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Complex Dynamics of a Model with R&D Competition

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Abstract: The paper analyzes a two-stage oligopoly game of semi-collusion in production described by a system with a symmetric structure. We examine the local stability of a Nash equilibrium and the presence of bifurcations. We discover that the model is capable of exhibiting extremely complicated dynamic behaviors.

Keywords: hopf bifurcation; R&D; time delay

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1. Introduction

Research and development (R&D) are critical to a company's growth. They enable the development of lower manufacturing costs, increase production productivity, and improve the quality of goods. The existing theories of oligopolies, particularly within R&D, have raised concerns about the position of R&D spillover in bringing about process innovation. Kamien et al. [1], and D'Aspremont and Jacquemin [2] are the most significant contributors to this sector. D'Aspremont and Jacquemin [2] first developed a two-stage game in which companies simultaneously aimed to reduce their R&D expenditures and obtain research development spillovers from their rivals.

Firms compete in a duopoly game in which both stages, each one with a spillover impact from the other in the outcome of R&D operation. Then, Kamien et al. [1] presented four possible versions. In these models, both companies reduce the expense of R&D acquisitions by the use of Research Joint Ventures (RJVs) or R&D cartelization. Following that, they commit themselves to compete in the product market through a Bertrand or Cournot game. The four models acknowledge that spillovers have impacted study works to varying degrees during the research and development period.

Joint venture establishment, which is linked to commodity export rivalry, is the most ideal and attractive strategy since it results in higher revenues and lower product costs. Qiu [3] and Amir et al. [4] had a similar approach to their studies. Shibata [5] recently investigated R&D investment spillovers in a variety of business systems. Specifically, he expanded Matsumura et al.'s [6] attempts to integrate R&D investment spillovers.

The variety of oligopoly games, in recent years, has grown substantially. Cavalli and Naimzada [7] analyzed oligopoly models with various decision-making structures and rationality degrees. Dubiel-Teleszynski [8] investigated heterogeneous Cournot games with adaptive and bounded participants. Peng et al. [9] developed a duopoly Stackelberg rivalry model and explored the presence, steadiness and bifurcation of equilibrium points. Ding et al. [10] presented a dynamical Cournot game associated with rationality and time delay for marginal benefit maximization. Many complex dynamical behaviors were found.

Overall, single-stage games are analytically tractable but not useful for simulations. However, concerning technical commodities, several dynamic factors are critical. Currently, a two-stage oligopoly game has increased the attention of academics. Bischi and Lamantia [11,12] suggested a two-stage oligopoly game to model firms' R&D networks. Matsumura et al. [6] assumed a two-stage Cournot model in which companies decide on R&D spending first and then on production amounts. Shibata [5] analyzed the spillover effects of R&D spending through a variety of industry systems. He expands the function of Matsumura et al. [6] in particular by including R&D investment spillovers.

Throughout history, companies operating in oligopolistic economies have been strongly advised by traditional theory that collaboration is preferable to competition on quantity or price. However, Fershtman and Gandal's research [13] indicated that collusion in the commodity sector might lower the total income. This is due mainly to the rivalry in various aspects of contact.

Zhang et al. [14] built a dynamical two-stage duopoly game based on Fershtman and Gandal's research results, assuming that the business has a linear inverse demand function and the firms are bounded rational. They believed that companies compete in the R&D stage and allowed for spillover at this stage. Both companies engage in collusive behavior in the product sector. That is, each company determines its amounts by the aggregate income. Cao et al. [15] revisited Zhang et al.'s [14] model by extending the related discrete time to the case of continuous time with delay.

In this paper, by taking the model of Zhang et al. [14] as a starting point, we propose an alternative modeling approach to transform the discrete-time model into a continuous delayed time model. Through the study of stability of the stationary equilibrium point, we observe switches from stability to instability to stability and characterize the birth of Hopf bifurcations.

The plan of this paper is as follows. Section 2 introduces the model with R&D competition augmented time delay. Section 3 provides an analysis of the stability properties of the dynamic system. Section 4 outlines our conclusions.

2. The Model

In this section, we revisit the model by Zhang et al. [14], where, instead of discrete time scales, continuous time scales are assumed. The model consists of a repeated two-stage game of semi-collusion in production with two companies in an oligopoly market, where the cost and inverse demand functions are linear, starting with a repetitive two-stage game. In the first point, we presume that both companies compete in R&D, but that spillover is permitted. In the second level, we presume that companies organize their output in accordance with the shared benefit. The model that emerges is made up of a two-dimensional map that defines the time evolution of the dynamic game:

$$\begin{aligned} x_1(t+1) &= x_1(t) + v_1 x_1(t)[A_1 + Bx_1(t) + Cx_2(t)], \\ x_2(t+1) &= x_2(t) + v_2 x_2(t)[A_2 + Bx_2(t) + Cx_1(t)], \end{aligned} \tag{1}$$

with x_1 and x_2 the R&D efforts of firm 1 and 2, respectively, $A_1 > 0, A_2 > 0$. A symmetry of parameters v_1 and v_2 exists in this system. System (1) has three boundary equilibria

$$E_0 = (0,0), \quad E_1 = \left(-\frac{A_1}{B}, 0\right), \quad E_2 = \left(0, -\frac{A_2}{B}\right)$$

and a Nash equilibrium

$$E_* = \left(\frac{A_2C - A_1B}{B^2 - C^2}, \frac{A_1C - A_2B}{B^2 - C^2}\right),$$

which is stable, provided that $(A_1, A_2, B, C) \in S_1$ where

$$S_1 = \{(A_1, A_2, B, C) : A_1 > 0, A_2 > 0, B < 0, |B| > |C|, \Delta_1 > 0, \Delta_2 > 0\},$$

with $\Delta_1 = A_2C - A_1B$ and $\Delta_2 = A_1C - A_2B$. In order to have a continuous version of system (1), we notice (1) is expressed as $x_l(t+1) = \phi(x_l(t))$, $l = 1, 2$. Rewriting this as

$$[x_l(t+1) - x_l(t)] = -x_l(t) + \phi(x_l(t)),$$

assuming that the terms in brackets approximates $\partial x_l(t)/\partial t$ and the existence of a time delay between the awareness of the right-hand side expression and the knowledge of the left-hand side expression, we obtain the following time-delayed model:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_1(t-\tau) + v_1 x_1(t-\tau)[A_1 + Bx_1(t-\tau) + Cx_2(t-\tau)] \\ \dot{x}_2(t) &= -x_2(t) + x_2(t-\tau) + v_2 x_2(t-\tau)[A_2 + Bx_2(t-\tau) + Cx_1(t-\tau)] \end{aligned} \tag{2}$$

It is clear that the steady states of the delay model (2) are the same of (1). Since we are interested in considering the effect of delay τ on the system dynamics, for this purpose, we will analyze the linearized version of (2) about the equilibrium keeping in mind that the Nash equilibrium will be locally asymptotically stable if and only if each of the characteristic roots has negative real parts (see, e.g., [13]).

3. Stability and Existence of Bifurcations

The linearization of (2) around E_* has a characteristic equation expressed by

$$\begin{vmatrix} -1 - \lambda + \left(Mv_1 + Bv_1x_1^*\right)e^{-\lambda\tau} & Cv_1x_1^*e^{-\lambda\tau} \\ Cv_2x_2^*e^{-\lambda\tau} & -1 - \lambda + (Nv_2 + Bv_2x_2^*)e^{-\lambda\tau} - \lambda \end{vmatrix} = 0,$$

where $M = A_1 + Bx_1^* + Cx_2^* + Bv_1x_1^*$ and $N = A_2 + Bx_2^* + Cx_1^*$. Calculating the determinant gives

$$\lambda^2 + 2\lambda + 1 + (m + m\lambda)e^{-\lambda\tau} + ne^{-2\lambda\tau} = 0, \tag{3}$$

where

$$m = -2 - B(v_1x_1^* + v_2x_2^*), \qquad n = 1 + B(v_1x_1^* + v_2x_2^*) + \left(B^2 - C^2\right)v_1v_2x_1^*x_2^*.$$

We now employ the method proposed in Chen et al. [16] to analyze the distribution of characteristic roots. It is clear that the marginal stability is determined by the equations $\lambda = 0$ and $\lambda = i\omega$ $(\omega > 0)$. The case $\lambda = 0$ is simple. Substituting $\lambda = 0$ into Equation (3), one obtains $1 + m + n = \left(B^2 - C^2\right)v_1v_2x_1^*x_2^* \neq 0$. Therefore, the characteristic Equation (3) has a zero root. The case $\lambda = i\omega$ $(\omega > 0)$ is instead more complicated. Substituting $\lambda = i\omega$ into Equation (3) yields

$$-\omega^2 + 2i\omega + 1 + (mi\omega + m)e^{-i\omega\tau} + ne^{-2i\omega\tau} = 0. \tag{4}$$

If $(\omega\tau)/2 \neq (\pi/2) + j\pi$, $j \in \mathbb{Z}$, then let $\theta = \tan[(\omega\tau)/2]$, and we have

$$e^{-i\omega\tau} = \frac{1 - i\theta}{1 + i\theta}. \tag{5}$$

Separating the real and imaginary parts, we obtain that θ satisfies

$$\begin{aligned} (\omega^2 - 1 + m - n)\theta^2 - 4\omega\theta &= \omega^2 - 1 - m - n, \\ (m-2)\omega\theta^2 - 2(\omega^2 - 1 + n)\theta &= -(2+m)\omega. \end{aligned} \tag{6}$$

Define

$$D(\omega) = \begin{vmatrix} \omega^2 - 1 + m - n & -4\omega \\ (m-2)\omega & -2(\omega^2 - 1 + n) \end{vmatrix},$$

$$E(\omega) = \begin{vmatrix} \omega^2 - 1 - m - n & -4\omega \\ -(2+m)\omega & -2(\omega^2 + 1 - n) \end{vmatrix}$$

and

$$F(\omega) = \begin{vmatrix} \omega^2 - 1 + m - n & \omega^2 - 1 - m - n \\ (m-2)\omega & -(2+m)\omega \end{vmatrix}.$$

We find that ω satisfies

$$D(\omega)E(\omega) = [F(\omega)]^2,$$

which is a polynomial equation for ω with degree 8 :

$$p(\omega) = \omega^8 + s_3\omega^6 + s_2\omega^4 + s_1\omega^2 + s_0 = 0, \tag{7}$$

with

$$\begin{aligned} s_3 &= 4 - m^2, \qquad s_2 = 6 - 2n^2 + (2n-3)m^2, \\ s_1 &= 4 - 4n^2 + 3(n-1)m^2, \qquad s_0 = (1-n)^2[-m^2 + (1+n)^2]. \end{aligned}$$

Moreover, it is easy to see that ω^2 is a positive root of the following equation:

$$z^4 + s_3z^3 + s_2z^2 + s_1z + s_0 = 0, \tag{8}$$

If, instead $(\omega\tau)/2 = (\pi/2) + j\pi$, $j \in \mathbb{Z}$, then $D(\omega) = F(\omega) = 0$, and thus ω^2 is still a positive root of (8).

Lemma 1.

(1) *If $\pm i\omega$ $(\omega > 0)$ is a pair of purely imaginary roots of the characteristic equation, then ω^2 is a positive root of the above quartic polynomial equation.*

(2) *There exists at least one positive solution to (7) provided that $n \neq 1$ and $|1+n| < |m|$.*

Proof. The first statement follows from the previous analysis. Since the leading term of the polynomial (7) is ω^8, the polynomial $p(\omega)$ tends to infinity as $\omega \to \infty$. Moreover, $p(0) = s_0 < 0$. Therefore, there is some $\omega > 0$ such that $p(\omega) = 0$. $\square$

The previous condition is only sufficient for $p(\omega)$ to have positive solutions. Even if the parameters do not satisfy it, we can always compute all positive solutions of $p(\omega) = 0$ since $p(\omega)$ is an 8-degree polynomial. Additionally, there are no more than eight positive solutions.

Proposition 1. *If the quartic Equation (8) has a positive root ω_*^2, $\omega_* > 0$, and $D(\omega_*) \neq 0$, then Equation (6) has a unique real root $\theta_* = F(\omega_*)/D(\omega_*)$. Hence, Equation (4) has a pair of purely imaginary roots $\pm i\omega_*$ when*

$$\tau = \tau_*^j = \frac{2\tan^{-1}(\theta_*) + 2j\pi}{\omega_*},\ j \in \mathbb{Z}. \tag{9}$$

Proof. From (6), we find $\theta_*^2 = E(\omega_*)/D(\omega_*)$ and $\theta_* = F(\omega_*)/D(\omega_*)$. Thus, $D(\omega_*)E(\omega_*) = [F(\omega_*)]^2$. Consequently, (3) has a pair of purely imaginary roots $\pm i\omega_*$ when τ is defined as in (9). This completes the proof of the Proposition. □

The next result establishes that a curve of simple root $\lambda(\tau)$ occurs and travels transversally around the imaginary axis under some conditions of transversality.

Proposition 2. *$\lambda = i\omega_*$ $(\omega_* > 0)$ is a simple root of (3) at $\tau = \tau_*^j, j \in \mathbb{Z}$. The crossing direction through the imaginary axis is determined by the sign of*

$$G(\omega_*, \theta_*) = \Big\{\Big[(m-2n)\tau_*^j - m + 2\Big]\theta_*^2 + 4\omega_*\theta_* + (m+2n)\tau_*^j - m - 2\Big\}\Big[m\omega_*\theta_*^2 - 4n\theta_* + m\omega_*] - \Big[(m\tau_*^j + 2)\omega_*\theta_*^2 - 4(n\tau_*^j + 1)\theta_* + (m\tau_*^j - 2)\omega_*\Big]\Big[(m-2n)\theta_*^2 + m + 2n\Big].$$

Proof. First, we differentiate (3) with respect to τ and obtain

$$P(\lambda, \tau)\frac{d\lambda}{d\tau} = \lambda Q(\lambda, \tau), \tag{10}$$

where we denote

$$P(\lambda, \tau) = 2\lambda + 2 + me^{-\lambda\tau} - (m\lambda + m)\tau e^{-\lambda\tau} - 2n\tau e^{-2\lambda\tau},$$
$$Q(\lambda, \tau) = (m\lambda + m)e^{-\lambda\tau} + 2ne^{-2\lambda\tau}.$$

The fact that $\lambda = i\omega_*$ is a simple root follows from (10). We argue by contradiction. If not, then $Q(i\omega_*, \tau_*^j) = 0$ holds true. Using (5), this yields

$$mi\omega_* + m + 2n(1 - i\theta_*)(1 + i\theta_*) = 0.$$

Thus, $m\omega_* - 2n\theta_* = 0$ and $-m\omega_*\theta_* + 2n = 0$, which leads to $\pm 1 = \theta_* = \tan\Big[(\omega\tau_*^j)/2\Big]$, i.e., the absurd $(\omega\tau_*^j)/2 = (\pi/4) + j\pi$.

We are now left to verify the transversality condition. For simplicity, we consider $d\tau/d\lambda$ instead of $d\lambda/d\tau$. In light of (10), we can find

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = i\omega_*} = \frac{P(i\omega_*, \tau_*^j)}{i\omega_* Q(i\omega_*, \tau_*^j)} = -\frac{iP(i\omega_*, \tau_*^j)\overline{Q(i\omega_*, \tau_*^j)}}{\omega_* Q(i\omega_*, \tau_*^j)\overline{Q(i\omega_*, \tau_*^j)}}.$$

By straightforward calculations, we can find

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = i\omega_*} = \frac{G(\omega_*, \theta_*)}{(1 + \theta_*^2)^2\omega_*\Big|Q(i\omega_*, \tau_*^j)\Big|^2}.$$

Finally, we can conclude

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda = i\omega_*}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = i\omega_*}\right\} = sign\{G(\omega_*, \theta_*)\}.$$

Therefore, we complete our proof. □

Remark 1. *A positive sign of $G(\omega_*, \theta_*)$ means that each crossing of the real part of characteristic roots at τ_*^j must be from left to right, whereas a negative sign indicates that the real part of a pair of conjugate roots of Equation (3) changes from positive value to negative value when τ_*^j is crossed.*

Due to the discussions and facts above, we have the following conclusions.

Theorem 1.

(1) *If Equation (7) has no positive root, then the equilibrium E_* of system (2) is locally asymptotically stable for $\tau \geq 0$.*

(2) *If Equation (7) has a unique positive root ω_*, then there exists a $\tau_* > 0$, where $\tau_* = \min\{\tau_*^j, j \in \mathbb{Z}\}$, such that the equilibrium E_* of system (2) is locally asymptotically stable when $\tau \in [0, \tau_*)$. As τ increases, the system dynamic may switch from stable to unstable, a Hopf bifurcation occurs, and then back to stable, and so on, according to sign$[G(\omega_*, \theta_*)]$.*

(3) *If Equation (7) has at least two positive roots, then there may exist many stability switches, with the occurrence of a Hopf bifurcation at each switch.*

4. Concluding Remarks

In this work, we considered a continuous-time version with delays of a discrete model with R&D competition. Our modification of the discrete model was suggested by real economic situations where there are always time delays between the times when information is obtained and when the decisions are implemented. Delay differential equations were used to model the resulting system.

Employing results on the distribution of the zeros of transcendental functions, we found a set of conditions to determine the stability of the Nash equilibrium point and the existence of Hopf bifurcations. More precisely, as the delay parameter increased, stability loss and gain emerged (which could repeat alternatively), and complicated, chaotic behavior dynamics emerged.

This scenario did not occur in the discrete version model. Chaotic behavior is undesirable in an economic system since chaos means market confusion or irregularity. When the market runs irregularly, product prices, firm earnings, and R&D spending all suffer. Therefore, all market players expect a stable market and hope to bring the market back to equilibrium when it is unbalanced. Hence, time delay has the important dual role of destabilizer and stabilizer.

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