$.

Notice $\lim_{n\to} \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 = 0$ and the assumption $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$, we have $\lim_{n\to} \|x_n - x_{n-1}\|^2 = 0$, which implies that

$$\|y_n - x_n\| = (1-\theta_n)\cdot\|x_n - x_{n-1}\| \to 0.$$

Therefore, there exists a subsequence $\{y_{nk}\}$ of $\{y_n\}$, which converges weakly to $\bar{x}$. It follows from the lower semicontinuity of $(I - J_\mu^B)T$ and (16) that

$$\|(I - J_\mu^B)T\bar{x}\|^2 = \lim_{k\to\infty} \inf \|(I - J_\mu^B)Ty_{nk}\|^2 = 0,$$

which means that $T\bar{x} \in B^{-1}(0)$.

On the other hand, according to (11) and (12), we have

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &= \|x_{n+1} - z - y_n + z\|^2 \\
&= \|y_n - z\|^2 + \|x_{n+1} - z\|^2 + 2\langle z - y_n, x_{n+1} - z\rangle \\
&= \|y_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\langle x_{n+1} - y_n, x_{n+1} - z\rangle \\
&= \theta_n\|x_n - z\|^2 + (1-\theta_n)\|x_{n-1} - z\|^2 - \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 \\
&\quad -\|x_{n+1} - z\|^2 + 2\langle x_{n+1} - y_n, x_{n+1} - z\rangle \\
&= \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\theta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2) \\
&\quad -\theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 + 2\langle x_{n+1} - y_n, x_{n+1} - z\rangle.
\end{aligned} \tag{17}$$

Using again the property (8), we have

$$\langle J_r^A z_n - z_n, J_r^A z_n - z\rangle \le 0, z \in A^{-1}(0).$$

If we take $z_n = (I - \tau_n T^*(I - J_\mu^B)T)y_n$ in the above inequality, then we have

$$\langle x_{n+1} - (I - \tau_n T^*(I - J_\mu^B)T)y_n, x_{n+1} - z\rangle \le 0, z \in A^{-1}(0),$$

which yields

$$\begin{aligned}
\langle x_{n+1} - y_n, x_{n+1} - z\rangle &\le \tau_n\langle T^*(J_\mu^B - I)Ty_n, x_{n+1} - z\rangle \\
&\le \tau_n\|T^*(J_\mu^B - I)Ty_n\| \cdot \|x_{n+1} - z\| \to 0.
\end{aligned} \tag{18}$$

Thus, it follows from (17) and (18) that

$$\|x_{n+1} - y_n\| \to 0.$$

Since the sequence (12) can be rewritten as

$$y_n - x_{n+1} - \tau_n T^*(I - J_\mu^B)Ty_n \in rAx_{n+1};$$

therefore, we have

$$\frac{1}{r}(y_n - x_{n+1} - \tau_n T^*(I - J_\mu^B)Ty_n) \in Ax_{n+1}. \tag{19}$$

In addition, it turns out from $\tau_n \to 0$ that

$$\begin{aligned} \|y_n - x_{n+1} - \tau_n T^*(I - J_\mu^B)Ty_n\| &\leq \|y_n - x_{n+1}\| + \tau_n \|T^*(I - J_\mu^B)Ty_n\| \\ &= \|y_n - x_{n+1}\| + \tau_n \|G(y_n)\| \to 0. \end{aligned}$$

Note that the graph of the maximal monotone operator A is weakly–strongly closed; by passing to the limit in (19), we obtain $0 \in A\bar{x}$, namely, $\bar{x} \in A^{-1}(0)$. Consequently, $\bar{x} \in \Omega$.

Since the choice of $\bar{x}$ is arbitrary,we conclude that $\omega_w(x_n) \subset \Omega$. Hence, it follows Lemma 1 that the result holds.

(II). Secondly, we consider the case of $\theta_n \equiv 0$. In this case, $y_n = x_{n-1}$. Similar to the proof of (15), we have that

$$\|x_{n+1} - z\|^2 \leq \|x_{n-1} - z\|^2 - \frac{3g^2(y_n)}{\|F(y_n)\|^2 + \|G(y_n)\|^2}, \tag{20}$$

and then

$$\frac{3g^2(x_{n-1})}{\|F(x_{n-1})\|^2 + \|G(x_{n-1})\|^2} \leq \|x_{n-1} - z\|^2 - \|x_{n+1} - z\|^2. \tag{21}$$

It may be assumed that the sequence $\{\|x_n - z\|\}$ is not decreasing at the infinity in the sense that there exists a subsequence $\{\sigma(n)\}$ of positive integers such that there exists a nondecreasing sequence $\sigma(n)$ for $n \geq N_1$ (for some N_1 large enough) such that $\sigma(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|,$$

for each $n \geq 0$.

Notice that (20) holds for each $\sigma(n)$, so from (20) with n replaced by $\sigma(n)$, we have

$$\begin{aligned} \|x_{\sigma(n)+1} - z\|^2 &\leq \|x_{\sigma(n)-1} - z\|^2 - \frac{3g^2(y_{\sigma(n)})}{\|F(y_{\sigma(n)})\|^2 + \|G(y_{\sigma(n)})\|^2} \\ &\leq \|x_{\sigma(n)-1} - z\|^2, \end{aligned}$$

which means that

$$\|x_{\sigma(n)+1} - z\|^2 - \|x_{\sigma(n)} - z\|^2 \leq \|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)} - z\|^2,$$

observe the relation $\|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|$ for each $n \geq N_1$, the above inequality concludes a contradiction.

So there exists an integer $N_0 \geq 0$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq N_0$. Then, we have the limit of the sequence $\{\|x_n - z\|^2\}$, denoted by $l = \lim_{n\to} \|x_n - z\|^2$, and so

$$\lim_{n\to\infty}(\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0, \ \sum_{n=1}^{\infty}(\|x_n - z\|^2 - \|x_{n+1} - z\|^2) < \infty.$$

Now, it remains to show that

$$\omega_w(x_n) \subset \Omega.$$

Since the sequence $\{x_n\}$ is bounded, let $\bar{x} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly converging to $\bar{x}$. It suffices to verify that $\bar{x} \in A^{-1}(0)$ and $T\bar{x} \in B^{-1}(0)$.

Next, we show that $\|x_n - x_{n-1}\| \to 0$. Indeed, it follows from the relation between the norm and inner product that

$$\begin{aligned}
\|x_n - x_{n-1}\|^2 &= \|x_n - z + z - x_{n-1}\|^2 \\
&= \|x_n - z\|^2 + \|z - x_{n-1}\|^2 + 2\langle x_n - z, z - x_{n-1}\rangle \\
&= \|x_n - z\|^2 + \|z - x_{n-1}\|^2 + 2\langle x_n - z, z - x_n + x_n - x_{n-1}\rangle \\
&\le \|x_{n-1} - z\|^2 - \|x_n - z\|^2 + 2\|x_n - z\| \cdot \|x_n - x_{n-1}\| \\
&\le \|x_{n-1} - z\|^2 - \|x_n - z\|^2 + 2(M+m) \cdot \|x_n - x_{n-1}\|,
\end{aligned}$$

where M is a constant such that $M \ge \|x_n - z\|$ for all n and $m > 0$ is a given constant, which means that

$$\|x_n - x_{n-1}\|^2 - 2(M+m) \cdot \|x_n - x_{n-1}\| \le \|x_{n-1} - z\|^2 - \|x_n - z\|^2,$$

and then

$$\sum_{n=0}^{\infty} [\|x_n - x_{n-1}\| - 2(M+m)] \cdot \|x_n - x_{n-1}\| \le \sum_{n=0}^{\infty} (\|x_{n-1} - z\|^2 - \|x_n - z\|^2) < \infty,$$

which implies $[\|x_n - x_{n-1}\| - 2(M+m)] \cdot \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$.

Since $\|x_n - x_{n-1}\| \le \|x_n - z\| + \|x_{n-1} - z\| \le 2M$, we have $\|x_n - x_{n-1}\| \to 0$ and then

$$\|x_{n+1} - x_{n-1}\| \to 0.$$

It follows from (21) that $g^2(x_{n-1}) \to 0$ and then

$$\|(I - J_\mu^B)Tx_{n-1}\|^2 \to 0.$$

By using the lower semicontinuity of $(I - J_\mu^B)T$, we have

$$\|(I - J_\mu^B)T\bar{x}\|^2 = \liminf_{k\to\infty} \|(I - J_\mu^B)Tx_{n_k-1}\|^2 = 0,$$

which means that $T\bar{x} \in B^{-1}(0)$.

Notice again that the sequence (12) can be rewritten as

$$x_{n-1} - x_{n+1} - \tau_n T^*(I - J_\mu^B)Tx_{n-1} \in rAx_{n+1};$$

therefore, we have

$$\frac{1}{r}(x_{n-1} - x_{n+1} - \tau_n T^*(I - J_\mu^B)Tx_{n-1}) \in Ax_{n+1}. \tag{22}$$

In addition, it turns out from $\tau_n \to 0$ that

$$\begin{aligned}
\|x_{n-1} - x_{n+1} - \tau_n T^*(I - J_\mu^B)Tx_{n-1}\| &\le \|x_{n-1} - x_{n+1}\| + \tau_n \|T^*(I - J_\mu^B)Tx_{n-1}\| \\
&= \|x_{n-1} - x_{n+1}\| + \tau_n \|G(x_{n-1})\| \to 0.
\end{aligned}$$

Note that the graph of the maximal monotone operator A is weakly–strongly closed, by passing to the limit in (22), we obtain $0 \in A\bar{x}$, namely, $\bar{x} \in A^{-1}(0)$. Consequently, $\bar{x} \in \Omega$.

Since the choice of $\bar{x}$ is arbitrary, we conclude that $\omega_w(x_n) \subset \Omega$. Hence, it follows Lemma 1 that the result holds.

(III). Finally, we consider the case of $\theta_n \equiv 1$. Indeed, we just need to replace x_{n-1} with x_n in the proof of **(II)** and then the desired result is obtained.

□

Next, we prove the strong convergence of Algorithm 2.

Algorithm 2 Update of self adaptive inertial-like algorithm

Initialization: Choose a sequence $\{\theta_n\} \subset [0,1]$ satisfying one of the three cases: **(I.)** $\theta_n \in (0,1)$ such that $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$; **(II.)** $\theta_n \equiv 0$; or **(III.)** $\theta_n \equiv 1$. Choose $\{\alpha_n\}$ and $\{\gamma_n\}$ in $(0,1)$ such that

$$\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=0}^{\infty} \gamma_n = \infty, \quad \lim_{n\to\infty} (1-\alpha_n-\gamma_n)\alpha_n > 0.$$

Select arbitrary initial points x_0, x_1.
Iterative Step: Given the iterate x_n, compute

$$y_n = x_{n-1} + \theta_n(x_n - x_{n-1}),$$

and define the $(n+1)$th iterate by

$$x_{n+1} = (1-\alpha_n-\gamma_n)y_n + \alpha_n J_r^A(I - \tau_n T^*(I - J_\mu^B)T)y_n, \tag{23}$$

where

$$\tau_n = \begin{cases} \frac{g(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2}, & \text{if } \|F(y_n)\|^2 + \|G(y_n)\|^2 \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2. *If the assumptions* (**A1**)–(**A3**) *are satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges in norm to* $z = P_\Omega(0)$ *(i.e., the minimum-norm element of the solution set* Ω*).*

Proof. Similar to the weak convergence, we consider the following three situations: **(I)**. $\theta_n \in (0,1)$ and $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$; **(II)**. $\theta_n \equiv 0$; and **(III)**. $\theta_n \equiv 1$.

(I). We first consider the strong convergence under the situation of $\theta_n \in (0,1)$ and $\underline{\lim}_{n\to\infty}\theta_n(1-\theta_n) > 0$.

Let us begin by showing the boundedness of the sequence $\{x_n\}$. To see this, we denote $z_n = J_r^A(I - \tau_n T^*(I - J_\mu^B)Ty_n$ and use the projection $z := P_\Omega(0)$ to obtain in a similar way to the proof of (13)–(15) of Theorem 1 that

$$\begin{aligned} \|y_n - z\| &= \theta_n\|x_n - z\| + (1-\theta_n)\|x_{n-1} - z\| \\ &\le \max\{\|x_n - z\|, \|x_{n-1} - z\|\}, \\ \|z_n - z\|^2 &\le \|y_n - z\|^2 - \frac{3g^2(y_n)}{\|F(y_n)\|^2 + \|G(y_n)\|^2}; \end{aligned} \tag{24}$$

hence, one can see $\|z_n - z\| \le \|y_n - z\|$.

It turns out from (23) that

$$\begin{aligned}
\|x_{n+1}-z\| &= \|(1-\alpha_n-\gamma_n)y_n+\alpha_n z_n - z\| \\
&= \|(1-\alpha_n-\gamma_n)(y_n-z)+\alpha_n(z_n-z)+\gamma_n(-z)\| \\
&\leq (1-\alpha_n-\gamma_n)\|y_n-z\|+\alpha_n\|z_n-z\|+\gamma_n\|z\| \\
&\leq (1-\gamma_n)(\theta_n\|x_n-z\|+(1-\theta_n)\|x_{n-1}-z\|)+\gamma_n\|z\| \\
&\leq \max\{\|x_n-z\|,\|x_{n-1}-z\|,\|z\|\} \\
&\leq \cdots \\
&\leq \max\{\|x_0-z\|,\|x_1-z\|,\|z\|\},
\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{z_n\}$.

Applying the identity (7), we deduce that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|(1-\alpha_n-\gamma_n)y_n+\alpha_n z_n - z\|^2 \\
&= \|(1-\alpha_n-\gamma_n)(y_n-z)+\alpha_n(z_n-z)+\gamma_n(-z)\|^2 \\
&\leq (1-\alpha_n-\gamma_n)\|y_n-z\|^2+\alpha_n\|z_n-z\|^2+\gamma_n\|z\|^2 \\
&\quad -(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2.
\end{aligned} \tag{25}$$

Substituting (13) and (24) into (25) and after some manipulations, we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq (1-\alpha_n-\gamma_n)\|y_n-z\|^2+\alpha_n(\|y_n-z\|^2-\frac{3g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2}) \\
&\quad +\gamma_n\|z\|^2-(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2 \\
&= (1-\gamma_n)\|y_n-z\|^2+\gamma_n\|z\|^2-(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2 \\
&\quad -\frac{3\alpha_n g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2}
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq (1-\gamma_n)[\theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2-\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2] \\
&\quad +\gamma_n\|z\|^2-(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2-\frac{3\alpha_n g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2} \\
&\leq \theta_n\|x_n-z\|^2+(1-\theta_n)\|x_{n-1}-z\|^2+\gamma_n\|z\|^2-(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2 \\
&\quad -(1-\gamma_n)\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2-\frac{3\alpha_n g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2}
\end{aligned} \tag{26}$$

Next we distinguish two cases.

Case 1. The sequence $\{\|x_n-z\|\}$ is nonincreasing at the infinity in the sense that there exists $n_0 \geq 0$ such that for each $n \geq n_0$, $\|x_{n+1}-z\| \leq \|x_n-z\|$. This particularly implies that $\lim_{n\to\infty}\|x_n-z\|$ exists and thus,

$$\lim_{n\to\infty}(\|x_n-z\|^2-\|x_{n-1}-z\|^2)=0, \quad \sum_{n=1}^{\infty}(\|x_n-z\|^2-\|x_{n-1}-z\|^2)<\infty.$$

For all $n > n_0$, it follows from (26) that

$$\begin{aligned}
&(1-\alpha_n-\gamma_n)\alpha_n\|z_n-y_n\|^2+(1-\gamma_n)\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2+\frac{3\alpha_n g^2(y_n)}{\|F(y_n)\|^2+\|G(y_n)\|^2} \\
\leq\ & \theta_n\|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2)+\gamma_n\|z\|^2 \\
\leq\ & \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2)+\gamma_n\|z\|^2.
\end{aligned}$$

Now, due to the assumptions on α_n, β_n, and γ_n and the boundedness of $\{x_n\}$ and $\{y_n\}$, we have

$$\lim_{n\to\infty} \|z_n - y_n\| = 0; \tag{27}$$

$$\lim_{n\to\infty} (1-\gamma_n)\theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 = 0; \tag{28}$$

$$\lim_{n\to\infty} \frac{3\alpha_n g^2(y_n)}{\|F(y_n)\|^2 + \|G(y_n)\|^2} = 0. \tag{29}$$

It turns out from (29) that $g(y_n) \to 0$ since F and G are Lipschitz continuous and so $\lim_{n\to\infty} \tau_n = 0$, and from (27) that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$, which in turn implies from (11) that

$$\begin{aligned} \lim_{n\to\infty} \|y_n - x_n\| &\leq \lim_{n\to\infty} (\|y_n - x_{n-1}\| + \|x_{n-1} - x_n\|) \\ &= \lim_{n\to\infty} (1+\theta_n)\|x_{n-1} - x_n\| = 0. \end{aligned}$$

Observing $\|x_{n+1} - y_n\| \leq \alpha_n\|z_n - y_n\| + \gamma_n\|y_n\| \to 0$, we obtain

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \to 0.$$

This proves the asymptotic regularity of $\{x_n\}$.

By repeating the relevant part of the proof of Theorem 1, we obtain $\omega_w(x_n) \subset \Omega$.

It is now at the position to prove the strong convergence of $\{x_n\}$. Rewriting $x_{n+1} = (1-\gamma_n)v_n + \gamma_n\alpha_n(z_n - y_n)$, where $v_n = (1-\alpha_n)y_n + \alpha_n z_n$, and making use of the inequality $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle$, which holds for all u, v in Hilbert spaces, we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\gamma_n)(v_n - z) + \gamma_n(\alpha_n(z_n - y_n) - z)\|^2 \\ &\leq (1-\gamma_n)^2\|v_n - z\|^2 + 2\gamma_n\langle \alpha_n(z_n - y_n) - z, x_{n+1} - z\rangle. \end{aligned}$$

It follows from (7) that

$$\|v_n - z\|^2 = (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z_n - z\|^2 - \alpha_n(1-\alpha_n)\|z_n - y_n\|^2,$$

and then

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq\ & (1-\gamma_n)^2((1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z_n - z\|^2 - \alpha_n(1-\alpha_n)\|z_n - y_n\|^2) \\ & +2\gamma_n\langle \alpha_n(z_n - y_n) - z, x_{n+1} - z\rangle. \end{aligned}$$

It turns out from (24) that $\|z_n - z\|^2 \leq \|y_n - z\|^2$; hence, we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq\ & (1-\gamma_n)\|y_n - z\|^2 - \alpha_n(1-\alpha_n)(1-\gamma_n)^2\|z_n - y_n\|^2 \\ & +2\gamma_n\langle \alpha_n(z_n - y_n) - z, x_{n+1} - z\rangle \end{aligned}$$

Submitting (13) into the above inequality, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq\ & (1-\gamma_n)(\theta_n\|x_n - z\|^2 + (1-\theta_n)\|x_{n-1} - z\|^2 \\ & -\theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2) - \alpha_n(1-\alpha_n)(1-\gamma_n)^2\|z_n - y_n\|^2 \\ & +2\gamma_n\Big\langle \alpha_n(z_n - y_n) - z, x_{n+1} - z\Big\rangle, \end{aligned} \tag{30}$$

which means that

$$\begin{aligned} \|x_{n+1}-z\|^2 &\leq (1-\gamma_n)\|x_n-z\|^2+(1-\gamma_n)(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2) \\ &\quad -\theta_n(1-\theta_n)(1-\gamma_n)\|x_n-x_{n-1}\|^2-\alpha_n(1-\alpha_n)(1-\gamma_n)^2\|z_n-y_n\|^2 \\ &\quad +2\gamma_n\Big\langle \alpha_n(z_n-y_n)-z, x_{n+1}-z\Big\rangle \\ &\leq (1-\gamma_n)\|x_n-z\|^2+(1-\gamma_n)(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2) \\ &\quad +2\gamma_n\Big\langle \alpha_n(z_n-y_n)-z, x_{n+1}-z\Big\rangle. \end{aligned}$$

For simplicity, we denote by

$$a_{n+1} \leq (1-\gamma_n)a_n+\gamma_n\delta_n+c_n, \tag{31}$$

where $a_n = \|x_n-z\|^2$, $\delta_n = 2\alpha_n\langle z_n-y_n, x_{n+1}-z\rangle+\langle -z, x_{n+1}-z\rangle$, and $c_n=(1-\gamma_n)(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2)$.

Since $\omega_w(x_n)\subset\Omega$ and $z=P_\Omega(0)$, which implies $\langle -z, q-z\rangle\leq 0$ for all $q\in\Omega$, we deduce that

$$\limsup_{n\to\infty}\langle -z, x_{n+1}-z\rangle = \max_{q\in\omega_w(x_n)}\langle -z, q-z\rangle \leq 0. \tag{32}$$

Combining (28) and (32) implies that

$$\begin{aligned} \limsup_{n\to\infty}\delta_n &= \limsup_{n\to\infty}\{\alpha_n\langle z_n-y_n, x_{n+1}-z\rangle+\langle -z, x_{n+1}-z\rangle\} \\ &= \limsup_{n\to\infty}\langle -z, x_{n+1}-z\rangle \leq 0. \end{aligned}$$

In addition, by the assumptions on θ_n and γ_n, we have

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}(1-\gamma_n)(1-\theta_n)(\|x_{n-1}-z\|^2-\|x_n-z\|^2)<\infty.$$

These enable us to apply Lemma 1 to (31) to obtain that $a_n\to 0$. Namely, $x_n\to z$ in norm, and the proof of Case 1 is complete.

Case 2. The sequence $\{\|x_n-z\|\}$ is not nonincreasing at the infinity in the sense that there exists a subsequence $\{\sigma(n)\}$ of positive integers such that $\sigma(n)\to\infty$ (as $n\to\infty$) and with the properties:

$$\|x_{\sigma(n)}-z\|<\|x_{\sigma(n)+1}-z\|, \quad \max\{\|x_{\sigma(n)}-z\|, \|x_n-z\|\}\leq\|x_{\sigma(n)+1}-z\|.$$

Notice the boundedness of the sequence $\{\|x_n-z\|\}$, which implies that there exists the limit of the sequence $\{\|x_{\sigma(n)}-z\|\}$ and, hence, we conclude that

$$\lim_{n\to\infty}(\|x_{\sigma(n)+1}-z\|^2-\|x_{\sigma(n)}-z\|^2)=0.$$

Observe that (26) holds for all $\sigma(n)$, so replacing n with $\sigma(n)$ in (26) and transposing, we obtain

$$
\begin{aligned}
& (1-\alpha_{\sigma(n)}-\gamma_{\sigma(n)})\alpha_{\sigma(n)}\|z_{\sigma(n)}-y_{\sigma(n)}\|^2+(1-\gamma_{\sigma(n)})\theta_{\sigma(n)}(1-\theta_{\sigma(n)})\|x_{\sigma(n)}-x_{\sigma(n)-1}\|^2 \\
& +\frac{3\alpha_{\sigma(n)}g^2(y_{\sigma(n)})}{\|F(y_{\sigma(n)})\|^2+\|G(y_{\sigma(n)})\|^2} \\
\leq\ & (1-\gamma_{\sigma(n)})(\theta_{\sigma(n)}\|x_{\sigma(n)}-z\|^2+(1-\theta_{\sigma(n)})\|x_{\sigma(n)-1}-z\|^2)-\|x_{\sigma(n)+1}-z\|^2+\gamma_{\sigma(n)}\|z\|^2 \\
\leq\ & \theta_{\sigma(n)}\|x_{\sigma(n)}-z\|^2+(1-\theta_{\sigma(n)})\|x_{\sigma(n)-1}-z\|^2+\|x_{\sigma(n)}-z\|^2-\|x_{\sigma(n)}-z\|^2 \\
& -\|x_{\sigma(n)+1}-z\|^2+\gamma_{\sigma(n)}\|z\|^2 \\
=\ & (1-\theta_{\sigma(n)})(\|x_{\sigma(n)-1}-z\|^2-\|x_{\sigma(n)}-z\|^2)+\|x_{\sigma(n)}-z\|^2-\|x_{\sigma(n)+1}-z\|^2+\gamma_{\sigma(n)}\|z\|^2.
\end{aligned}
$$

Now, taking the limit by letting $n\to\infty$ yields

$$\lim_{n\to\infty}\|z_{\sigma(n)}-y_{\sigma(n)}\|=0; \tag{33}$$

$$\lim_{n\to\infty}g(y_{\sigma(n)})=0; \tag{34}$$

$$\lim_{n\to\infty}\|x_{\sigma(n)}-x_{\sigma(n)-1}\|^2=0. \tag{35}$$

Note that we still have $\|x_{\sigma(n)+1}-x_{\sigma(n)}\|\to 0$ and that the relations (33)–(35) are sufficient to guarantee that $\omega_w(x_{\sigma(n)})\subset\Omega$.

Next, we prove $x_{\sigma(n)}\to z$.

As a matter of fact, observe that (30) holds for each $\sigma(n)$. So replacing n with $\sigma(n)$ in (30) and using the relation $\|x_{\sigma(n)}-z\|^2<\|x_{\sigma(n)+1}-z\|^2$, we obtain

$$
\begin{aligned}
\|x_{\sigma(n)+1}-z\|^2 =\ & (1-\gamma_{\sigma(n)})(\theta_{\sigma(n)}\|x_{\sigma(n)}-z\|^2+(1-\theta_{\sigma(n)})\|x_{\sigma(n)-1}-z\|^2 \\
& -\theta_{\sigma(n)}(1-\theta_{\sigma(n)})\|x_{\sigma(n)}-x_{\sigma(n)-1}\|^2) \\
& -\alpha_{\sigma(n)}(1-\alpha_{\sigma(n)})(1-\gamma_{\sigma(n)})^2\|z_{\sigma(n)}-y_{\sigma(n)}\|^2 \\
& +2\gamma_{\sigma(n)}\left\langle\alpha_{\sigma(n)}(z_{\sigma(n)}-y_{\sigma(n)})-z,x_{\sigma(n)+1}-z\right\rangle \\
\leq\ & (1-\gamma_{\sigma(n)})\|x_{\sigma(n)}-z\|^2+2\gamma_{\sigma(n)}\left\langle\alpha_{\sigma(n)}(z_{\sigma(n)}-y_{\sigma(n)})-z,x_{\sigma(n)+1}-z\right\rangle;
\end{aligned}
$$

therefore, we have

$$\gamma_{\sigma(n)}\|x_{\sigma(n)}-z\|^2\leq\|x_{\sigma(n)}-z\|^2-\|x_{\sigma(n)+1}-z\|^2+2\gamma_{\sigma(n)}\left\langle\alpha_{\sigma(n)}(z_{\sigma(n)}-y_{\sigma(n)})-z,x_{\sigma(n)+1}-z\right\rangle,$$

Notice again the relation $\|x_{\sigma(n)}-z\|^2<\|x_{\sigma(n)+1}-z\|^2$, we obtain

$$
\begin{aligned}
\|x_{\sigma(n)}-z\|^2 &\leq 2\langle\alpha_{\sigma(n)}(z_{\sigma(n)}-y_{\sigma(n)})-z,x_{\sigma(n)+1}-z\rangle \\
&\leq M\|z_{\sigma(n)}-y_{\sigma(n)}\|+2\langle -z,x_{\sigma(n)+1}-z\rangle.
\end{aligned} \tag{36}
$$

[Here M is a constant such that $M\geq 2\|x_n-z\|$ for all n.]

Now, since $\|x_{\sigma(n)+1}-x_{\sigma(n)}\|\to 0$, $z=P_\Omega(0)$ and $\omega(x_{\sigma(n)})\subset\Omega$, we have

$$
\begin{aligned}
\limsup_{n\to\infty}\langle -z,x_{\sigma(n)+1}-z\rangle &= \limsup_{n\to\infty}\langle -z,x_{\sigma(n)}-z\rangle \\
&= \max_{q\in\omega_w(x_{\sigma(n)})}\langle -z,q-z\rangle\leq 0.
\end{aligned}
$$

Consequently, the relation (36) and $\|z_{\sigma(n)}-y_{\sigma(n)}\|\to 0$ assure that $x_{\sigma(n)}\to z$, which follows from Lemma 3 that

$$\|x_n-z\|\leq\|x_{\sigma(n)+1}-z\|\leq\|x_{\sigma(n)+1}-x_{\sigma(n)}\|+\|x_{\sigma(n)}-z\|\to 0.$$

Namely, $x_n \to z$ in norm, and the proof of Case 2 is complete.

(II). Now, we consider the case of $\theta_n \equiv 0$. In this case, we have $y_n = x_{n-1}$ and $x_{n+1} = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n J_r^A(I-\tau_n T^*(I-J_\mu^B)T)x_{n-1}$. Denote by $z_{n-1} = J_r^A(I-\tau_n T^*(I-J_\mu^B)T)x_{n-1}$, similar to the proof of (24)–(26), we obtain that the sequence $\{x_n\}$ is bounded and

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq\ & (1-\gamma_n)\|x_{n-1}-z\|^2 + \gamma_n\|z\|^2 - (1-\alpha_n-\gamma_n)\alpha_n\|z_{n-1}-x_{n-1}\|^2 \\ & -\frac{3\alpha_n g^2(x_{n-1})}{\|F(x_{n-1})\|^2+\|G(x_{n-1})\|^2},\end{aligned}$$

which implies that

$$\begin{aligned} & (1-\alpha_n-\gamma_n)\alpha_n\|z_{n-1}-x_{n-1}\|^2 + \frac{3\alpha_n g^2(x_{n-1})}{\|F(x_{n-1})\|^2+\|G(x_{n-1})\|^2} \\ \leq\ & (1-\gamma_n)\|x_{n-1}-z\|^2 + \gamma_n\|z\|^2 - \|x_{n+1}-z\|^2 \\ =\ & (\|x_{n-1}-z\|^2 - \|x_{n+1}-z\|^2) + \gamma_n(\|z\|^2 - \|x_{n-1}-z\|^2). \end{aligned} \tag{37}$$

Next, we distinguish two cases.

Case 1. There exists $n_0 \geq 0$ such that for each $n > n_0$, $\|x_{n+1}-z\| \leq \|x_n - z\|$, which implies that $\lim_{n\to\infty}\|x_n - z\|$ exists, and thus

$$\lim_{n\to\infty}(\|x_n-z\|^2 - \|x_{n-1}-z\|^2) = 0, \quad \sum_{n=1}^{\infty}(\|x_n-z\|^2 - \|x_{n-1}-z\|^2) < \infty.$$

Since $\gamma_n \to 0$ and $\underline{\lim}_{n\to\infty}(1-\alpha_n-\gamma_n)\alpha_n > 0$, it follows from (37) that

$$\|z_{n-1}-x_{n-1}\|^2 \to 0; \ \frac{3\alpha_n g^2(x_{n-1})}{\|F(x_{n-1})\|^2+\|G(x_{n-1})\|^2} \to 0,$$

which means that $g(x_{n-1}) \to 0$ since F and G are Lipschitz continuous and so $\tau_n \to 0$. Similar to the proof of $\|x_n - x_{n-1}\| \to 0$ in the weak convergence Theorem 1, we still have the asymptotic regularity of $\{x_n\}$ and $\omega_w(x_n) \subset \Omega$.

Similar to the proof of (30), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq\ & (1-\gamma_n)\|x_{n-1}-z\|^2 - \alpha_n(1-\alpha_n)(1-\gamma_n)^2\|z_{n-1}-x_{n-1}\|^2 \\ & +2\gamma_n\left\langle \alpha_n(z_{n-1}-x_{n-1}) - z, x_{n+1}-z\right\rangle \\ =\ & (1-\gamma_n)\|x_n-z\|^2 + (1-\gamma_n)(\|x_{n-1}-z\|^2 - \|x_n-z\|^2) \\ & -\alpha_n(1-\alpha_n)(1-\gamma_n)^2\|z_{n-1}-x_{n-1}\|^2 + 2\gamma_n\left\langle \alpha_n(z_{n-1}-x_{n-1}) - z, x_{n+1}-z\right\rangle \\ \leq\ & (1-\gamma_n)\|x_n-z\|^2 + (1-\gamma_n)(\|x_{n-1}-z\|^2 - \|x_n-z\|^2) \\ & +2\gamma_n\left\langle \alpha_n(z_{n-1}-x_{n-1}) - z, x_{n+1}-z\right\rangle \\ =\ & (1-\gamma_n)a_n + \gamma_n\delta_n + c_n,\end{aligned}$$

where $a_n = \|x_n - z\|^2$, $\delta_n = 2\langle \alpha_n(z_{n-1}-x_{n-1}) - z, x_{n+1}-z\rangle$ and $c_n = (1-\gamma_n)(\|x_{n-1}-z\|^2 - \|x_n-z\|^2)$.

Obviously, γ_n, δ_n and c_n are satisfying the conditions in Lemma 1, so we can conclude that $x_n \to z$.

Case 2. The sequence $\{\|x_n - z\|\}$ is not nonincreasing at the infinity in the sense that there exists a subsequence $\{\sigma(n)\}$ of positive integers such that $\sigma(n) \to \infty$ (as $n \to \infty$) and with the properties:

$$\|x_{\sigma(n)}-z\| < \|x_{\sigma(n)+1}-z\|, \quad \max\{\|x_{\sigma(n)}-z\|, \|x_n-z\|\} \leq \|x_{\sigma(n)+1}-z\|.$$

Since the sequence $\{\|x_n - z\|\}$ is bounded, there exists the limit of the sequence $\{\|x_{\sigma(n)} - z\|\}$ and, hence, we conclude that

$$\lim_{n\to\infty}(\|x_{\sigma(n)+1} - z\|^2 - \|x_{\sigma(n)} - z\|^2) = 0.$$

Notice (37) holds for all $\sigma(n)$, so replacing n with $\sigma(n)$ in (37) and using the relation $\|x_{\sigma(n)} - z\| < \|x_{\sigma(n)+1} - z\|$, we have

$$\begin{aligned}
&(1 - \alpha_{\sigma(n)} - \gamma_{\sigma(n)})\alpha_{\sigma(n)}\|z_{\sigma(n)-1} - x_{\sigma(n)-1}\|^2 + \frac{3\alpha_{\sigma(n)}g^2(x_{\sigma(n)-1})}{\|F(x_{\sigma(n)-1})\|^2 + \|G(x_{\sigma(n)-1})\|^2} \\
\leq\ & (\|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)+1} - z\|^2) + \gamma_{\sigma(n)}(\|z\|^2 - \|x_{\sigma(n)-1} - z\|^2) \\
=\ & (\|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)} - z\|^2) + (\|x_{\sigma(n)} - z\|^2 - \|x_{\sigma(n)+1} - z\|^2) + \gamma_{\sigma(n)}(\|z\|^2 - \|x_{\sigma(n)-1} - z\|^2) \\
\leq\ & \gamma_{\sigma(n)}(\|z\|^2 - \|x_{\sigma(n)-1} - z\|^2).
\end{aligned}$$

Since $\gamma_{\sigma(n)} \to 0$ and $\underline{\lim}_{n\to\infty}(1 - \alpha_{\sigma(n)} - \gamma_{\sigma(n)})\alpha_{\sigma(n)} > 0$, we obtain

$$\|z_{\sigma(n)-1} - x_{\sigma(n)-1}\|^2 \to 0; \quad \frac{3\alpha_{\sigma(n)}g^2(x_{\sigma(n)-1})}{\|F(x_{\sigma(n)-1})\|^2 + \|G(x_{\sigma(n)-1})\|^2} \to 0.$$

Similarly, we still have the asymptotic regularity of $\{x_{\sigma(n)}\}$ and $\omega_w(x_{\sigma(n)}) \subset \Omega$.

In addition, similar to the inequality above (31), we obtain the following

$$\begin{aligned}
\|x_{\sigma(n)+1} - z\|^2 \leq\ & (1 - \gamma_{\sigma(n)})\|x_{\sigma(n)} - z\|^2 + (1 - \gamma_{\sigma(n)})(\|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)} - z\|^2) \\
& + 2\gamma_{\sigma(n)}\Big\langle \alpha_{\sigma(n)}(z_{\sigma(n)-1} - x_{\sigma(n)-1}) - z, x_{\sigma(n)+1} - z\Big\rangle,
\end{aligned}$$

which means that

$$\begin{aligned}
\gamma_{\sigma(n)})\|x_{\sigma(n)} - z\|^2 \leq\ & \|x_{\sigma(n)} - z\|^2 - \|x_{\sigma(n)+1} - z\|^2 + (1 - \gamma_{\sigma(n)})(\|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)} - z\|^2) \\
& + 2\gamma_{\sigma(n)}\Big\langle \alpha_{\sigma(n)}(z_{\sigma(n)-1} - x_{\sigma(n)-1}) - z, x_{\sigma(n)+1} - z\Big\rangle,
\end{aligned}$$

notice again the relation $\|x_{\sigma(n)} - z\|^2 \leq \|x_{\sigma(n)+1} - z\|^2$ for all $\sigma(n)$, we have

$$\begin{aligned}
\|x_{\sigma(n)} - z\|^2 \leq\ & 2\Big\langle \alpha_{\sigma(n)}(z_{\sigma(n)-1} - x_{\sigma(n)-1}) - z, x_{\sigma(n)+1} - z\Big\rangle \\
\leq\ & M\|z_{\sigma(n)-1} - x_{\sigma(n)-1}\| + 2\langle -z, x_{\sigma(n)+1} - z\rangle.
\end{aligned} \tag{38}$$

[Here M is a constant such that $M \geq 2\|x_n - z\|$ for all n.]

Again, since $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0$, $z = P_\Omega(0)$ and $\omega(x_{\sigma(n)}) \subset \Omega$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle -z, x_{\sigma(n)+1} - z\rangle &= \limsup_{n\to\infty}\langle -z, x_{\sigma(n)} - z\rangle \\
&= \max_{q\in\omega_w(x_{\sigma(n)})}\langle -z, q - z\rangle \leq 0.
\end{aligned}$$

Consequently, the inequality (38) and $\|z_{\sigma(n)-1} - x_{\sigma(n)-1}\| \to 0$ assure that $x_{\sigma(n)} \to z$, which follows from Lemma 3 that

$$\|x_n - z\| \leq \|x_{\sigma(n)+1} - z\| \leq \|x_{\sigma(n)+1} - x_{\sigma(n)}\| + \|x_{\sigma(n)} - z\| \to 0.$$

Namely, $x_n \to z$ in norm, and the proof of the second situation **(II)** is complete.

(III.) Finally, we consider the case of $\theta_n \equiv 1$. Indeed, we just need to replace x_{n-1} with x_n in the proof of **(II)**, and then the desired result is obtained.

□

4. Numerical Examples and Experiments

Example 1. *We consider the numerical Let* $H_1 = H_2 = L^2[0,1]$*. Define the mappings* A*,* B*, and* T *by* $Tx(t) := x(t)$*,* $Ax(t) := \frac{x(t)}{2}$*, and* $B(x)(t) := \frac{2x(t)}{3}$ *for all* $x(t) \in L^2[0,1]$*. Then it can be shown that* A *and* B *are monotone operators, respectively, and the adjoint* T^* *of* T *is* $T^*x(t) := x(t)$*. For simplicity, we choose* $\alpha_n = \frac{n-1}{n+1}$*,* $\gamma_n = \frac{1}{n+1}$ *in Algorithm 2 for all* $n \geq 1$*. We consider different choices of initial functions* $x_0(t), x_1(t)$ *and* $\theta_n = 0.5 + \frac{1}{(n+1)}$*; 0; 1. In addition,* $\|x_{n+1} - x_n\| < 10^{-10}$ *is used as stopping criterion.*

Case I: $x_0(t) = t, x_1(t) = 2t;$
Case II: $x_0(t) = e^{-t}, x_1(t) = 2sin(5t);$
Case III: $x_0(t) = e^{-t}, x_1(t) = 2t.$

It is clear that our algorithm is fast, efficient, stable, and simple to implement. All the numerical results are presented in Figures 1–3 under different initial functions, and the number of iterations and CPU run time remain almost consistent, which are shown in Table 1.

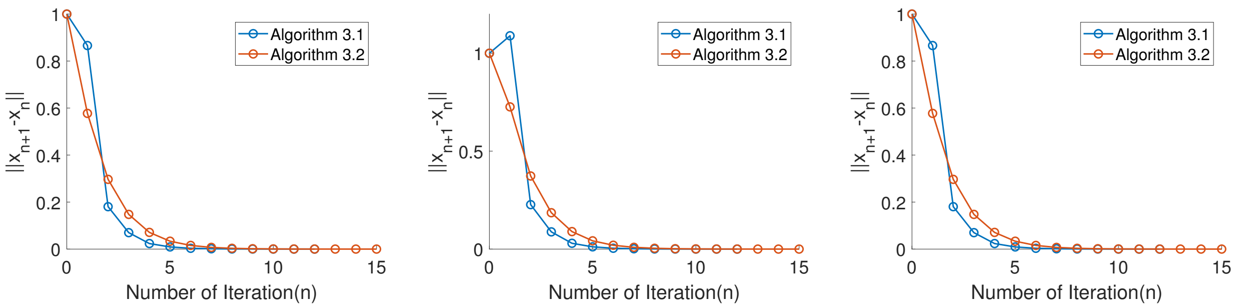


Figure 1. Three initial cases for $\theta_n = 0.5 + 1/(n+1)$.

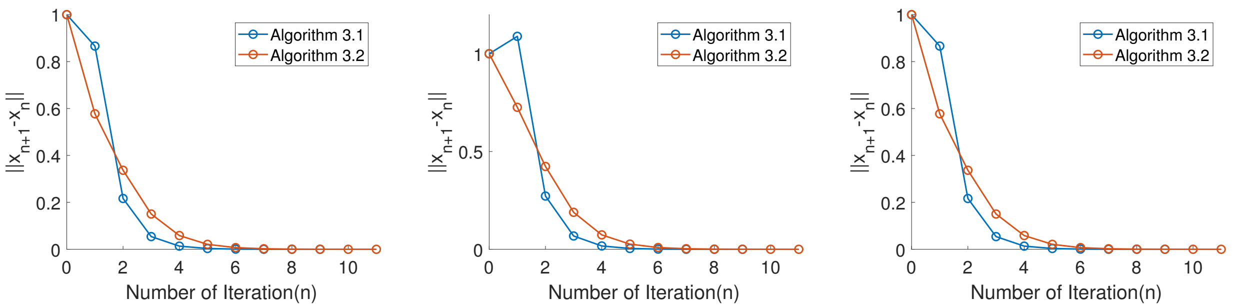


Figure 2. Three initial cases for $\theta_n = 1$.

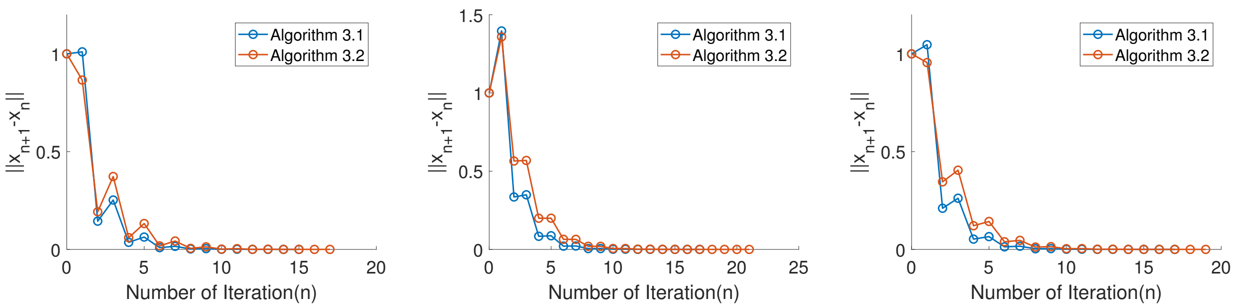


Figure 3. Three initial cases for $\theta_n = 0$.

Table 1. Time and iterations of Algorithms 1 and 2 in Ex.1.

	Algorithm	Case I (sec.)/(n)	Case II (sec.)/(n)	Case III (sec.)/(n)
$\theta_n = 0$	Algorithm 1	4.26/16	4.75/18	4.78/18
	Algorithm 2	4.80/18	9.35/20	1.67/22
$\theta_n = 0.5 + \frac{1}{(n+1)}$	Algorithm 1	4.19/12	4.96/12	4.27/12
	Algorithm 2	4.23/16	16.37/18	10.69/18
$\theta_n = 1/(\text{n})$	Algorithm 1	2.56/10	2.63/10	2.60/10
	Algorithm 2	3.17/12	3.25/12	3.25/12

Example 2. *We consider an examplewhich is from the realm of compressed sensing. More specifically, we try to recover the K-sparse original signal x_0 from the observed signal b.*

Here, matrix $T \in R^{m*n}$, $m << n$ would be involved and created by standard Gaussian distribution. The observed signal $b = Tx + \epsilon$, where ϵ is noise. For more details on signal recovery, one can consult Nguyen and Shin [31].

Conveniently, solving the above sparse signal recovery problem is usually equivalent to solving the following LASSO problem (see Gibali et al. [32] and Moudafi et al. [33]):

$$\begin{gathered} \min_{x \in R^n} \|Tx - b\|^2 \\ s.t. \ \|x\|_1 \le t, \end{gathered}$$

where t is a given positive constant. If we define

$$A(x) = \begin{cases} \{u : \sup_{\|x\|_1 \le t} \langle x - y, u \rangle \le 0\}, & \text{if } y \in R^n, \\ \emptyset, & \text{otherwise}, \end{cases} \quad B(x) = \begin{cases} R^m, & \text{if } x = b, \\ \emptyset, & \text{otherwise}, \end{cases}$$

then one can see that the LASSO problem coincides with the problem of finding $x^* \in R^n$ such that

$$0 \in A(x^*) \text{ and } 0 \in B(Tx^*).$$

During the operation, $T \in R^{m*n}$ is generated randomly with $m = 2^{15}, 2^7$, $n = 2^{16}, 2^8$, $x_0 \in R^n$ is K-spikes $(K = 100, 50)$ with amplitude ± 1 distributed throughout the region randomly. In addition, the signal to noise ratio (SNR) is chosen as $SNR = 40$, $\alpha_n = 0.5 + 1/(10n + 2)$ in two algorithms and $\gamma_n = 1/n$ in Algorithm 2. The recovery simulation results are illustrated in Figure 4.

Moreover, we also compare our algorithms with the results of Sitthithakerngkiet et al. [21], Kazimi and Riviz. [22], Byrne et al. [5] which have no inertial item and Tang [34] with a general inertial method.

For simplicity, for Algorithm 3.1 in Sitthithakerngkiet et al. [21], the nonexpansive mappings S_n are defined as $S_n = I$, $D = I$, $\xi = 0.5$, and $u = 0.1$, the parameters $\alpha_n = \frac{1}{n+1}$ and $\beta_n = 0.5 - \frac{1}{10n+2}$. For Algorithm 3.1 in Sitthithakerngkiet et al. [21], Kazimi and Riviz [22], and Algorithm 3.1 in Byrne et al. [5], the step size $\gamma = \frac{1}{L}$, where $L = \|T^*T\|$. For Algorithms 3.1 and 3.2 in Tang [34], the step size is self-adaptive, and $\alpha_n = 0.5 + 1/(10n + 2)$, $\gamma_n = 1/n$. The experiment results are illustrated in Figure 5 and Table 2.

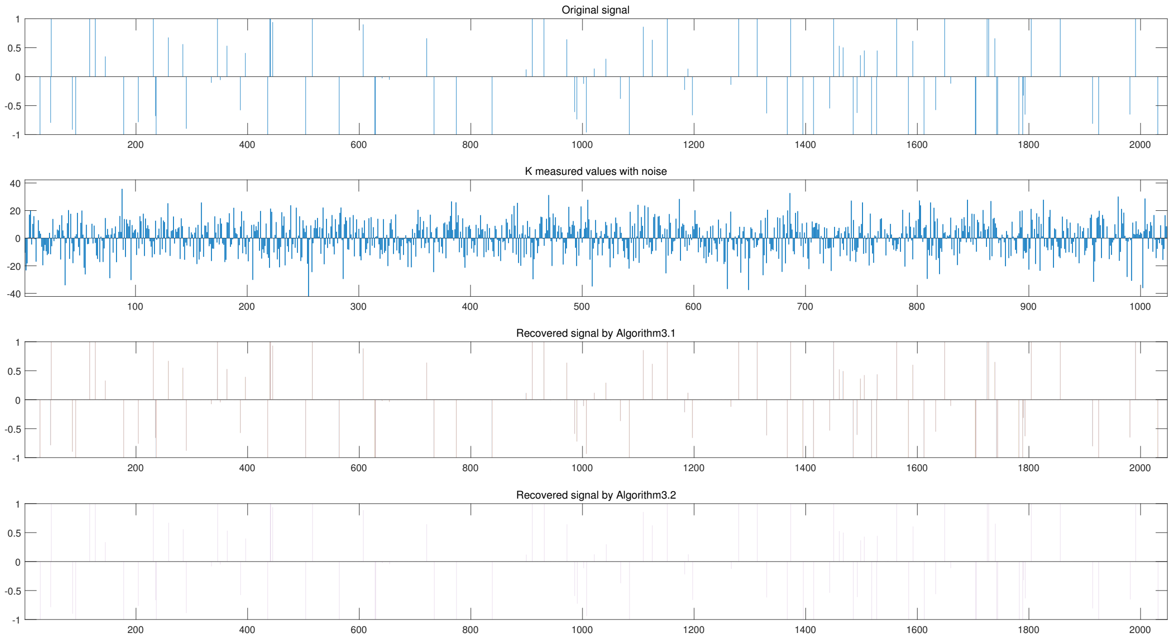


Figure 4. Numerical results for $m = 2^{15}$, $m = 2^{16}$, and $K = 100$.

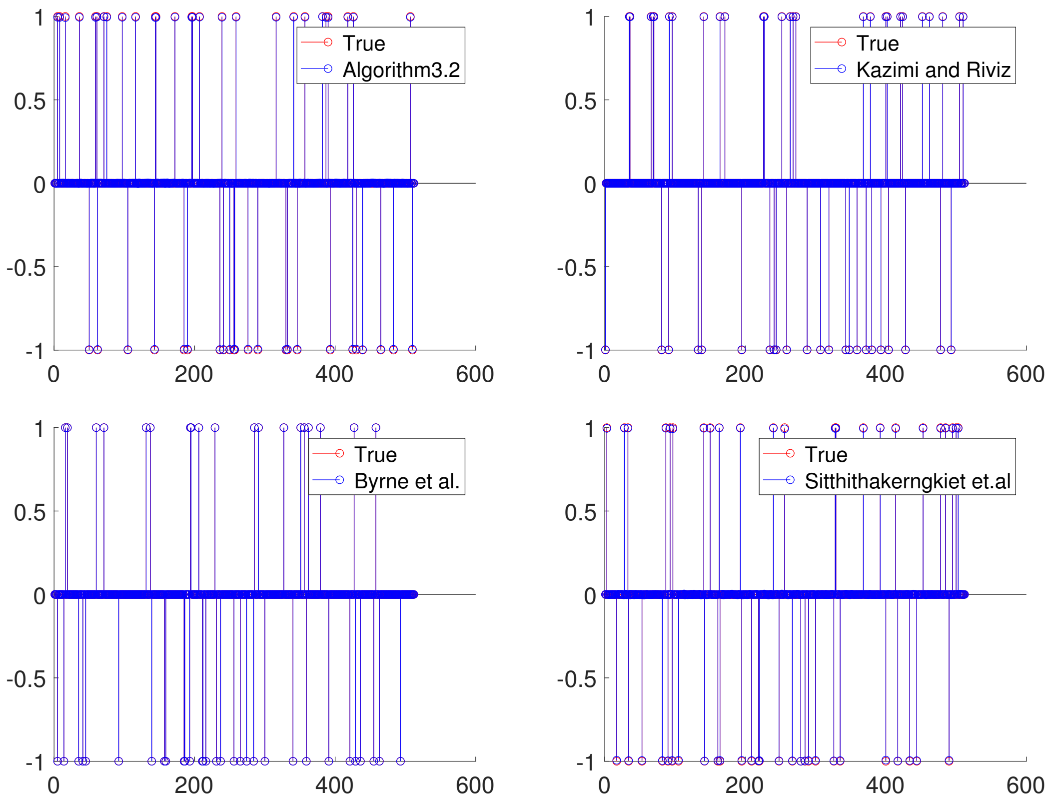
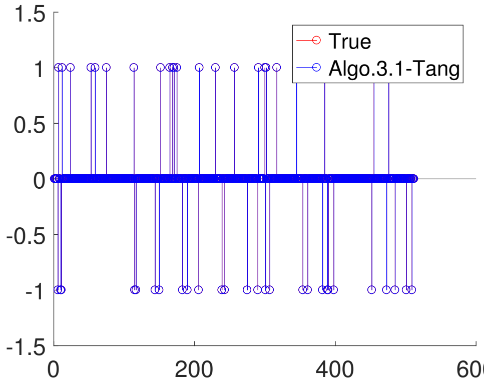


Figure 5. *Cont.*



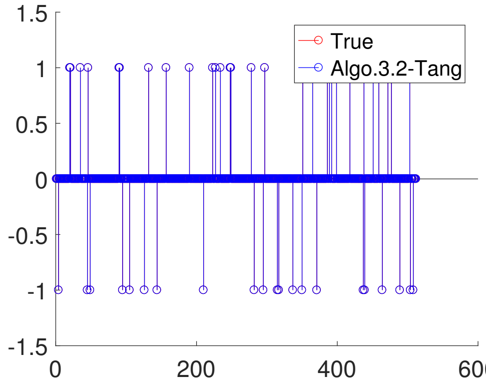


Figure 5. Numerical results for $m = 2^7, m = 2^9$, and $K = 50$.

Table 2. Comparisons of Algorithm 3.1, Algorithm 3.2, and Algorithm 3.1 in Sitthithakerngkiet [21], Algorithm 3.1 in Byrne [5], Algorithm 3.1 in Kazimi and Riviz [22], and Algorithms 3.1 and 3.2 in Tang [34].

DOL	Method	Iter (n)	CPU Time (s)
10^{-4}	Algorithm 1	3	0.019
	Algorithm 2	65	0.19
	Algorithm 3.1-Tang [34]	3	0.14
	Algorithm 3.2-Tang [34]	35	2.26
	Sitthithakerngkiet [21]	78	0.12
	Byrne et al. [5]	2	0.01
	Kazimi and Riviz [22]	48	0.08
10^{-5}	Algorithm 1	3	0.017
	Algorithm 2	102	0.24
	Algorithm 3.1-Tang [34]	8	2.37
	Algorithm 3.2-Tang [34]	76	2.78
	Sitthithakerngkiet [21]	1272	3.03
	Byrne et al. [5]	3	0.013
	Kazimi and Riviz [22]	503	0.74

From Table 2, we can see that our Algorithms 1 and 2 seem to have some competitive advantages.

Compared with the general inertial methods, the main advantage of our Algorithms 1 and 2 in this paper, as mentioned in the previous sections, is that they have no constraint on the norm of the difference between x_n and x_{n-1} in advance, and no assumption on the inertial parameter θ_n, so it is extremely natural, attractive, and user friendly.

Moreover, when we test Algorithm 3.1 of Sitthithakerngkiet et al. [21], Algorithm 3.1 of Byrne et al. [5], and Kazimi and Riviz [22], we find that the convergence rate depends strongly on the step size γ, which depends on the norm of linear operator T, so another advantage of our Algorithms 1 and 2 in this paper is the self-adaptive step size.

5. Conclusions

We proposed two new self-adaptive inertial-like proximal point algorithms (Algorithms 1 and 2) for the split common null point problem (SCNPP). Under more general conditions, the weak and strong convergences to a solution of SCNPP are obtained. The new inertial-like proximal point algorithms listed are novel in the following ways:

(1) Different from the average inertia technique, the convergence of the proposed algorithms remain even if without the term below:

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty.$$

They do not need to calculate the values of $\|x_n - x_{n-1}\|$ in advance if one chooses the coefficients θ_n, which means that the algorithms are easy to use.

(2) The inertial factors θ_n can be chosen in $[0,1]$, which means that θ_n is a possible equivalent to 1 and opens a wider path for parameter selection.

(3) The step sizes of our inertial proximal algorithms are self-adaptive and are independent of the cocoercive coefficients, which means that they do not use any prior knowledge of the operator norms.

In addition, two numerical examples involving comparison results have been expressed to show the efficiency and reliability of the listed algorithms.

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