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Nonlocal Neumann Boundary Value Problem for Fractional Symmetric Hahn Integrodifference Equations

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Abstract: In this article, we present a nonlocal Neumann boundary value problems for separate sequential fractional symmetric Hahn integrodifference equation. The problem contains five fractional symmetric Hahn difference operators and one fractional symmetric Hahn integral of different orders. We employ Banach fixed point theorem and Schauder's fixed point theorem to study the existence results of the problem.

Keywords: fractional symmetric Hahn integral; Riemann-Liouville fractional symmetric Hahn difference; Neumann boundary value problems; existence

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1. Introduction

Quantum calculus is a study of calculus without limit that deals with a set of non-differentiable functions. It has been used in many studies such as approximation problems, particle physics problems, quantum mechanics, and calculus of variations. The q-calculus, one type of quantum calculus initiated by Jackson [1–5] has been employed in several fields of applied sciences and engineering such as physical problems, dynamical system, control theory, electrical networks, economics and so on [6–14].

Later, the motivation of quantum calculus based on two parameters q, ω was presented in 1949. W. Hahn [15] introduced the Hahn difference operator which is a combination of two well-known difference operators, the forward difference operator and the Jackson q-difference operator. In 2009, Aldwoah [16,17] defined the right inverse of $D_{q,\omega}$ in the terms of both the Jackson q-integral containing the right inverse of D_q and Nörlund sum contaning the right inverse of Δ_ω [18]. Moreover, Fractional Hahn operators [19] was introduced in 2017. These calculus are also employed in many research works [20–31] including the studies of initial and boundary value problems [32–40].

For symmetry of Hahn calculus, Artur et al. [41] introduced symmetric Hahn difference operator in 2013. Recently, Patanarapeelert and Sitthiwirattham [42] introduced fractional symmetric Hahn difference operator. However, the study of the boundary value problems for fractional symmetric Hahn difference equation in the beginning, there exists only one paper on this subject [43].

The main motivation for this paper is to enrich the literature on the boundary value problems for fractional symmetric Hahn difference equations. We study the boundary value problem involving functions F and H which separate fractional symmetric Hahn integral and fractional symmetric Hahn difference, and the boundary condition is a Neumann boundary condition that is assigned values at two non-local points. In this paper, we aim to employ this recent work to study solutions to a boundary value problem for fractional symmetric Hahn integrodifference equations. Our problem is a nonlocal Neumann boundary value problems for sequential fractional symmetric Hahn integrodifference equation of the form

$$\begin{aligned} \tilde{D}^{\alpha}_{q,\omega}\tilde{D}^{\beta}_{q,\omega}u(t) &= \lambda F\big(t,u(t),(\tilde{\Psi}^{\gamma}_{q,\omega}u)(t)\big)+\mu H\Big(t,u(t),(\tilde{\Upsilon}^{\nu}_{q,\omega}u)(t)\Big), \quad t\in I^{T}_{q,\omega}, \\ \tilde{D}^{\theta_1}_{q,\omega}g_1(\eta_1)u(\eta_1) &= \phi_1(u), \\ \tilde{D}^{\theta_2}_{q,\omega}g_2(\eta_2)u(\eta_2) &= \phi_2(u), \qquad \eta_1,\ \eta_2\in I^{T}_{q,\omega}-\{\omega_0,T\} \end{aligned} \tag{1}$$

where $I^{T}_{q,\omega}:=\{q^kT+\omega[k]_q : k\in\mathbb{N}_0\}\cup\{\omega_0\},\ \alpha,\beta,\gamma,\nu,\theta_1,\theta_2\in(0,1],\ \omega>0;\ q\in(0,1);$ $\lambda,\mu\in\mathbb{R}^+;$ $F,H\in C\big(I^{T}_{q,\omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\big)$ and $g_1,g_2\in C\big(I^{T}_{q,\omega},\mathbb{R}^+\big)$ are given functions, $\phi_1,\phi_2 : C\big(I^{T}_{q,\omega},\mathbb{R}\big)\to\mathbb{R}$ are given functionals; and for $\varphi,\psi\in\big(I^{T}_{q,\omega}\times I^{T}_{q,\omega},[0,\infty)\big)$, we define the operators

$$\Big(\tilde{\Psi}^{\gamma}_{q,\omega}u\Big)(t):=\Big(\tilde{\mathcal{I}}^{\gamma}_{q,\omega}u\Big)(t)=\frac{q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_q(\gamma)}\int_{\omega_0}^{t}\widetilde{\Big(t-s\Big)}_{q,\omega}^{\underline{\gamma-1}}\varphi(t,\sigma^{\gamma-1}_{q,\omega}(s))\,u(\sigma^{\gamma-1}_{q,\omega}(s))\,\tilde{d}_{q,\omega}s,$$

$$\Big(\tilde{\Upsilon}^{\nu}_{q,\omega}u\Big)(t):=\Big(\tilde{D}^{\nu}_{q,\omega}u\Big)(t)=\frac{q^{\binom{-\nu}{2}}}{\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{t}\widetilde{\Big(t-s\Big)}_{q,\omega}^{\underline{-\nu-1}}\psi(t,\sigma^{-\nu-1}_{q,\omega}(s))\,u(\sigma^{-\nu-1}_{q,\omega}(s))\,\tilde{d}_{q,\omega}s.$$

We first transform this nonlinear problem (1) into a fixed point problem, by in view of a linear variant of (1). When the fixed point operator is accessible, we use the classical fixed point theorems to find existence results. To study the solution of problem (1), we recall some definitions and basic knowledge, and we also study some properties of fractional symmetric Hahn integral that will be used in our main results in Section 2. In Section 3, we present the existence and uniqueness of a solution of problem (1) by using the Banach fixed point theorem. The existence of at least one solution of problem (1) is also investigated by using the Schuader's fixed point theorem. In the last section, we give an example to illustrate our results.

2. Preliminaries

2.1. Basic Notions and Results

In this section, we introduce the definitions of fractional symmetric Hahn difference calculus and its properties [41–45] as follows.

For $0<q<1$, $\omega>0$, $\omega_0=\frac{\omega}{1-q}$ and $[k]_q=\frac{1-q^k}{1-q}$, we define

$$\widetilde{[k]}_q := \begin{cases} \dfrac{1-q^{2k}}{1-q^2}=[k]_{q^2}, & k\in\mathbb{N} \\ 1, & k=0, \end{cases}$$

$$\widetilde{[k]}_q! := \begin{cases} \widetilde{[k]}_q\widetilde{[k-1]}_q\cdots\widetilde{[1]}_q=\displaystyle\prod_{i=1}^{k}\frac{1-q^{2i}}{1-q^2}, & k\in\mathbb{N} \\ 1, & k=0. \end{cases}$$

The q,ω-forward jump operator is defined by

$$\sigma^{k}_{q,\omega}(t):=q^kt+\omega[k]_q,$$

and the q,ω-backward jump operator is defined by

$$\rho^{k}_{q,\omega}(t):=\frac{t-\omega[k]_q}{q^k},$$

where $k\in\mathbb{N}$.

For $n\in\mathbb{N}_0:=\{0,1,2,\dots\}$, and $a,b\in\mathbb{R}$, the q-analogue of the power function is defined as

$$(a-b)_q^{\underline{0}}:=1, \qquad (a-b)_q^{\underline{n}}:=\prod_{i=0}^{n-1}(a-bq^i),$$

the q-symmetric analogue of the power function is defined as

$$\widetilde{(a-b)}_q^{\underline{0}} := 1, \qquad \widetilde{(a-b)}_q^{\underline{n}} := \prod_{i=0}^{n-1}(a - bq^{2i+1}),$$

and, the q,ω-symmetric analogue of the power function is defined as

$$\widetilde{(a-b)}_{q,\omega}^{\underline{0}} := 1, \qquad \widetilde{(a-b)}_{q,\omega}^{\underline{n}} := \prod_{i=0}^{n-1}\Big[a - \sigma_{q,\omega}^{2i+1}(b)\Big].$$

Generally, for $\alpha \in \mathbb{R}$, the power functions are defined as

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}},\ a \neq 0,$$

$$\widetilde{(a-b)}_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^{2i+1}}{1-\left(\frac{b}{a}\right)q^{2(\alpha+i)+1}},\ a \neq 0,$$

$$\widetilde{(a-b)})_{q,\omega}^{\underline{\alpha}} = \Big(\widetilde{(a-\omega_0)-(b-\omega_0)}\Big)_q^{\underline{\alpha}} = (a-\omega_0)^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2i+1}}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2(\alpha+i)+1}},\ a \neq \omega_0.$$

Particularly, $a_q^{\underline{\alpha}} = \widetilde{a}_q^{\underline{\alpha}} = a^\alpha$ and $\widetilde{(a-\omega_0)}_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$ if $b=0$. If $a=b$, $(0)_q^{\underline{\alpha}} = \widetilde{(0)}_q^{\underline{\alpha}} = \widetilde{(\omega_0)}_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

The q-symmetric gamma and q-symmetric beta functions are defined as

$$\begin{aligned}
\tilde{\Gamma}_q(x) &:= \begin{cases} \frac{(1-q^2)_q^{\underline{x-1}}}{(1-q^2)^{x-1}} = \frac{\widetilde{(1-q)}_q^{\underline{x-1}}}{(1-q^2)^{x-1}}, & x \in \mathbb{R}\setminus\{0,-1,-2,\dots\} \\ \widetilde{[x-1]}_q!, & x \in \mathbb{N}, \end{cases} \\
\tilde{B}_q(x,y) &:= \int_0^1 \left(q^{-1}s\right)^{x-1} \widetilde{(1-s)}_q^{\underline{y-1}} \tilde{d}_q s = \frac{\tilde{\Gamma}_q(x)\tilde{\Gamma}_q(y)}{\tilde{\Gamma}_q(x+y)},
\end{aligned}$$

respectively.

Lemma 1 ([42]). *For $m,n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$,*

(a) $\widetilde{(x - \sigma_{q,\omega}^n(x))}_{q,\omega}^{\underline{\alpha}} = (x-\omega_0)^k \widetilde{(1-q^n)}_q^{\underline{\alpha}}$,

(b) $\widetilde{(\sigma_{q,\omega}^m(x)) - \sigma_{q,\omega}^n(x))}_{q,\omega}^{\underline{\alpha}} = q^{m\alpha}(x-\omega_0)^\alpha \widetilde{(1-q^{n-m})}_q^{\underline{\alpha}}$.

Definition 1 ([41]). *For $q \in (0,1)$, $\omega > 0$, and f is a function defined on $I_{q,\omega}^T \subseteq \mathbb{R}$, the symmetric Hahn difference of f is defined by*

$$\begin{aligned}
\tilde{D}_{q,\omega} f(t) &:= \frac{f(\sigma_{q,\omega}(t)) - f(\rho_{q,\omega}(t))}{\sigma_{q,\omega}(t) - \rho_{q,\omega}(t)} \quad t \in I_{q,\omega}^T - \{\omega_0\}, \\
\tilde{D}_{q,\omega} f(\omega_0) &= f'(\omega_0) \ \text{ where } f \text{ is differentiable at } \omega_0.
\end{aligned}$$

$\tilde{D}_{q,\omega} f$ is called q,ω-symmetric derivative of f, and f is q,ω-symmetric differentiable on $I_{q,\omega}^T$. For $N \in \mathbb{N}$, $\tilde{D}_{q,\omega}^N f(x) = \tilde{D}_{q,\omega}\tilde{D}_{q,\omega}^{N-1} f(x)$ where $\tilde{D}_{q,\omega}^0 f(x) = f(x)$.

Remark 1. *If f and g are q,ω-symmetric differentiable on $I_{q,\omega}^T$,*

(a) $\tilde{D}_{q,\omega}[f(t)+g(t)] = \tilde{D}_{q,\omega} f(t) + \tilde{D}_{q,\omega} g(t)$,

(b) $\tilde{D}_{q,\omega}[f(t)g(t)] = f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega}g(t) + g(\sigma_{q,\omega}(t))\tilde{D}_{q,\omega}f(t)$,

(c) $\tilde{D}_{q,\omega}\left[\dfrac{f(t)}{g(t)}\right] = \dfrac{g(\rho_{q,\omega}(t))\tilde{D}_{q,\omega}f(t) - f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega}g(t)}{g(\rho_{q,\omega}(t))g(\sigma_{q,\omega}(t))}$, $g(\rho_{q,\omega}(t))g(\sigma_{q,\omega}(t)) \neq 0$,

(d) $\tilde{D}_{q,\omega}[C] = 0$ *where C is constant.*

Definition 2 ([41]). *Let I be any closed interval of $\mathbb{R}$ containing a, b and ω_0 and $f : I \to \mathbb{R}$ be a given function. The symmetric Hahn integral of f from a to b is defind by*

$$\int_a^b f(t)\tilde{d}_{q,\omega}t := \int_{\omega_0}^b f(t)\tilde{d}_{q,\omega}t - \int_{\omega_0}^a f(t)\tilde{d}_{q,\omega}t,$$

and

$$\tilde{\mathcal{I}}_{q,\omega}f(t) = \int_{\omega_0}^x f(t)\tilde{d}_{q,\omega}t := (1-q^2)(x-\omega_0)\sum_{k=0}^{\infty} q^{2k} f\left(\sigma_{q,\omega}^{2k+1}(x)\right), \quad x \in I,$$

where the above series converges at $x = a$ and $x = b$. For $N \in \mathbb{N}$, $\tilde{\mathcal{I}}^N_{q,\omega}f(x) = \tilde{\mathcal{I}}_{q,\omega}\tilde{\mathcal{I}}^{N-1}_{q,\omega}f(x)$ where $\tilde{\mathcal{I}}^0_{q,\omega}f(x) = f(x)$.

The following is the relation between the symmetric Hahn difference and integral.

$$\tilde{D}_{q,\omega}\tilde{\mathcal{I}}_{q,\omega}f(x) = f(x) \text{ and } \tilde{\mathcal{I}}_{q,\omega}\tilde{D}_{q,\omega}f(x) = f(x) - f(\omega_0).$$

Remark 2 ([41]). *Let $a, b \in I^T_{q,\omega}$ and f, g be symmetric Hahn integrable on $I^T_{q,\omega}$. Then,*

(a) $\int_a^a f(t)\tilde{d}_{q,\omega}t = 0$,

(b) $\int_a^b f(t)\tilde{d}_{q,\omega}t = -\int_b^a f(t)\tilde{d}_{q,\omega}t$,

(c) $\int_a^b f(t)\tilde{d}_{q,\omega}t = \int_c^b f(t)\tilde{d}_{q,\omega}t + \int_a^c f(t)\tilde{d}_{q,\omega}t$, $c \in I^T_{q,\omega}$, $a < c < b$,

(d) $\int_a^b [\alpha f(t) + \beta g(t)]\tilde{d}_{q,\omega}t = \alpha\int_a^b f(t)\tilde{d}_{q,\omega}t + \beta\int_a^b g(t)\tilde{d}_{q,\omega}t$, $\alpha, \beta \in \mathbb{R}$,

(e) $\int_a^b \left[f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega}g(t)\right]\tilde{d}_{q,\omega}t = \left[f(t)g(t)\right]_a^b - \int_a^b \left[g(\sigma_{q,\omega}(t))\tilde{D}_{q,\omega}f(t)\right]\tilde{d}_{q,\omega}t$.

Lemma 2 ([41]). *[Fundamental theorem of symmetric Hahn calculus]*

Let $f : I \to \mathbb{R}$ be continuous at ω_0. Then,

$$F(x) := \int_{\omega_0}^x f(t)\tilde{d}_{q,\omega}t, \quad x \in I$$

is continuous at ω_0 and $\tilde{D}_{q,\omega}F(x)$ exists for every $x \in \sigma_{q,\omega}(I) := \{qt + \omega : t \in I\}$ where

$$\tilde{D}_{q,\omega}F(x) = f(x).$$

In addition,

$$\int_a^b \tilde{D}_{q,\omega}f(t)\tilde{d}_{q,\omega}t = f(b) - f(a) \text{ for all } a, b \in I.$$

Lemma 3 ([42]). *Let $0 < q < 1$, $\omega > 0$ and $f : I \to \mathbb{R}$ be continuous at ω_0. Then,*

$$\int_{\omega_0}^t \int_{\omega_0}^r f(s)\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}r \quad = \quad q\int_{\omega_0}^t \int_{qs+\omega}^t f(qs+\omega)\,\tilde{d}_{q,\omega}r\,\tilde{d}_{q,\omega}s.$$

Definition 3 ([42]). *Let $\alpha, \omega > 0$, $0 < q < 1$, and f be a function defined on $I^T_{q,\omega}$. The fractional symmetric Hahn integral is defined by*

$$\tilde{\mathcal{I}}^{\alpha}_{q,\omega}f(t) \quad := \quad \frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_{\omega_0}^t \widetilde{(t-s)}^{\,\alpha-1}_{q,\omega} f\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\tilde{d}_{q,\omega}s$$

$$
\begin{aligned}
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)}{\widetilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\left(t-\sigma_{q,\omega}^{2k+1}(t)\right)_{q,\omega}^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^{2k+\alpha}(t)\right)\\
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)^{\alpha}}{\widetilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\left(\widetilde{1-q^{2k+1}}\right)_{q}^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^{2k+\alpha}(t)\right)
\end{aligned}
$$

and $\widetilde{\mathcal{I}}^{0}_{q,\omega}f)(t) = f(t)$.

Definition 4 ([42]). *For $\alpha,\omega > 0$, $0 < q < 1$ and f defined on $I^T_{q,\omega}$, the fractional symmetric Hahn difference operator of Riemann-Liouville type of order α is defined by*

$$
\begin{aligned}
\widetilde{D}^{\alpha}_{q,\omega}f(t) &:= \widetilde{D}^{N}_{q,\omega}\widetilde{\mathcal{I}}^{N-\alpha}_{q,\omega}f(t)\\
&= \frac{q^{\binom{-\alpha}{2}}}{\widetilde{\Gamma}_q(-\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\alpha-1}} f\left(\sigma_{q,\omega}^{-\alpha-1}(s)\right)\tilde{d}_{q,\omega}s\\
\widetilde{D}^{0}_{q,\omega}f(t) &= f(t)
\end{aligned}
$$

where $N-1 < \alpha < N$, $N \in \mathbb{N}$.

Lemma 4 ([42]). *Let $\alpha,\omega > 0$, $0 < q < 1$ and $f : I^T_{q,\omega} \to \mathbb{R}$. Then,*

$$
\tilde{\mathcal{I}}^{\alpha}_{q,\omega}\tilde{D}^{\alpha}_{q,\omega}f(t) = f(t) + C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + \cdots + C_N(t-\omega_0)^{\alpha-N}
$$

for some $C_i \in \mathbb{R}, i = 1,2,\ldots,N$ and $N-1 < \alpha < N$ for $N \in \mathbb{N}$.

Lemma 5 ([46] Arzelá-Ascoli theorem). *A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 6 ([46]). *If a set is closed and relatively compact then it is compact.*

Lemma 7 ([47] Schauder's fixed point theorem). *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

2.2. Auxiliary Lemmas

In this part, we establish some lemmas that are used to prove our main results.

Lemma 8. *Let $q \in (0,1), \omega > 0$ and $n > 0$. Then,*

$$
\int_{\omega_0}^{t}\tilde{d}_{q,\omega}s = t-\omega_0 \quad \textit{and} \quad \int_{\omega_0}^{t}(s-\omega_0)^n\,\tilde{d}_{q,\omega}s = \frac{q^n}{\widetilde{[n+1]}_q}(t-\omega_0)^{n+1}.
$$

Lemma 9. *Let $\alpha,\beta > 0$, $q \in (0,1)$ and $\omega > 0$. Then,*

(i) $\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\,\tilde{d}_{q,\omega}s = \frac{(t-\omega_0)^{\alpha}}{\widetilde{[\alpha]}_q}$,

(ii) $\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\left(\sigma_{q,\omega}^{\alpha-1}(s)-\omega_0\right)^{\beta}\tilde{d}_{q,\omega}s = q^{\alpha\beta}(t-\omega_0)^{\alpha+\beta}B_q(\beta+1,\alpha)$,

(iii) $\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\alpha-1}(s)}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\left(\widetilde{\sigma_{q,\omega}^{\alpha-1}(s)-r}\right)_{q,\omega}^{\underline{\beta-1}}\tilde{d}_{q,\omega}r\,\tilde{d}_{p,\omega}s = \frac{q^{\alpha\beta}}{\widetilde{[\beta]}_q}(t-\omega_0)^{\alpha+\beta}B_q(\beta+1,\alpha)$.

Lemma 10. *Let $\alpha,\beta,\theta > 0$, $q \in (0,1)$ and $\omega > 0$. Then,*

(a) $\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(s)-\omega_0\right)^{\beta-1}\tilde{d}_{q,\omega}s = \frac{(t-\omega_0)^{\beta-\theta-1}}{q^{\theta(\beta-1)}}\widetilde{B}_q(\beta,-\theta)$,

(b) $\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(x)} \widetilde{(t-x)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(x)-s\right)_{q,\omega}^{\underline{\beta-1}}\left(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x$
$= \frac{(t-\omega_0)^{\alpha+\beta-\theta-1}}{q^{\theta(\alpha+\beta-1)-\beta(\alpha-1)}}\widetilde{B}_q(\alpha,\beta)\widetilde{B}_q(\alpha+\beta,-\theta),$

(c) $\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(t-y)}_{q,\omega}^{\underline{-\theta-1}}\left(\widetilde{\sigma_{q,\omega}^{-\theta-1}(y)}-x\right)_{q,\omega}^{\underline{\beta-1}}\left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x)}-s\right)_{q,\omega}^{\underline{\alpha-1}} \times$
$\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}x\,\tilde{d}_{q,\omega}y$

$= \frac{(t-\omega_0)^{\alpha+\beta-\theta}}{q^{\theta(\alpha+\beta)-\alpha\beta}[\widetilde{\alpha}]_q}\widetilde{B}_q(\alpha+1,\beta)\widetilde{B}_q(\alpha+\beta+1,-\theta).$

Proof. We use the definition of q,ω-symmetric analogue of the power function, Lemma 1 and Definition 2 to obtain

$$
\begin{aligned}
(a)\quad & \int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(s)-\omega_0\right)^{\beta-1}\tilde{d}_{q,\omega}s \\
&= (1-q^2)(t-\omega_0)\sum_{j=0}^{\infty}q^{2j}\left(t-\widetilde{\sigma_{q,\omega}^{2j+1}(t)}\right)_{q,\omega}^{\underline{-\theta}}\left(q^{-\theta-1}\left(\sigma_{q,\omega}^{2j+1}(t)-\omega_0\right)\right)^{\beta-1} \\
&= q^{-\theta(\beta-1)}(1-q^2)(t-\omega_0)^{\beta-\theta-1}\sum_{j=0}^{\infty}q^{2j}\widetilde{(1-q^{2j+1})}_{q}^{\underline{-\theta-1}}\left(q^{-1}q^{2j+1}\right)^{\beta-1} \\
&= q^{-\theta(\beta-1)}(t-\omega_0)^{\beta-\theta-1}\int_{\omega_0}^{t}\widetilde{(1-s)}_{q,\omega}^{\underline{-\theta-1}}(q^{-1}s)^{\beta-1}\tilde{d}_{q,\omega}s \\
&= \frac{(t-\omega_0)^{\beta-\theta-1}}{q^{\theta(\beta-1)}}\widetilde{B}_q(\beta,-\theta), \\
(b)\quad & \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(x)-s\right)_{q,\omega}^{\underline{\beta-1}}\left(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x \\
&= \int_{\omega_0}^{t}\widetilde{(t-x)}_{q,\omega}^{\underline{-\theta-1}}\left[\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(x)}\left(\widetilde{\sigma_{q,\omega}^{-\theta-1}(x)}-s\right)_{q,\omega}^{\underline{-\beta-1}}\left(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\right)^{\alpha-1}\tilde{d}_{q,\omega}s\right]\tilde{d}_{q,\omega}x \\
&= q^{\beta(\alpha-1)}B_q(\alpha,\beta)\int_{\omega_0}^{t}\widetilde{(t-x)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(x)-\omega_0\right)^{\alpha+\beta-1}\tilde{d}_{q,\omega}x \\
&= \frac{(t-\omega_0)^{\alpha+\beta-\theta-1}}{q^{\theta(\alpha+\beta-1)-\beta(\alpha-1)}}\widetilde{B}_q(\alpha,\beta)\widetilde{B}_q(\alpha+\beta,-\theta), \\
(c)\quad & \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\widetilde{(t-y)}_{q,\omega}^{\underline{-\theta-1}}\left(\widetilde{\sigma_{q,\omega}^{-\theta-1}(y)}-x\right)_{q,\omega}^{\underline{\beta-1}}\left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x)}-s\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}x\,\tilde{d}_{q,\omega}y \\
&= \int_{\omega_0}^{t}\widetilde{(t-y)}_{q,\omega}^{\underline{-\theta-1}}\left[\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\left(\widetilde{\sigma_{q,\omega}^{-\theta-1}(y)}-x\right)_{q,\omega}^{\underline{\beta-1}}\left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x)}-s\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\right]\tilde{d}_{q,\omega}y \\
&= \frac{q^{\alpha\beta}}{[\widetilde{\alpha}]_q}B_q(\alpha+1,\beta)\int_{\omega_0}^{t}\widetilde{(t-y)}_{q,\omega}^{\underline{-\theta-1}}\left(\sigma_{q,\omega}^{-\theta-1}(x)-\omega_0\right)^{\alpha+\beta}\tilde{d}_{q,\omega}y \\
&= \frac{(t-\omega_0)^{\alpha+\beta-\theta}}{q^{\theta(\alpha+\beta)-\alpha\beta}[\widetilde{\alpha}]_q}\widetilde{B}_q(\alpha+1,\beta)\widetilde{B}_q(\alpha+\beta+1,-\theta).
\end{aligned}
$$

□

2.3. Lemma for Linear Variant Form

In this part, a solution of a linear variant form of the problem (1) is investigated as shown in the following lemma.

Lemma 11. *Let $\Omega \neq 0$; $\omega > 0$; $q \in (0,1)$; $\alpha, \beta, \theta_1, \theta_2 \in (0,1]$; $h \in C\big(I_{q,\omega}^T, \mathbb{R}\big)$ and $g_1, g_2 \in C\big(I_{q,\omega}^T, \mathbb{R}^+\big)$ be given functions; $\phi_1, \phi_2 \in C\big(I_{q,\omega}^T, \mathbb{R}\big) \to \mathbb{R}$ be given functionals. Then the problem*

$$\begin{aligned} \tilde{D}_{q,\omega}^{\alpha} \tilde{D}_{q,\omega}^{\beta} u(t) &= h(t), \quad t \in I_{q,\omega}^T, \\ \tilde{D}_{q,\omega}^{\theta_i} g_i(\eta_i) u(\eta_i) &= \phi_i(u), \quad \eta_i \in I_{q,\omega}^T - \{\omega_0, T\}, \ i = 1,2, \end{aligned} \tag{2}$$

has the unique solution

$$\begin{aligned} u(t) = \ & [\boldsymbol{A}_2 \boldsymbol{O}_1[\phi_1, h] - \boldsymbol{A}_1 \boldsymbol{O}_2[\phi_2, h]] \frac{(t-\omega_0)^{\beta-1}}{\Omega} \\ & -[\boldsymbol{B}_2 \boldsymbol{O}_1[\phi_1, h] - \boldsymbol{B}_1 \boldsymbol{O}_2[\phi_2, h]] \frac{q^{\binom{\beta}{2}}}{\Omega \tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\beta-1}} \Big(\sigma_{q,\omega}^{\beta-1}(s) - \omega_0\Big)^{\alpha-1} \tilde{d}_{q,\omega} s \\ & + \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t} \int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(t-x)}_{q,\omega}^{\underline{\beta-1}} \Big(\widetilde{\sigma_{q,\omega}^{\beta-1}(x) - s}\Big)_{q,\omega}^{\underline{\alpha-1}} h(\sigma_{q,\omega}^{\alpha-1}(s)) \tilde{d}_{q,\omega} s \tilde{d}_{q,\omega} x. \end{aligned} \tag{3}$$

where the functionals $\boldsymbol{O}_1[\phi_1, h]$, $\boldsymbol{O}_2[\phi_2, h]$ are defined by

$$\begin{aligned} \boldsymbol{O}_1[\phi_1, h] := \ & \phi_1(u) - \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_1)} \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_1-1}(y)} \int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(\eta_1 - y)}_{q,\omega}^{\underline{-\theta_1-1}} \times \\ & \Big(\widetilde{\sigma_{q,\omega}^{-\theta_1-1}(y) - x}\Big)_{q,\omega}^{\underline{\beta-1}} \Big(\widetilde{\sigma_{q,\omega}^{\beta-1}(x) - s}\Big)_{q,\omega}^{\underline{\alpha-1}} g_1\Big(\sigma_{q,\omega}^{-\theta_1-1}(y)\Big) \times \\ & h(\sigma_{q,\omega}^{\alpha-1}(s)) \, \tilde{d}_{q,\omega} s \tilde{d}_{q,\omega} x \tilde{d}_{q,\omega} y, \end{aligned} \tag{4}$$

$$\begin{aligned} \boldsymbol{O}_2[\phi_2, h] := \ & \phi_2(u) - \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_2)} \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_2-1}(y)} \int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(\eta_2 - y)}_{q,\omega}^{\underline{-\theta_2-1}} \times \\ & \Big(\widetilde{\sigma_{q,\omega}^{-\theta_2-1}(y) - x}\Big)_{q,\omega}^{\underline{\beta-1}} \Big(\widetilde{\sigma_{q,\omega}^{\beta-1}(x) - s}\Big)_{q,\omega}^{\underline{\alpha-1}} g_2\Big(\sigma_{q,\omega}^{-\theta_2-1}(y)\Big) \times \\ & h(\sigma_{q,\omega}^{\alpha-1}(s)) \, \tilde{d}_{q,\omega} s \tilde{d}_{q,\omega} x \tilde{d}_{q,\omega} y, \end{aligned} \tag{5}$$

and the constants $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_1, \boldsymbol{B}_2$ and Ω are defined by

$$\boldsymbol{A}_1 := \frac{1}{\tilde{\Gamma}_q(-\theta_1)} \int_{\omega_0}^{\eta_1} \widetilde{(\eta_1 - s)}_{q,\omega}^{\underline{-\theta_1-1}} \Big(\sigma_{q,\omega}^{\underline{-\theta_1-1}}(s) - \omega_0\Big)^{\alpha-1} g_1\Big(\sigma_{q,\omega}^{-\theta_1-1}(s)\Big) \tilde{d}_{q,\omega} s, \tag{6}$$

$$\boldsymbol{A}_2 := \frac{1}{\tilde{\Gamma}_q(-\theta_2)} \int_{\omega_0}^{\eta_2} \widetilde{(\eta_2 - s)}_{q,\omega}^{\underline{-\theta_2-1}} \Big(\sigma_{q,\omega}^{\underline{-\theta_2-1}}(s) - \omega_0\Big)^{\alpha-1} g_2\Big(\sigma_{q,\omega}^{-\theta_2-1}(s)\Big) \tilde{d}_{q,\omega} s, \tag{7}$$

$$\begin{aligned} \boldsymbol{B}_1 := \ & \frac{q^{\binom{\beta}{2}+\binom{-\theta_1}{2}}}{\tilde{\Gamma}_q(-\theta_1)} \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_1-1}(x)} \widetilde{(\eta_1 - x)}_{q,\omega}^{\underline{-\theta_1-1}} \Big(\sigma_{q,\omega}^{-\theta_1-1}(x) - s\Big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \Big(\sigma_{q,\omega}^{-\theta_1-1}(s) - \omega_0\Big)_{q,\omega}^{\alpha-1} g_1\Big(\sigma_{q,\omega}^{-\theta_1-1}(x)\Big) \tilde{d}_{q,\omega} s \tilde{d}_{q,\omega} x, \end{aligned} \tag{8}$$

$$\begin{aligned} \boldsymbol{B}_2 := \ & \frac{q^{\binom{\beta}{2}+\binom{-\theta_2}{2}}}{\tilde{\Gamma}_q(-\theta_2)} \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_2-1}(x)} \widetilde{(\eta_2 - x)}_{q,\omega}^{\underline{-\theta_2-1}} \Big(\sigma_{q,\omega}^{-\theta_2-1}(x) - s\Big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \Big(\sigma_{q,\omega}^{-\theta_2-1}(s) - \omega_0\Big)_{q,\omega}^{\alpha-1} g_2\Big(\sigma_{q,\omega}^{-\theta_2-1}(x)\Big) \tilde{d}_{q,\omega} s \tilde{d}_{q,\omega} x, \end{aligned} \tag{9}$$

$$\Omega := \boldsymbol{A}_2 \boldsymbol{B}_1 - \boldsymbol{A}_1 \boldsymbol{B}_2. \tag{10}$$

Proof. From (2), we take a fractional symmetric Hahn integral of order α and find that

$$\tilde{D}_{q,\omega}^{\beta} u(t) = C_0 (t-\omega_0)^{\alpha-1} + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}} h\Big(\sigma_{q,\omega}^{\alpha-1}(s)\Big) \tilde{d}_{q,\omega} s. \tag{11}$$

Taking fractional symmetric Hahn integral of order β for (11), we obtain

$$u(t) = C_1(t-\omega_0)^{\beta-1} + \frac{C_0 q^{\binom{\beta}{2}}}{\tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\beta-1}} \left(\sigma_{q,\omega}^{\beta-1}(s) - \omega_0\right)^{\alpha-1} \tilde{d}_{q,\omega}s$$
$$+ \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(t-x)}_{q,\omega}^{\underline{\beta-1}} \left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x) - s}\right)_{q,\omega}^{\underline{\alpha-1}} h\left(\sigma_{q,\omega}^{\alpha-1}(s)\right) \tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x. \tag{12}$$

Next, we take fractional symmetric Hahn difference of order $\theta_i,\ i = 1,2$ for (12). Thus, we have

$$\tilde{D}_{q,\omega}^{\theta_i} u(t) = \frac{C_1 q^{\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(-\theta_i)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{-\theta_i-1}} \left(\sigma_{q,\omega}^{-\theta-1}(s) - \omega_0\right)^{\beta-1} \tilde{d}_{q,\omega}s + \frac{C_0 q^{\binom{\beta}{2}+\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_i)} \times$$
$$\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_i-1}(x)} \widetilde{(t-x)}_{q,\omega}^{\underline{-\theta_i-1}} \left(\widetilde{\sigma_{q,\omega}^{-\theta_i-1}(x) - s}\right)_{q,\omega}^{\underline{\beta-1}} \left(\sigma_{q,\omega}^{\beta-1}(s) - \omega_0\right)^{\alpha-1} \tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x$$
$$+ \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_i)} \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\theta_i-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(t-y)}_{q,\omega}^{\underline{-\theta_i-1}} \left(\widetilde{\sigma_{q,\omega}^{-\theta_i-1}(y) - x}\right)_{q,\omega}^{\underline{\beta-1}} \times$$
$$\left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x) - s}\right)^{\underline{\alpha-1}} h(\sigma_{q,\omega}^{\alpha-1}(s))\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y. \tag{13}$$

Substituting $i = 1,2$ into (13) and employing the condition of (2), we have

$$\boldsymbol{A}_1 C_0 + \boldsymbol{B}_1 C_1 = \boldsymbol{O}_1[\phi_1, h], \tag{14}$$
$$\boldsymbol{A}_2 C_0 + \boldsymbol{B}_2 C_1 = \boldsymbol{O}_2[\phi_2, h]. \tag{15}$$

Solving the system of Equations (14) and (15), we get

$$C_0 = \frac{\boldsymbol{B}_1\boldsymbol{O}_2[\phi_2,h] - \boldsymbol{B}_2\boldsymbol{O}_1[\phi_1,h]}{\Omega} \quad \text{and} \quad C_2 = \frac{\boldsymbol{A}_2\boldsymbol{O}_1[\phi_1,h] - \boldsymbol{A}_1\boldsymbol{O}_2[\phi_2,h]}{\Omega}$$

where $\boldsymbol{O}_1[\phi_1,h], \boldsymbol{O}_2[\phi_2,h], \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_1, \boldsymbol{B}_2$ and Ω are defined as (4)–(10), respectively. Substituting C_1, C_2 into (12), we obtain the solution (3). The converse can be proved by direct computation. The proof is complete. □

3. Existence and Uniqueness of Solution of the Problem (1)

To consider the existence and uniqueness of solution to the problem (1), we use Banach fixed point theorem.

Let $\mathcal{C} = C\left(I_{q,\omega}^T, \mathbb{R}\right)$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \|\tilde{D}_{q,\omega}^{\nu}\|,$$

where $\|u\| = \max_{t\in I_{q,\omega}^T}\left\{|u(t)|\right\}$ and $\|\tilde{D}_{q,\omega}^{\nu}u\| = \max_{t\in I_{q,\omega}^T}\|(\tilde{D}_{q,\omega}^{\nu}u)(t)\|$. An operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is defined by

$$(\mathcal{A}u)(t)$$
$$:= [\boldsymbol{A}_2\boldsymbol{O}_1^*[\phi_1, F_u + H_u] - \boldsymbol{A}_1\boldsymbol{O}_2^*[\phi_2, F_u + H_u]]\frac{(t-\omega_0)^{\beta-1}}{\Omega}$$
$$-[\boldsymbol{B}_2\boldsymbol{O}_1^*[\phi_1, F_u + H_u] - \boldsymbol{B}_1\boldsymbol{O}_2^*[\phi_2, F_u + H_u]]\frac{q^{\binom{\beta}{2}}}{\Omega\tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\beta-1}} \left(\sigma_{q,\omega}^{\beta-1}(s) - \omega_0\right)^{\alpha-1} \tilde{d}_{q,\omega}s$$
$$+ \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)} \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)} \widetilde{(t-x)}_{q,\omega}^{\underline{\beta-1}} \left(\sigma_{q,\omega}^{\beta-1}(x) - s\right)^{\underline{\alpha-1}} \Big[\lambda F\left(\sigma_{q,\omega}^{\alpha-1}(s), u\left(\sigma_{q,\omega}^{\alpha-1}(s)\right),\right.$$

$$\tilde{\Psi}^{\gamma}_{q,\omega}u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big)\Big) + \mu H\Big(\sigma^{\alpha-1}_{q,\omega}(s), u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big), \tilde{\Upsilon}^{\nu}_{q,\omega}u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big)\Big)\Big]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x \tag{16}$$

where the functionals $\boldsymbol{O}^*_i[\phi_i, F_u + H_u],\ i = 1,2$ are given by

$$\begin{aligned}
&\boldsymbol{O}^*_i[\phi_i, F_u + H_u] \\
&:= \phi_i(u) - \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(\theta_i)}\int_{u_0}^{\eta_i}\int_{\omega_0}^{\sigma^{-\theta_i-1}_{q,\omega}(y)}\int_{\omega_0}^{\sigma^{\beta-1}_{q,\omega}(x)}\widetilde{(\eta_i - y)}^{\underline{-\theta_i-1}}_{q,\omega}\Big(\widetilde{\sigma^{-\theta_i-1}_{q,\omega}(y) - x}\Big)^{\underline{\beta-1}}_{q,\omega} \\
&\Big(\widetilde{\sigma^{\beta-1}_{q,\omega}(x) - s}\Big)^{\underline{\alpha-1}}_{q,\omega} g_1\Big(\sigma^{-\theta_i-1}_{q,\omega}(y)\Big)\Big[\lambda F\Big(\sigma^{\alpha-1}_{q,\omega}(s), u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big), \tilde{\Psi}^{\gamma}_{q,\omega}u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big)\Big) \\
&+ \mu H\Big(\sigma^{\alpha-1}_{q,\omega}(s), u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big), \tilde{\Upsilon}^{\nu}_{q,\omega}u\Big(\sigma^{\alpha-1}_{q,\omega}(s)\Big)\Big)\Big]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y,
\end{aligned} \tag{17}$$

and the constants $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_1, \boldsymbol{B}_2, \Omega$ are given in Lemma 11.

To prove the existence results to the problem (1), we first convert the given nonlinear problem (1) into a fixed point problem. If the operator $\mathcal{A}$ has fixed point, then the problem (1) has the solution.

Theorem 1. *Assume that $F, H : I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $g_1, g_2 : I^T_{q,\omega} \to \mathbb{R}^+$ are continuous, and $\varphi, \psi : I^T_{q,\omega} \times I^T_{q,\omega} \to [0,\infty)$ are continuous with $\varphi_0 = \max\{\varphi(t,s)\} : (t,s)$ and $\psi_0 = \max\{\psi(t,s)\}$. Suppose that the following conditions hold:*

(H_1) *There exist positive constants M_i such that for each $t \in I^T_{q,\omega}$ and $u_i, v_i \in \mathbb{R},\ i = 1,2$,*

$$\Big|F[t,u_1,u_2] - F[t,v_1,v_2]\Big| \le M_1|u_1 - v_1| + M_2|u_2 - v_2|.$$

(H_2) *There exist positive constants N_i such that for each $t \in I^T_{q,\omega}$ and $u_i, v_i \in \mathbb{R},\ i = 1,2$,*

$$\Big|H[t,u_1,u_2] - H[t,v_1,v_2]\Big| \le N_1|u_1 - v_1| + N_2|u_2 - v_2|.$$

(H_3) *There exist positive constants ω_i such that for each $u_i, v_i \in \mathcal{C},\ i = 1,2$,*

$$\Big|\phi_1(u) - \phi_1(v)\Big| \le \omega_1\|u - v\|_{\mathcal{C}} \ \text{ and } \ \Big|\phi_2(u) - \phi_2(v)\Big| \le \omega_2\|u - v\|_{\mathcal{C}}.$$

(H_4) *For each $t \in I^T_{q,\omega}$, $g_i < g_i(t) < G_i,\ i = 1,2$.*
(H_5) *$\Xi = \mathcal{L}\chi + \omega_1\Theta^*_2 + \omega^*_2\Theta^*_1 < 1$,*
where

$$\mathcal{L} := \lambda\Big[M_1 + M_2\varphi_0 q^{\binom{\gamma}{2}}\frac{(T-\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\Big] + \mu\Big[N_1 + N_2\psi_0 q^{\binom{-\nu}{2}}\frac{(T-\omega_0)^{-\nu}}{\tilde{\Gamma}_q(1-\nu)}\Big] \tag{18}$$

$$\chi := q^{\binom{\alpha+\beta}{2}}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}_q(\alpha+\beta+1)} + q^{\binom{\alpha+\beta-\nu}{2}}\frac{(T-\omega_0)^{\alpha+\beta-\nu}}{\tilde{\Gamma}_q(\alpha+\beta-\nu+1)} + G_1\Phi_1\Theta^*_2 + G_2\Phi_2\Theta^*_1 \tag{19}$$

$$\Phi_i := q^{\binom{\alpha+\beta-\theta_i}{2}}\frac{(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_i+1)},\ i = 1,2 \tag{20}$$

$$\Theta^*_1 := \Theta_1 + \overline{\Theta}_1 \tag{21}$$

$$\Theta^*_2 := \Theta_2 + \overline{\Theta}_2 \tag{22}$$

$$\Theta_1 := (T-\omega_0)^{\beta-1}\max|A_1| + \frac{q^{\binom{\beta}{2}+(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)(T-\omega_0)^{\alpha+\beta-1}}{\tilde{\Gamma}(\alpha+\beta)}\max|B_1| \tag{23}$$

$$\overline{\Theta}_1 := (T-\omega_0)^{\beta-\nu-1}\frac{\tilde{\Gamma}_q(\beta)}{\tilde{\Gamma}_q(\beta-\nu)}q^{\binom{-\nu}{2}-\nu(\beta-1)}\max|A_1|$$
$$+(T-\omega_0)^{\alpha+\beta-\nu-1}\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha+\beta-\nu)}q^{\binom{\alpha+\beta-\nu-1}{2}-(\alpha-1)}\max|B_1| \tag{24}$$
$$\Theta_2 := (T-\omega_0)^{\beta-1}\max|A_2| + \frac{q^{\binom{\beta}{2}+(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)(T-\omega_0)^{\alpha+\beta-1}}{\tilde{\Gamma}(\alpha+\beta)}\max|B_2| \tag{25}$$
$$\overline{\Theta}_2 := (T-\omega_0)^{\beta-\nu-1}\frac{\tilde{\Gamma}_q(\beta)}{\tilde{\Gamma}_q(\beta-\nu)}q^{\binom{-\nu}{2}-\nu(\beta-1)}\max|A_2|$$
$$+(T-\omega_0)^{\alpha+\beta-\nu-1}\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha+\beta-\nu)}q^{\binom{\alpha+\beta-\nu-1}{2}-(\alpha-1)}\max|B_2|. \tag{26}$$

Then, the problem (1) *has a unique solution in* $I^T_{q,\omega}$.

Proof. For each $t\in I^T_{q,\omega}$ and $u,v\in\mathcal{C}$, we find that

$$\begin{aligned}\left|\tilde{\Psi}^\nu_{q,\omega}u-\tilde{\Psi}^\nu_{q,\omega}v\right| &\le \frac{\varphi_0 q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_q(\gamma)}\int_{\omega_0}^t \widetilde{(t-s)}^{\underline{\gamma-1}}_{q,\omega}\left|u\left(\sigma^{\gamma-1}_{q,\omega}(s)\right)-v\left(\sigma^{\gamma-1}_{q,\omega}(s)\right)\right|\tilde{d}_{q,\omega}s\\ &\le \frac{\varphi_0\|u-v\|q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_q(\gamma)}\int_{\omega_0}^T\widetilde{(T-s)}^{\underline{\gamma-1}}_{q,\omega}\tilde{d}_{q,\omega}s\\ &= \varphi_0 q^{\binom{\gamma}{2}}\frac{(T-\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}.\end{aligned} \tag{27}$$

Similary,

$$\left|\tilde{\Upsilon}^\nu_{q,\omega}u-\tilde{\Upsilon}^\nu_{q,\omega}v\right| = \varphi_0 q^{\binom{-\nu}{2}}\frac{(T-\omega_0)^{-\nu}}{\tilde{\Gamma}_q(1-\nu)}. \tag{28}$$

Denote that

$$\begin{aligned}\mathcal{F}|u-v|(t) &= \left|F[t,u(t),(\tilde{\Psi}^\gamma_{q,\omega}u)(t)]-F[t,v(t),(\tilde{\Psi}^\gamma_{q,\omega}v)(t)]\right|\\ \mathcal{H}|u-v|(t) &= \left|H[t,u(t),(\tilde{\Upsilon}^\nu_{q,\omega}u)(t)]-H[t,v(t),(\tilde{\Upsilon}^\nu_{q,\omega}v)(t)]\right|,\end{aligned}$$

we obtain

$$\begin{aligned}&\left|\boldsymbol{O}^*_i[\phi_i,F_u+H_u]-\boldsymbol{O}^*_i[\phi_i,F_v+H_v]\right|\\ \le\ & |\phi_i(u)-\phi_i(v)| + \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_i)}\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma^{-\theta_i-1}_{q,\omega}(y)}\int_{\omega_0}^{\sigma^{\beta-1}_{q,\omega}(x)}\widetilde{(\eta_i-y)}^{\underline{-\theta_i-1}}_{q,\omega}\times\\ &\left(\sigma^{-\theta_i-1}_{q,\omega}(y)-x\right)^{\underline{\beta-1}}_{q,\omega}\left(\sigma^{\beta-1}_{q,\omega}(x)-s\right)^{\underline{\alpha-1}}_{q,\omega}g_i\left(\sigma^{-\theta_i-1}_{q,\omega}(y)\right)\times\\ &\left[\lambda\mathcal{F}|u-v|\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)+\mu\mathcal{H}|u-v|\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\right]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y\\ \le\ & \omega_i\|u-v\|_C\\ &+\left(\lambda\left[M_1|u-v|+M_2|\tilde{\Psi}^\gamma_{q,\omega}u-\tilde{\Psi}^\gamma_{q,\omega}v|\right]+\mu\left[N_1|u-v|+N_2|\tilde{\Upsilon}^\nu_{q,\omega}u-\tilde{\Upsilon}^\nu_{q,\omega}v|\right]\right)\times\\ &\frac{G_i q^{\binom{\alpha+\beta-\theta_i}{2}}(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_i+1)}\\ \le\ & \left(\omega_i+\mathcal{L}\left(\lambda\left[M_1+M_2\varphi_0 q^{\binom{\gamma}{2}}\frac{(T-\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}\right]\mu\left[N_1+N_2\Psi_0 q^{\binom{-\nu}{2}}\frac{(T-\omega_0)^{-\nu}}{\tilde{\Gamma}_q(1-\nu)}\right]\right)\right)\times\end{aligned}$$

$$
\begin{aligned}
&\frac{G_i q^{\binom{\alpha+\beta-\theta_i}{2}}(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_i+1)}\|u-v\|_{\mathcal{C}} \\
=\;& (\omega_i+\mathcal{L})\frac{G_i q^{\binom{\alpha+\beta-\theta_i}{2}}(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_i+1)}\|u-v\|_{\mathcal{C}},
\end{aligned} \tag{29}
$$

for $i=1,2$. Therefore,

$$
\begin{aligned}
&\Big|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\Big| \\
\leq\;& \frac{(T-\omega_0)^{\beta-1}}{|\Omega|}\Big\{|\boldsymbol{A}_2|\Big|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]-\boldsymbol{O}_1^*[\phi_1,F_v+H_v]+|\boldsymbol{A}_1|\Big|\boldsymbol{O}_2^*[\phi_2,F_u+H_u] \\
&-\boldsymbol{O}_2^*[\phi_2,F_v+H_v]\Big|\Big\}+\frac{q^{\binom{\beta}{2}}}{|\Omega|\tilde{\Gamma}_q(\beta)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\beta-1}\Big(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\Big)^{\alpha-1}\tilde{d}_{q,\omega}s\times \\
&\Big\{|\boldsymbol{B}_2|\Big|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]-\boldsymbol{O}_1^*[\phi_1,F_v+H_v]+|\boldsymbol{B}_1|\Big|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]-\boldsymbol{O}_2^*[\phi_2,F_v+H_v]\Big|\Big\} \\
&+\frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\tilde{\Gamma}_q(\alpha)+\tilde{\Gamma}_q(\beta)}\int_{\omega_0}^{T}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\widetilde{(T-x)}_{q,\omega}^{\underline{\beta-1}}\Big(\sigma_{q,\omega}^{\beta-1}(x)-s\Big)_{q,\omega}^{\underline{\alpha-1}}\times \\
&\Big[\lambda\mathcal{F}|u-v|(\sigma_{q,\omega}^{\alpha-1}(s))+\mu\mathcal{H}|u-v|(\sigma_{q,\omega}^{\alpha-1}(s))\Big]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x \\
\leq\;& \Bigg\{\mathcal{L}q^{\binom{\alpha}{2}+\binom{\beta}{2}+\alpha\beta}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}_q(\alpha+\beta-1)}+\frac{(T-\omega_0)^{\beta-1}}{|\Omega|}\Bigg[|\boldsymbol{A}_2|\Bigg(\omega_1+\mathcal{L}G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)}\Bigg) \\
&+|\boldsymbol{A}_1|\Bigg(\omega_2+\mathcal{L}G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\Bigg]+\frac{q^{\binom{\beta}{2}+(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)(T-\omega_0)^{\beta-1}}{|\Omega|\tilde{\Gamma}(\alpha+\beta)}\times \\
&\Bigg[|\boldsymbol{B}_2|\Bigg(\omega_1+\mathcal{L}G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)}\Bigg) \\
&+|\boldsymbol{B}_1|\Bigg(\omega_2+\mathcal{L}G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\Bigg]\Bigg\}\|u-v\|_C \\
=\;& \Bigg\{\mathcal{L}\Bigg[q^{\binom{\alpha+\beta}{2}}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}_q(\alpha+\beta+1)}+\frac{(T-\omega_0)^{\beta-1}}{|\Omega|}\Bigg(|\boldsymbol{A}_2|G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)} \\
&+|\boldsymbol{A}_1|G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\Bigg]+\Bigg[\frac{q^{\binom{\beta}{2}+(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)(T-\omega_0)^{\alpha+\beta-1}}{|\Omega|\tilde{\Gamma}_q(\alpha+\beta)}\times \\
&\Bigg(|\boldsymbol{B}_2|G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)}+|\boldsymbol{B}_1|G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\Bigg] \\
&+\frac{(t-\omega_0)^{\beta-1}}{|\Omega|}(\omega_1|\boldsymbol{A}_2|+\omega_2|\boldsymbol{A}_1|)+\frac{q^{\binom{\beta}{2}+(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)(T-\omega_0)^{\alpha+\beta-1}}{|\Omega|\tilde{\Gamma}_q(\alpha+\beta)}(\omega_1|\boldsymbol{B}_2|+\omega_2|\boldsymbol{B}_1|)\Bigg\}\times \\
&\|u-v\|_{\mathcal{C}} \\
=\;& \Bigg\{\mathcal{L}\Bigg[q^{\binom{\alpha+\beta}{2}}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}_q(\alpha+\beta+1)}+G_1\Phi_1\Theta_2+G_2\Phi_2\Theta_1\Bigg]+\omega_1\Theta_2+\omega_2\theta_1\Bigg\}\|u-v\|_{\mathcal{C}}.
\end{aligned} \tag{30}
$$

Take fractional Hahn difference of order ν for (16). Then, we have

$$
\begin{aligned}
&(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}u)(t) \\
=\;& [\boldsymbol{A}_2\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+\boldsymbol{A}_1\boldsymbol{O}_2^*[\phi_2,F_v+H_v]]\frac{q^{\binom{-\nu}{2}}}{\Omega\tilde{\Gamma}_q(-\nu)}\times
\end{aligned}
$$

$$\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\nu-1}}\Big(\sigma_{q,\omega}^{-\nu-1}(s)-\omega_0\Big)^{\beta-1}\tilde{d}_{q,\omega}s$$
$$-\big[\boldsymbol{B}_2\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+\boldsymbol{B}_1\boldsymbol{O}_2^*[\phi_2,F_v+H_v]\big]\frac{q^{\binom{\beta}{2}+\binom{-\nu}{2}}}{\Omega\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\nu)}\times$$
$$\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\nu-1}}\Big(\sigma_{q,\omega}^{-\nu-1}(x)-s\Big)_{q,\omega}^{\underline{\beta-1}}\Big(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\Big)^{\alpha-1}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x$$
$$+\frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\nu}{2}}}{\tilde{d}_q(\alpha)\tilde{d}_q(\beta)\tilde{d}_q(-\nu)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\widetilde{(t-y)}_{q,\omega}^{\underline{-\nu-1}}\Big(\sigma_{q,\omega}^{-\nu-1}(y)-x\Big)_{q,\omega}^{\underline{\beta-1}}\times$$
$$\Big(\sigma_{q,\omega}^{\beta-1}(x)-s\Big)_{q,\omega}^{\underline{\alpha-1}}\Big\{\lambda F[\sigma_{q,\omega}^{\alpha-1}(s),u(\sigma_{q,\omega}^{\alpha-1}),(\tilde{\Psi}_{q,\omega}^{\gamma}u)(\sigma_{q,\omega}^{\alpha-1})]$$
$$+\mu H[\sigma_{q,\omega}^{\alpha-1}(s),u(\sigma_{q,\omega}^{\alpha-1}),(\tilde{\Upsilon}_{q,\omega}^{\nu}u)(\sigma_{q,\omega}^{\alpha-1})]\Big\}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y. \tag{31}$$

Similary,

$$\begin{aligned}
&\big|(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}u)(t)-(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}v)(t)\big|\\
<\;&\Big\{|\boldsymbol{A}_2||\boldsymbol{O}_1^*[\phi_1,F_u+H_u]-\boldsymbol{O}_1^*[\phi_1,F_v+H_v]|+|\boldsymbol{A}_1||\boldsymbol{O}_2^*[\phi_2,F_u+H_u]-\boldsymbol{O}_2^*[\phi_2,F_v+H_v]|\Big\}\times\\
&\frac{q^{\binom{-\nu}{2}}}{|\Omega|\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{-\nu-1}}\Big(\sigma_{q,\omega}^{-\nu-1}(s)-\omega_0\Big)^{\beta-1}\tilde{d}_{q,\omega}s\\
&+\Big\{|\boldsymbol{B}_2||\boldsymbol{O}_1^*[\phi_1,F_u+H_u]-\boldsymbol{O}_1^*[\phi_1,F_v+H_v]|+|\boldsymbol{B}_1||\boldsymbol{O}_2^*[\phi_2,F_u+H_u]-\boldsymbol{O}_2^*[\phi_2,F_v+H_v]|\Big\}\times\\
&\frac{q^{\binom{\beta}{2}+\binom{-\nu}{2}}}{|\Omega|\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{T}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-1}(x)}\widetilde{(T-x)}_{q,\omega}^{\underline{-\nu-1}}\Big(\widetilde{\sigma_{q,\omega}^{-\nu-1}(x)}-s\Big)_{q,\omega}^{\underline{\beta-1}}\Big(\sigma_{q,\omega}^{\beta-1}(s)-\omega_0\Big)^{\alpha-1}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\\
&+\frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\nu}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{T}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-1}(y)}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\widetilde{(T-y)}_{q,\omega}^{\underline{-\nu-1}}\Big(\widetilde{\sigma_{q,\omega}^{-\nu-1}(y)}-x\Big)_{q,\omega}^{\underline{\beta-1}}\Big(\sigma_{q,\omega}^{\beta-1}(x)-s\Big)_{q,\omega}^{\underline{\alpha-1}}\\
&[\lambda\mathcal{F}|u-v|\Big(\sigma_{q,\omega}^{\alpha-1}(s)\Big)+\mu\mathcal{H}|u-v|\Big(\sigma_{q,\omega}^{\alpha-1}(s)\Big)]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y\\
\le\;&\Bigg\{\mathcal{L}\Bigg[q^{\binom{\alpha+\beta-\nu}{2}}\frac{(T-\omega_0)^{\alpha+\beta-\nu}}{\tilde{\Gamma}_q(\alpha+\beta-\nu+1)}+\frac{(T-\omega_0)^{\beta-\nu-1}}{|\Omega|}\frac{\tilde{\Gamma}_q(\beta)}{\tilde{\Gamma}_q(\beta-\nu)}q^{\binom{-\nu}{2}-\nu(\beta-1)}\times\\
&\Bigg(|\boldsymbol{A}_2|G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)}+|\boldsymbol{A}_1|G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\\
&+\frac{(T-\omega_0)^{\alpha+\beta-\nu-1}}{|\Omega|}\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha+\beta-\nu)}q^{\binom{\alpha+\beta-\nu-1}{2}-(\alpha-1)}\times\\
&\Bigg(|\boldsymbol{B}_2|G_1q^{\binom{\alpha+\beta-\theta_1}{2}}\frac{(\eta_1-\omega_0)^{\alpha+\beta-\theta_1}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_1+1)}+|\boldsymbol{B}_1|G_2q^{\binom{\alpha+\beta-\theta_2}{2}}\frac{(\eta_2-\omega_0)^{\alpha+\beta-\theta_2}}{\tilde{\Gamma}_q(\alpha+\beta-\theta_2+1)}\Bigg)\Bigg]\\
&+\frac{(T-\omega_0)^{\beta-\nu-1}}{|\Omega|}\frac{\tilde{\Gamma}_q(\beta)}{\tilde{\Gamma}_q(\beta-\nu)}q^{\binom{-\nu}{2}-\nu(\beta-1)}(\omega_1|\boldsymbol{A}_2|+\omega_2|\boldsymbol{A}_1|)+\frac{(T-\omega_0)^{\alpha+\beta-\nu-1}}{|\Omega|}\times\\
&\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha+\beta-\nu)}q^{\binom{\alpha+\beta-\nu-1}{2}-(\alpha-1)}(\omega_1|\boldsymbol{B}_2|+\omega_2|\boldsymbol{B}_1|)\Bigg\}\|u-v\|_{\mathcal{C}}\\
=&\Big\{\mathcal{L}\Big[q^{\binom{\alpha+\beta-\nu}{2}}\frac{(T-\omega_0)^{\alpha+\beta-\nu}}{\tilde{\Gamma}(\alpha+\beta-\nu+1)}+G_1\Phi_1\overline{\Theta}_2+G_2\Phi_2\overline{\Theta}_1\Big]+\omega_1\overline{\Theta}_2+\omega_2\overline{\Theta}_1\Big\}\|u-v\|_{\mathcal{C}}.
\end{aligned}\tag{32}$$

From (30) and (32), we find that

$$\|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\|_{\mathcal{C}}$$

$$
\begin{aligned}
&\leq \left\{\mathcal{L}\left[q^{\binom{\alpha+\beta}{2}}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}(\alpha+\beta+1)}+q^{\binom{\alpha+\beta-\nu}{2}}\frac{(T-\omega_0)^{\alpha+\beta-\nu}}{\tilde{\Gamma}()\alpha+\beta-\nu+1}+G_1\Phi_1(\Theta_2+\overline{\Theta}_2)+G_2\Phi_2(\Theta_1+\overline{\Theta}_1)\right]\right.\\
&\quad\left.+\,\omega_1(\Theta_2+\overline{\Theta}_2)+\omega_2(\Theta_1+\overline{\Theta}_1)\right\}\|u-v\|_{\mathcal{C}}\\
&=\ [\mathcal{L}\chi+\omega_1\Theta_2^*+\omega_2\Theta_1^*]\|u-v\|_{\mathcal{C}}\\
&=\ \Xi\,\|u-v\|_{\mathcal{C}}. \qquad (33)
\end{aligned}
$$

From (H_3), we can conclude that $\mathcal{A}$ is a contraction. Hence, from Banach fixed point theorem, $\mathcal{A}$ has a fixed point which is a unique solution of problem (1) on $I^T_{q,\omega}$. □

4. Existence of at Least One Solution of Problem (1)

In this section, we further consider the existence of at least one solution of (1) by using the Schauder's fixed point theorem as follows.

Theorem 2. *Suppose that (H_1),(H_2),(H_4) and (H_5) defined in Theorem 1 hold. Then, problem (1) has at least one solution on $I^T_{q,\omega}$.*

Proof. We split the proof into several steps.

Step I. Verify $\mathcal{A}$ map bounded sets into bounded sets in B_R. Let $B_R := \{u \in C(I^T_{q,\omega}) : \|u\|_{\mathcal{C}} \leq R\}$, $\max\limits_{t\in I^T_{q,\omega}}|F(t,0,0)| = F$, $\max\limits_{t\in I^T_{q,\omega}}|H(t,0,0)| = H$, $\sup\limits_{u\in C}|\phi_i(u)| = P_i$ and choose a constant

$$
R \geq \frac{[\lambda F+\mu H]\chi+P_1\Theta_2^*+P_2\Theta_1^*}{1-\mathcal{L}\chi}. \qquad (34)
$$

Denote that

$$
\big|\mathcal{F}(t,u,0)\big| = \Big|F[t,u(t),(\tilde{\Psi}^{\gamma}_{q,\omega}u)(t)]-F[t,0,0]\Big|+\big|F[t,0,0]\big|,
$$

$$
\big|\mathcal{H}(t,u,0)\big| = \Big|H[t,u(t),(\tilde{\Upsilon}^{\nu}_{q,\omega}u)(t)]-H[t,0,0]\Big|+\big|H[t,0,0]\big|.
$$

Consider that

$$
\begin{aligned}
&\Big|\boldsymbol{O}_i^*[\phi_i,F_u+H_u]\Big|\\
&\leq\ P_i+\frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)\tilde{\Gamma}_q(-\theta_i)}\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma^{-\theta_i-1}_{q,\omega}(y)}\int_{\omega_0}^{\sigma^{\beta-1}_{q,\omega}(x)}\widetilde{(\eta_i-y)}^{\underline{-\theta_i-1}}_{q,\omega}\left(\widetilde{\sigma^{-\theta_i-1}_{q,\omega}(y)-x}\right)^{\underline{\beta-1}}_{q,\omega}\times\\
&\quad\left(\widetilde{\sigma^{\beta-1}_{q,\omega}(x)-s}\right)^{\underline{\alpha-1}}_{q,\omega}g_i\left(\sigma^{-\theta_i-1}_{q,\omega}(y)\right)\left[\lambda\left|\mathcal{F}\left(\sigma^{\alpha-1}_{q,\omega}(s),u,0\right)\right|+\mu\left|\mathcal{H}\left(\sigma^{\alpha-1}_{q,\omega}(s),u,0\right)\right|\right]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y\\
&\leq\ P_i+G_i\Big[\lambda\big(M_1|u|+M_2|\Psi^{\gamma}_{q,\omega}u|+F\big)+\mu\Big(N_1|u|+N_2|\Upsilon^{\nu}_{q,\omega}u|+H\Big)\Big]\frac{q^{\binom{\alpha+\beta-\theta_i}{2}}(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}(\alpha+\beta-\theta_i+1)}\\
&\leq\ P_i+G_i\Big[\lambda F+\mu H+\mathcal{L}\|u\|_C\Big]\frac{q^{\binom{\alpha+\beta-\theta_i}{2}}(\eta_i-\omega_0)^{\alpha+\beta-\theta_i}}{\tilde{\Gamma}(\alpha+\beta-\theta_i+1)}, \qquad (35)
\end{aligned}
$$

where $i = i, 2$, and

$$
\begin{aligned}
\big|(\mathcal{A}u)(t)\big| \leq\ &\frac{(T-\omega_0)^{\beta-1}}{|\Omega|}\Big\{|\boldsymbol{A}_2|\big|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]\big|+|\boldsymbol{A}_1|\big|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\big|\Big\}\\
&+\frac{q^{\binom{\beta}{2}}}{|\Omega|\tilde{\Gamma}_q(\beta)}\int_{\omega_0}^{T}\widetilde{(T-s)}^{\underline{\beta-1}}_{q,\omega}\left(\sigma^{\beta-1}_{q,\omega}(s)-\omega_0\right)^{\alpha-1}d_{q,\omega}s\times\\
&\Big\{|\boldsymbol{B}_2|\big|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]\big|+|\boldsymbol{B}_1|\big|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\big|\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\beta)}\int_{\omega_0}^{T}\int_{\omega_0}^{\sigma_{q,\omega}^{\beta-1}(x)}\widetilde{(T-x)}_{q,\omega}^{\underline{\beta-1}}\left(\widetilde{\sigma_{q,\omega}^{\beta-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\times\\
&\quad\left[\lambda\left|\mathcal{F}\left(\sigma_{q,\omega}^{\alpha-1}(s),u,0\right)\right|+\mu\left|\mathcal{H}\left(\sigma_{q,\omega}^{\alpha-1}(s),u,0\right)\right|\right]\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\tilde{d}_{q,\omega}y\\
&\leq\ [\lambda F+\mu H+\mathcal{L}\|u\|_C]\left[q^{\binom{\alpha+\beta}{2}}\frac{(T-\omega_0)^{\alpha+\beta}}{\tilde{\Gamma}(\alpha+\beta+1)}+G_1\Phi_1\Theta_2+G_2\Phi_2\Theta_1\right]+P_1\Theta_2+P_2\Theta_1.
\end{aligned}
\tag{36}
$$

Next,

$$
\begin{aligned}
\left|\left(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}u\right)(t)\right|\leq\ &[\lambda F+\mu H+\mathcal{L}\|u\|_C]\left[q^{\binom{\alpha+\beta-\nu}{2}}\frac{(T-\omega_0)^{\alpha+\beta-\nu}}{\tilde{\Gamma}_q(\alpha+\beta-\nu+1)}+G_1\Phi_1\overline{\Theta}_2+G_2\Phi_2\overline{\Theta}_1\right]\\
&+P_1\overline{\Theta}_2+P_2\overline{\Theta}_1.
\end{aligned}
\tag{37}
$$

From (36) and (37), we have

$$
\begin{aligned}
\|(\mathcal{A}u)(t)\|\ &\leq\ [\lambda F+\mu H+\mathcal{L}\|u\|_C]\chi+P_1\Theta_2^*+P_2\Theta_1^*\\
&\leq\ R.
\end{aligned}
\tag{38}
$$

Step II. Since $\mathcal{F}$ and $\mathcal{H}$ are continuous, the operator $\mathcal{A}$ is the continuous on B_R.
Step III. Examine that $\mathcal{A}$ is equicontinuous on B_R.
For any $t_1,t_2\in I_{q,\omega}^T$ with $t_2>t_1$, we obtain

$$
\begin{aligned}
&\left|(\mathcal{A}u)(t_2)-(\mathcal{A}u)(t_1)\right|\\
&\leq\ \left|\lambda\|F\|+\mu\|H\|\right|\frac{q^{\binom{\alpha+\beta}{2}}}{\tilde{\Gamma}_q(\alpha+\beta+1)}\left|(t_2-\omega_0)^{\alpha+\beta}-(t_1-\omega_0)^{\alpha+\beta}\right|\\
&\quad+\frac{\left|(t_2-\omega_0)^{\beta-1}-(t_1-\omega_0)^{\beta-1}\right|}{|\Omega|}\left\{|\boldsymbol{A}_2|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+|\boldsymbol{A}_1|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\right\}\\
&\quad+\frac{q^{\binom{\beta}{2}-(\alpha-1)\beta}\tilde{\Gamma}_q(\alpha)}{|\Omega|\tilde{\Gamma}_q(\alpha+\beta)}\left|(t_2-\omega_0)^{\alpha+\beta-1}-(t_1-\omega_0)^{\alpha+\beta-1}\right|\times\\
&\quad\left\{|\boldsymbol{B}_2|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+|\boldsymbol{B}_1|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\right\}
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
&\left|\left(\tilde{D}_{q,\omega}^{\nu}u\right)(t_2)-\left(\tilde{D}_{q,\omega}^{\nu}u\right)(t_1)\right|\\
&\leq\left|\lambda\|F\|+\mu\|H\|\right|\frac{q^{\binom{\alpha+\beta-\nu}{2}}}{\tilde{\Gamma}_q(\alpha+\beta-\nu+1)}\left|(t_2-\omega_0)^{\alpha+\beta-\nu}-(t_1-\omega_0)^{\alpha+\beta-\nu}\right|\\
&\quad+\frac{q^{\binom{-\nu}{2}-\nu(\beta-1)}\tilde{\Gamma}_q(\beta)}{|\Omega|\tilde{\Gamma}_q(\beta-\nu)}\left|(t_2-\omega_0)^{\beta-1}-(t_1-\omega_0)^{\beta-1}\right|\times\\
&\quad\left\{|\boldsymbol{A}_2|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+|\boldsymbol{A}_1|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\right\}\\
&\quad+\frac{q^{\binom{\alpha+\beta-\nu-1}{2}-(\alpha-1)}\tilde{\Gamma}_q(\alpha)}{|\Omega|\tilde{\Gamma}_q(\alpha+\beta-\nu)}\left|(t_2-\omega_0)^{\alpha+\beta-\nu-1}-(t_1-\omega_0)^{\alpha+\beta-\nu-1}\right|\times\\
&\quad\left\{|\boldsymbol{B}_2|\boldsymbol{O}_1^*[\phi_1,F_u+H_u]+|\boldsymbol{B}_1|\boldsymbol{O}_2^*[\phi_2,F_u+H_u]\right\}.
\end{aligned}
\tag{40}
$$

We find that the right-hand side of (40) tends to be zero when $|t_2-t_1|\to 0$. Hence, $\mathcal{A}$ is relatively compact on B_R. Therefore, the set $\mathcal{A}(B_R)$ is an equicontinuous set. From Steps I to III and the Arzelá-Ascoli theorem, $\mathcal{A}:\mathcal{C}\to\mathcal{C}$ is completely continuous. By Schauder fixed point theorem, our problem (1) has at least one solution. $\square$

5. Example

In this section, we provide an example to show our results. We let $\alpha = \frac{3}{4}$, $\beta = \frac{1}{3}, \gamma = \frac{1}{2}, \nu = \frac{1}{4}$, $\theta_1 = \frac{1}{3}$, $\theta_2 = \frac{2}{3}$, $q = \frac{1}{2}$, $\omega = \frac{2}{3}$, $\omega_0 = \frac{4}{3}$, $T = 10$, $\lambda = e^{-4}$, $\mu = e^{-\pi}$, $\eta_1 = \sigma^4_{\frac{1}{2},\frac{2}{3}}(10) = \frac{15}{8}$, $\eta_2 = \sigma^5_{\frac{1}{2},\frac{2}{3}}(10) = \frac{77}{48}$, $g_1(t) = (\pi + \sin t)^2$, $g_2(t) = (e + \cos t)^2$, $F\left[t, u(t), (\tilde{\Psi}^{\gamma}_{q,\omega}u)(t)\right] = \frac{e^{-\left[\sin^2(2\pi t)\right]}}{100+e^{\cos^2(2\pi t)}} \cdot \frac{|u(t)|+e^{-\pi}\left|\left(\tilde{\Psi}^{\frac{1}{2}}_{\frac{1}{2},\frac{2}{3}}u\right)(t)\right|}{1+|u(t)|}$ and $H\left[t, u(t), \left(\tilde{\Upsilon}^{\nu}_{q,\omega}u\right)(t)\right] = \frac{e^{-\left[\cos^2(2\pi t+\pi)\right]}}{(t+\pi)^3} \cdot \frac{|u(t)|+e^{-\frac{\pi}{2}}\left|\left(\tilde{\Upsilon}^{\frac{1}{4}}_{\frac{1}{2},\frac{2}{3}}u\right)(t)\right|}{1+|u(t)|}$ which are satisfied with the conditions of the problem (1). Therefore the problem (1) is represented by

$$
\begin{aligned}
\tilde{D}^{\frac{3}{4}}_{\frac{1}{2},\frac{2}{3}}\tilde{D}^{\frac{1}{3}}_{\frac{1}{2},\frac{2}{3}}u(t) &= \frac{e^{-\left[\sin^2(2\pi t)+4\right]}}{100+e^{\cos^2(2\pi t)}} \cdot \frac{|u(t)|+e^{-\pi}\left|\tilde{\Psi}^{\frac{1}{2}}_{\frac{1}{2},\frac{2}{3}}u(t)\right|}{1+|u(t)|} \\
&\quad + \frac{e^{-\left[\cos^2(2\pi t+\pi)\right]}}{(t+\pi)^3} \cdot \frac{|u(t)|+e^{-\frac{\pi}{2}}\left|\tilde{\Upsilon}^{\frac{1}{4}}_{\frac{1}{2},\frac{2}{3}}u(t)\right|}{1+|u(t)|}, \ t \in I^{10}_{\frac{1}{2},\frac{2}{3}} \\
\tilde{D}^{\frac{1}{3}}_{\frac{1}{2},\frac{2}{3}}\left(\pi + \sin\left(\frac{15}{8}\right)\right)^2 u\left(\frac{15}{8}\right) &= \frac{|u(t_i)|}{1000e^2}\sin^2|\pi u(t_i)|, \ t_i \in \sigma^i_{\frac{1}{2},\frac{2}{3}}(10) \\
\tilde{D}^{\frac{2}{3}}_{\frac{1}{2},\frac{2}{3}}\left(e + \cos\left(\frac{77}{48}\right)\right)^2 u\left(\frac{77}{48}\right) &= \frac{|u(t_i)|}{1000\pi^2}\cos^2|\pi u(t_i)|, \ t_i \in \sigma^i_{\frac{1}{2},\frac{2}{3}}(10) \qquad (41)
\end{aligned}
$$

where $\varphi(t,s) = \frac{e^{-|t-s|}}{(t+e)^3}$ and $\psi(t,s) = \frac{e^{-|t-s|}}{(t+\pi)^3}$.

To investigate the values of $M_1, M_2, N_1, N_2, \omega_1, \omega_2, g_1, g_2, G_1, G_2, \mathcal{L}, \Phi_1, \Phi_2, \Theta_1, \Theta_2, \Theta_1^*$, Θ_2^*, χ and Ξ, we employ the assumptions (H_1)–(H_5) to get the results as follows.

For all $t \in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $u,\ v \in \mathbb{R}$, we find that

$$
\begin{aligned}
\left|F[t,u,(\tilde{\Psi}^{\gamma}_{q,\omega}u)] - F[t,v,(\tilde{\Psi}^{\gamma}_{q,\omega}u)]\right| &\le \frac{1}{101}|u-v| + \frac{1}{101e\pi}\left|\tilde{\Psi}^{\gamma}_{q,\omega}u - \tilde{\Psi}^{\gamma}_{q,\omega}v\right|, \\
\left|H[t,u,(\tilde{\Upsilon}^{\nu}_{q,\omega}u)] - H[t,v,(\tilde{\Upsilon}^{\nu}_{q,\omega}v)]\right| &\le \frac{1}{\left(\frac{4}{3}+\pi\right)^3}|u-v| + \frac{1}{\left(\frac{4}{3}+\pi\right)^3 e^{\frac{\pi}{2}}}\left|\tilde{\Psi}^{\nu}_{q,\omega}u - \tilde{\Psi}^{\nu}_{q,\omega}v\right|.
\end{aligned}
$$

Thus, (H_1) and (H_2) hold with $M_1 = 0.0099, M_2 = 0.000428, N_1 = 0.01116$ and $N_2 = 0.00232$.

For all $u,\ v \in \mathcal{C}$, we obtain

$$
|\phi_1(u) - \phi_1(v)|| \le \frac{1}{1000e^2}||u-v||_{\mathsf{C}}, \quad |\phi_2(u) - \phi_2(v)|| \le \frac{1}{1000\pi^2}||u-v||_{\mathsf{C}}.
$$

Then, (H_3) holds with $\omega_1 = 0.0001353$, $\omega_2 = 0.0001013$.

The condition (H_4) holds with $g_1 = 4.5864$, $g_2 = 2.9525$, $G_1 = 17.1528$, $G_2 = 13.8256$.

We next find that

$$
\varphi_0 = 0.01504, \ \psi_0 = 0.01116, \ |A_1| \le 8.6795, \ |A_2| \le 1.83867,
$$

$$
|B_1| \le 41.5268, \ |B_2| \le 11.7567 \text{ and } |\Omega| \ge 1.4668,
$$

and,

$$
\mathcal{L} = 0.000665, \ \Phi_1 = 0.69789, \ \Phi_2 = 4.29227^{-13}, \ \Theta_1 = 45.7791, \ \Theta_2 = 12.8606,
$$

$$
\bar{\Theta}_1 = 16.4749, \ \bar{\Theta}_2 = 4.65093, \ \Theta_1^* = 62.2540, \ \Theta_2^* = 17.5115 \text{ and } \chi = 253.8093.
$$

So, (H_5) hold with $\Xi = 0.17746 < 1$.

Therefore, by Theorem 1 problem (41) has a unique solution.

6. Conclusions

We present the new problem involving five fractional symmetric Hahn difference operators and one fractional symmetric Hahn integral of different orders where the new concepts of fractional symmetric Hanh calculus were used. By using the Schauder and Banach fixed point theorems we found conditions under which this problem, respectively, has a solution and has a unique solution. In addition, some properties of symmetric Hahn integral are also studied. The results of this article are new and enrich the field of boundary value problems for fractional symmetric Hahn integrodifference equations. In the future, we may expand this work by considering a new boundary value problem.

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