

Symmetry Problems for PDE

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Abstract: The results of this paper allow one to derive several results of general interest: to prove the Schiffer's conjecture, to solve the Pompeiu problem, to prove two symmetry results in harmonic analysis and to give a new method for solving an old symmetry problem.

Keywords: symmetry problems

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1. Introduction

Let D be a bounded connected domain in $\mathbb{R}^m$, $m \geq 2$, with a smooth boundary S , N is the outer unit normal to S , $k > 0$ is a constant, $u_N := u_N|_S$ is the limiting value of the normal derivative of u on S from D . Denote by $|D|$ the volume of D , by $|S|$ the area of S , and by $J_p(r)$ the Bessel function regular at $r = 0$. Various symmetry problems were considered by the author in [1–6].

Consider the problem

$$\nabla^2 u + k^2 u = c_0 \text{ in } D, \quad u|_S = c_1, \quad u_N = c_2, \quad c_j = \text{const}, \quad j = 0, 1, 2, \quad k = \text{const} > 0. \quad (1)$$

This problem is over-determined and, in general, does not have a solution. Suppose it has a solution. Our first result is the following theorem:

Theorem 1. *If $|C_1| + |c_2| > 0$, $C_1 := c_1 - \frac{c_0}{k^2}$, and problem (1) is solvable, then D is a ball.*

Special cases of this result were proved earlier by the author:

(a) $c_0 = 0$, $c_1 = 1$, $c_2 = 0$.

This result contains the solution to the Pompeiu problem, see [1,3,4],

(b) $c_0 = 0$, $c_1 = 0$, $c_2 = 1$.

This result contains the Schiffer's conjecture, see [5].

(c) two symmetry problems in harmonic analysis are reduced to Theorem 1, see [6]; these results are formulated in Theorem A in this paper in Section 4.

(d) $c_0 = 1$, $k = 0$, $c_1 = 0$, $c_2 = \frac{|D|}{|S|}$, where $|D|$ is the volume of D and $|S|$ is the area of S .

By different methods, this problem was treated in [2], and in [7].

The proofs in this paper are shorter than in the author's earlier works. This paper contains some new ideas.

This paper is organized as follows: In Section 2, reduction of problem (1) is given to the case when $c_0 = 0$. In Section 3, a proof of Theorem 1 is given. In Section 4, a proof of symmetry properties in harmonic analysis is given, see Theorem A. In Section 5, a proof of the Schiffer's conjecture is given, See Theorem B. In Section 6, the Pompeiu problem is solved, see Theorem C. In Section 7, the set of possible values of the radii of the balls in the Pompeiu problem and in the Schiffer's conjecture are given. In Section 8, a symmetry problem for the Laplace equation is considered, see Theorem 2. In Section 9, conclusions are briefly stated.



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2. Reduction of Problem (1) to the Case $c_0 = 0$

Define $v := u + C$, $C = \text{const.}$ Then,

$$\nabla^2 v + k^2 v = c_0 + Ck^2; \quad v|_S = c_1 + C; \quad v_N|_S = c_2. \quad (2)$$

Choose $C = -\frac{c_0}{k^2}$. Then, v solves the problem

$$\nabla^2 v + k^2 v = 0 \quad \text{in } D, \quad v|_S = c_1 - \frac{c_0}{k^2} := C_1, \quad v_N|_S = c_2. \quad (3)$$

Let $L \subset S$ be the boundary curve of a normal section P of D , that is, the plane passing through the normal $N = N_{s_0}$ to S at a point s_0 .

3. Proof of Theorem 1

Let us start with $m = 3$. The proof is the same for $m > 3$. Let P_{j3} be the normal sections of D along the principal directions $j = 1, 2$. These directions are orthogonal; they correspond to the sections for which the curvatures at s_0 are minimal and maximal for $m = 3$. The section P_{j3} is the plane $x_j x_3$ passing through the normal N to S at the origin s_0 . The x_3 axis is directed along N . Let L_1 be the boundary of the intersection of P_{13} with S . The directions x_j , $1 \leq j \leq 2$, are the principal directions, so that the curvature k_j of the normal section P_{j3} at the point s_0 is $k_j(s_0)$. If S is a smooth convex surface, then the Gauss curvature of the surface S at a point s_0 is $k(s_0) := k_1(s_0)k_2(s_0)$ and the mean curvature $\kappa := \frac{k_1 + k_2}{2}$ for $m = 3$. A point $s_0 \in S$ is called *umbilic* if all the normal sections have the same curvatures at this point. The surface S is locally spherical at an umbilic point.

The following results, formulated in Lemma 1, are known, see, for example, [8,9].

Lemma 1. *If the Gauss curvature of a smooth closed surface $S \subset \mathbb{R}^m$ is a positive constant, then S is a sphere. If the mean curvature of a closed smooth surface $S \subset \mathbb{R}^m$ is a positive constant, then S is a sphere. If all the points of a smooth closed surface $S \subset \mathbb{R}^m$ are umbilic, then S is a sphere.*

Consider now the case $m = 2$. Let s be the curve length of L_1 , $\mathbf{s}$ be the point on L_1 , corresponding to the parameter s , $x(s) := x_1(s)$, $y(s) := x_3(s)$, be the parametric representation of L_1 , $\mathbf{s} = x(s)e_1 + y(s)e_2$, where $\{e_1, e_2\}$ is a Cartesian basis in P_{13} , e_1 is directed along x_1 -axis, e_2 is directed along x_3 -axis, and $s_0 := \mathbf{s}(s_0)$ plays the role of the origin. It is known that the vector $\mathbf{t}(s) = \frac{d\mathbf{s}}{ds}$ is the unit vector, tangential to L_1 at the point $\mathbf{s}$, and

$$\frac{dt}{ds} = k_1(s)v(s), \quad (4)$$

where $k_1(s) \geq 0$ is the curvature of L_1 and $v(s)$ is the unit normal to L . Because P_{13} is the normal section, one has $N = -v$ if S is convex. Since $u_N = \nabla u \cdot N = c_2 = \text{const}$ on S , the convexity of S does not change sign, so $v = v(s)$ does not change sign. We have

$$k_1(s) > 0, \quad N(s) = -v, \quad (5)$$

on L_1 . If problem (1) has a solution, then problem (3) has a solution and vice versa.

We prove that, if $|C_1| + |c_2| > 0$ and (3) has a solution, then D is a ball. This is equivalent to Theorem 1.

Remark 1. *If $C_1 = c_2 = 0$, then $v = 0$ in D regardless of the shape of D . Therefore, the condition $|C_1| + |c_2| > 0$ is necessary for the solvability of problem (1) to imply that D is a ball. Theorem 1 shows that this condition and the solvability of problem (1) are sufficient for D to be a ball.*

Let us continue with the proof. Differentiate the identity $v(x(s), y(s)) = C_1$ with respect to s and get $\nabla v \cdot \mathbf{t} = 0$, $|\mathbf{t}| = 1$. Differentiate this identity with respect to s and

obtain: $\nabla v \cdot t = 0$, where $\mathbf{t} := t_1 e_1 + t_2 e_2$, $k_1(s) > 0$ and $\nabla v \cdot \nu = -v_N = -c_2$. Differentiate formula $\nabla v \cdot t = 0$ with respect to s again and use Formula (4) to get:

$$v_{xx}t_1^2(s) + 2v_{xy}t_1(s)t_2(s) + v_{yy}t_2^2(s) + \nabla v \cdot k_1(s)\nu(s) = 0, \quad (6)$$

The restrictions on t_1, t_2 , are the relations $t_1^2 + t_2^2 = 1$ and $\nabla v \cdot \mathbf{t} = 0$. Otherwise, the t_1, t_2 are arbitrary. They depend on the orientation of the coordinate axes. Using the condition $u_N = c_2$ on S , rewrite (6) as

$$v_{xx}t_1^2(s) + 2v_{xy}t_1(s)t_2(s) + v_{yy}t_2^2(s) = c_2k_1(s). \quad (7)$$

Let us prove Theorem 1 for $m = 2$.

Let $s = s_0$ in (7), where s_0 is an initial value of s .

If $m = 2$, then Equation (3) implies:

$$v_{xx} + v_{yy} = -k^2C_1 \text{ on } S. \quad (8)$$

Take the coordinate system in which $t_2 = 0$, $t_1 = 1$ in (7), and get:

$$v_{xx}|_{s=s_0} = c_2k_1(s_0). \quad (9)$$

Take the coordinate system in which $t_2 = 1$, $t_1 = 0$ in (7), and get:

$$v_{yy}|_{s=s_0} = c_2k_1(s_0). \quad (10)$$

Therefore,

$$v_{xx}|_{s=s_0} + v_{yy}|_{s=s_0} = 2c_2k_1(s_0). \quad (11)$$

If one uses Equation (8), then one gets $\Delta v = -k^2C_1$. From (11), one obtains $\Delta v = 2c_2k_1(s_0)$. Therefore, $-k^2C_1 = 2c_2k_1(s_0)$, so

$$k_1(s_0) = -\frac{k^2C_1}{2c_2}, \quad c_2 \neq 0, \quad C_1 \neq 0. \quad (12)$$

The right side of (12) does not depend on s_0 . Therefore, $k_1(s_0)$ is a constant. If the curvature of a closed smooth curve S in $\mathbb{R}^2$ is a constant, then S is a circle.

Let us give a different argument. Its advantage is the choice of the equation with coefficients invariant with respect to rotations of the coordinate system. Let us write the left side of Equation (7) as

$$(At, t) = k_1(s_0)c_2,$$

where A is the 2×2 symmetric real-valued matrix defined by this equation and $t = \{t_1, t_2\}$. Let W_j , $j = 1, 2$, be the normalized eigenvectors of A , $AW_j = \lambda_j W_j$, $(W_j, W_m) = \delta_{jm}$. One has $t = h_j W_j$, here and below summation is understood over the repeated indices, $t_j = (t, W_j)$, $j = 1, 2$,

$$W_1 = \{1, \gamma\}/\Delta^{1/2}, \quad \Delta := 1 + \gamma^2, \quad W_2 = \{-\gamma, 1\}/\Delta^{1/2},$$

$$h_1 = (t_1 + \gamma t_2)/\Delta^{1/2}, \quad h_2 = (-\gamma t_1 + t_2)/\Delta^{1/2}.$$

Clearly,

$$(At, t) = (Ah_j W_j, h_m W_m) = \lambda_j h_j h_m \delta_{jm} = [\lambda_1(t_1 + \gamma t_2)^2 + \lambda_2(-\gamma t_1 + t_2)^2]/\Delta. \quad (13)$$

Remember that $t_1^2 + t_2^2 = 1$ and choose the coordinate system in which $t_1 = 1, t_2 = 0$. With this choice of t , Equation (13) yields:

$$(At, t) = (\lambda_1 + \lambda_2 \gamma^2)/\Delta = k_1(s_0)c_2.$$

Take $t_1 = 0$, $t_2 = 1$. Then, Equation (13) yields:

$$(At, t) = (\lambda_2 + \lambda_1 \gamma^2) / \Delta = k_1(s_0) c_2.$$

Consequently,

$$(At, t) = (\lambda_1 + \lambda_2) / 2 = k_1(s_0) c_2.$$

Since

$$\lambda_1 + \lambda_2 = \text{Tr}(A) = (u_{xx} + u_{yy})|_S = -k^2 C_1,$$

we obtain Formula (12) again. The trace $\text{Tr}(A)$ of a matrix is invariant with respect to rotations of the coordinate system.

Theorem 1 is proved for $m = 2$.

Remark 2. If $c_2 = 0$ in (12) and $C_1 = 0$, then problem (3) has a unique solution $v = 0$ by the uniqueness of the solution to the Cauchy problem for linear elliptic equations. In general, problem (3) does not have a solution, it is over-determined. For the special values of k^2 , the eigenvalues of the Neumann Laplacian in D , problem (3) with $c_2 = 0$ and $C_1 \neq 0$ may have a solution for the balls if the radii of the balls have special values. This happens because there is a solution $v = v(r)$, $r := |x|$, of the Neumann problem with $c_2 = 0$ for a ball, and if $\frac{\partial v}{\partial r} = 0$ at a special value r_j of r , then $v(r_j) \neq 0$, see Section 7.

If $C_1 = 0$ and $c_2 \neq 0$, then problem (3) for a ball has a solution if k^2 is the eigenvalue of the Dirichlet Laplacian in this ball, with an eigenfunction depending only on r , and if the radius of the ball has special value, see Section 7 below.

If k^2 is not a Neumann eigenvalue of the Laplacian in D , then the relation $c_2 = 0$ implies $v = 0$ in D and $C_1 = 0$. If k^2 is not a Dirichlet eigenvalue of the Laplacian in D , then the relation $C_1 = 0$ implies $v = 0$ in D and $c_2 = 0$.

One can prove that, if problem (3) has a solution and $c_2 = 0$, then k^2 is simultaneously a Neumann and a Dirichlet eigenvalue of the Laplacian in D , see ([1], p. 408).

To prove the last statement, we note that, if $c_2 = 0$ and (3) is solvable, then k^2 is a Neumann eigenvalue. If $v = \text{const}$ on S , then $\nabla v = 0$ on S because the derivative in the direction N , orthogonal to S , equals to zero, and the derivative in the direction tangential to S equals to zero because $v = \text{const}$ on S . Since ∇v is a linear combination of derivatives along N and perpendicular to N (tangential to S), it is equal to zero on S . Therefore, each of the components of ∇v is a Dirichlet eigenfunction of the Laplacian in D with the eigenvalue k^2 . Thus, k^2 is the Dirichlet eigenvalue of the Laplacian in D . By the Laplacian, we mean here $-\nabla^2$.

Consider a new argument, based on the Green's formula, for proving that, if v solves problem (3), then S has to be a sphere.

Let $g = g(x, y)$ be the Green's function of the operator $\Delta + k^2$ in $\mathbb{R}^m$, $m \geq 2$, satisfying the radiation condition. Since $k^2 > 0$, the differential equation and the radiation condition determine g uniquely. By the Green's formula, one gets an integral representation of the solution to problem (3):

$$v(x) = c_2 \int_S g(x, s) ds - C_1 \int_S \frac{\partial g(x, s)}{\partial N_s} ds, \quad x \in D, \quad (14)$$

$$0 = c_2 \int_S g(x, s) ds - C_1 \int_S \frac{\partial g(x, s)}{\partial N_s} ds, \quad x \in \Omega := D' := \mathbb{R}^3 \setminus D. \quad (15)$$

Let $x \rightarrow t \in S$, $x \in D$, in Formula (14). Then, using Formula (1.1.3) (a) from [10], the limiting value on S of the potential of double layer, one derives:

$$C_1 = c_2 \int_S g(t, s) ds - C_1 \left[\int_S \frac{\partial g(t, s)}{\partial N_s} ds - \frac{1}{2} \right]. \quad (16)$$

Equation (16) is equivalent to

$$C_1 \left[\frac{1}{2} + \int_S \frac{\partial g(t, s)}{\partial N_s} ds \right] = c_2 \int_S g(t, s) ds, \quad \forall t \in S. \quad (17)$$

If one proves that Equation (17) may hold only if S is a sphere, then Theorem 1 is proved. Without loss of generality, one may take $C_1 = 1$ if $c_2 = 0$, and, if $C_1 = 0$, then one may take $c_2 = 1$.

Using the known formulas for the limiting values on S of the potential of the double layer and for the normal derivative of the potential of the single layer (see, e.g., [10], p. 18, $x \rightarrow t$ means convergence in non-tangential to S directions):

$$\lim_{x \rightarrow t} \int_S \frac{\partial g(x, s)}{\partial N_s} \mu(s) ds = \int_S \frac{\partial g(t, s)}{\partial N_s} \mu(s) ds - \frac{\mu(t)}{2}, \quad x \in D, \quad t \in S,$$

$$\lim_{x \rightarrow t} \frac{\partial \left(\int_S g(x, s) \mu(s) ds \right)}{\partial N_t} = \int_S \frac{\partial g(t, s)}{\partial N_t} \mu(s) ds + \frac{\mu(t)}{2}, \quad x \in D, \quad t \in S,$$

one derives Formulas (18) and (19) below.

For $c_2 = 0$ and $C_1 = 1$, one obtains from (17) the following:

$$\int_S \frac{\partial g(t, s)}{\partial N_s} ds = -\frac{1}{2}, \quad v(x) = - \int_S \frac{\partial g(x, s)}{\partial N_s} ds, \quad x \in D, \quad v_{N_t} = 0, \quad t \in S. \quad (18)$$

For $c_2 = 1$ and $C_1 = 0$, one obtains from (14) the relations:

$$\int_S \frac{\partial g(t, s)}{\partial N_t} ds = \frac{1}{2}, \quad v(x) = \int_S g(x, s) ds, \quad x \in D, \quad v_{N_t} = 1, \quad t \in S. \quad (19)$$

Formula (18) implies that S is a sphere and so does Formula (19). In both cases, the proof is essentially the same. Let us give a proof for Formula (18).

Proof. If S is not a sphere, then the behavior of $g_{N_s}(t, s)$, as $s \rightarrow t$, depends on the location of t on S , it depends on the curvature at the point t of a normal section of S which contains the normal to S at the point $t \in S$. Only for a sphere (all the points of a sphere are umbilic) there is no such dependence and the first integral (18) can be a constant on S .

A consequence of Theorem 1 is that the integral operator $Qu := \int_S g(t, s) \phi(s) ds$ can have an eigenvalue $\lambda = 0$ with constant eigenfunction $\phi(s) = 1$ only if S is a sphere, k^2 is the eigenvalue of the Dirichlet Laplacian in D and $u_N|_S = \text{const} \neq 0$, where $u(x) := \int_S g(x, s) ds$, $u|_S = 0$.

If $c_2 \neq 0$ and $C_1 \neq 0$, then Theorem 1 was proved above. $\square$

Let us give another method for proving Theorem 1.

Write Equation (7) as $(At, t) = c_2 k_1(s)$, where, for $m = 2$, $t_1^2 + t_2^2 = 1$ and A is a real-valued symmetric matrix, $A = A'$, with the elements: $A_{11} = v_{xx}$, $A_{22} = v_{yy}$, $A_{12} = A_{21} = v_{xy}$. Such a matrix can be diagonalized by an orthogonal matrix $U' = U^{-1}$, a rotation, $U = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$. In the diagonal form, $U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

The eigenvalues λ_1 and λ_2 of A are invariant with respect to the rotation of the coordinate axes.

Indeed, the eigenvalues are the roots of the equation $\det |A - \lambda I| = 0$. This equation is $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$, where $\text{Tr}(A)$ is the trace of A and $\det(A)$ is the determinant of A . Both coefficients are invariant with respect to the rotation of the coordinate axes. The trace of A is defined by the formula: $\text{Tr}(A) := \sum_{j=1}^m A_{jj}$. It is known that $\text{Tr}(AB) = \text{Tr}(BA)$. Indeed,

$$\text{Tr}(AB) = \sum_{i=1}^m \sum_{j=1}^m A_{ij} B_{ji} = \sum_{j=1}^m \sum_{i=1}^m B_{ji} A_{ij} = \text{Tr}(BA).$$

Claim 1: $Tr(A)$ is invariant with respect to rotations.

Proof. $Tr(U^{-1}AU) = Tr(UU^{-1}A) = Tr(A)$.

Claim 2: The $\det(A)$ is invariant with respect to rotations. $\square$

Proof. Here, we have used the known formulas: $\det(AB) = \det(BA) = \det A \det B$.

Choose a vector τ such that $t = U\tau$. Then,

$$(At, t) = (AU\tau, U\tau) = (U^{-1}AU\tau, \tau),$$

where the matrix $U^{-1}AU$ is diagonal. Since the matrix U describes a rotation, it preserves the length of vectors, and $1 = t_1^2 + t_2^2 = \tau_1^2 + \tau_2^2$, if $m = 2$. In this case,

$$(At, t) = (U^{-1}AU\tau, \tau) = \lambda_1\tau_1^2 + \lambda_2\tau_2^2 = c_2k_1(s_0). \quad (20)$$

One may find a rotation (by the angle $\pi/2$) that sends vector (τ_1, τ_2) onto $(\tau_2, -\tau_1)$. Then, Formula (20) takes the form:

$$\lambda_1\tau_2^2 + \lambda_2\tau_1^2 = c_2k_1(s_0). \quad (21)$$

Add Equations (20) and (21) and use the formula $\tau_1^2 + \tau_2^2 = 1$ to get:

$$\lambda_1 + \lambda_2 = 2c_2k_1(s_0). \quad (22)$$

The sum $\lambda_1 + \lambda_2 = (u_{xx} + u_{yy})|_S = -k^2C_1$ is the trace of the symmetric real-valued matrix A . One has:

$$Tr(A) = \lambda_1 + \lambda_2 = u_{xx} + u_{yy} = -k^2C_1 = 2c_2k_1(s_0).$$

Consequently, we obtain Formula (12) again:

$$k_1(s_0) = -\frac{k^2C_1}{2c_2}, \quad c_2 \neq 0, \quad C_1 \neq 0. \quad (23)$$

Since C_1 and c_2 are constants, the curvature $k_1(s_0)$ does not depend on $s_0 \in S$. It is constant on $S = L_1$ for $m = 2$. By Lemma 1, a closed smooth curve with a constant curvature is a circle (a ball in $\mathbb{R}^2$) and its curvature $k_1 > 0$ is positive. This is a different method to prove Theorem 1 for $m = 2$. Its ideas can be used when $m > 2$. $\square$

Assume now that $m > 2$. By considering the normal sections P_{jm} , one derives, as in the case $m = 2$, the formulas:

$$v_{x_jx_j}|_{s=s_0} = c_2k_j(s_0); \quad v_{x_jx_j}|_{s=s_0} + v_{x_mx_m}|_{s=s_0} = 2c_2k_j(s_0); \quad 1 \leq j \leq m-1. \quad (24)$$

Since $v_{x_mx_m}|_{s=s_0}$ does not depend on the normal section, it does not depend on j . Therefore, Formula (24) implies that $v_{x_jx_j}|_{s=s_0} = c_2k_j(s_0)$, $1 \leq j \leq m-1$. The function $v_{x_mx_m}|_{s=s_0}$ is a continuous function of s_0 on S . It does not depend on j . Consequently, $k_j(s_0)$ does not depend on j , so $k_j(s_0) = k(s_0)$, $1 \leq j \leq m-1$. Therefore, the point s_0 is umbilic. Since s_0 is an arbitrary point of S , all the points of S are umbilic. By Lemma 1, S is a sphere.

Theorem 1 is proved for $m > 2$.

The quantity $k(s_0)$ is equal to the mean curvature $\kappa(s_0)$ of S :

$$k(s_0) = \kappa(s_0) := \frac{\sum_{j=1}^{m-1} k_j(s_0)}{m-1}.$$

Let us derive a formula for the mean curvature $\kappa(s)$ in terms of C_1, c_2 , and m .

Put the x_p -axis along the outer unit normal N at the point $s_0 \in S$ and let P_{jp} , $j \neq p$, be the normal sections. The normal sections P_{jp} , $j \neq p$, depend on j but do not depend on p . We can use $m - 1$ normal sections. As in the case $m = 2$, one derives:

$$v_{x_j x_j}|_{s=s_0} + v_{x_p x_p}|_{s=s_0} = 2c_2 k_j(s_0), \quad 1 \leq j \leq m-1. \quad (25)$$

Equation (3) on the boundary S yields:

$$\sum_{j=1}^m v_{x_j x_j}|_S = -k^2 C_1. \quad (26)$$

Using (25), one obtains (compare with Equation (11)):

$$\sum_{j \neq p} v_{x_j x_j}|_{s=s_0} + \sum_{j=1}^{m-1} c_2 k_j(s_0) = 2c_2 \sum_{j=1}^{m-1} k_j(s_0). \quad (27)$$

Therefore,

$$\sum_{j \neq p} v_{x_j x_j}|_{s=s_0} = c_2 \sum_{j=1}^{m-1} k_j(s_0) := (m-1)c_2 \kappa(s_0), \quad (28)$$

where $\kappa(s_0)$ is the mean curvature of S at the point s_0 . The right side of (28) does not depend on p . Therefore, its left side does not depend on p and from (28) we get:

$$\sum_{j=1}^m v_{x_j x_j}|_{s=s_0} = mc_2 \kappa(s_0). \quad (29)$$

On the other hand, from Equations (26) and (29), we have: $-k^2 C_1 = mc_2 \kappa(s_0)$ on S . Since c_2 and C_1 are constants independent of s_0 , we conclude, assuming that $c_2 \neq 0$, that $\kappa(s)$ is constant on S :

$$\kappa(s) = -\frac{k^2 C_1}{mc_2} \quad \text{on } S, \quad c_2 \neq 0, \quad C_1 \neq 0. \quad (30)$$

If $\kappa(s) = \text{const}$ on S and S is a smooth closed surface, then, by Lemma 1, S is a sphere. Remark 2 is also valid for Formula (30). This is another proof of Theorem 1 for $c_2 \neq 0$ and $C_1 \neq 0$.

4. Symmetry Problems in Harmonic Analysis

One can derive from Theorem 1 the following symmetry results in harmonic analysis. Let $D \subset \mathbb{R}^3$ be a bounded connected domain with a closed Lipschitz surface S , $\Omega := \mathbb{R}^3 \setminus D$, S^2 be the unit sphere in $\mathbb{R}^3$, $k = \text{const} > 0$.

Theorem A. (a) If $\int_D e^{ik\alpha \cdot s} ds = 0$ for all $\alpha \in S^2$, then D is a ball.

(b) If $\int_S e^{ik\alpha \cdot s} ds = 0$ for all $\alpha \in S^2$, then S is a sphere.

Proof of Theorem A. (a) Let

$$u := \int_D g(x, y) dy, \quad x, y \in \mathbb{R}^3, \quad g = g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (31)$$

Since $(\nabla^2 + k^2)g(x, y) = -\delta(x - y)$, one concludes that

$$(\nabla^2 + k^2)u = -\chi_D(x), \quad (32)$$

where $\chi := \chi_D(x)$ is the characteristic function of D , $\chi := 1$, $x \in D$, $\chi = 0$, $x \in \Omega$. We need the following lemma to prove Theorem A, part (a). $\square$

Lemma 2. From the assumption

$$\int_D e^{ik\alpha \cdot x} dx = 0 \quad \forall \alpha \in S^2, \quad (33)$$

it follows that $u = 0$ in Ω , where u is defined in (31).

Proof of Lemma 2. One has: $(\nabla^2 + k^2)u = 0$ in Ω , and our assumption implies

$$u = \frac{e^{ik|x|}}{4\pi|x|} \int_D e^{ik\alpha \cdot y} dy + O(|x|^{-2}) = O(|x|^{-2}) \quad |x| \gg 1, \quad \alpha = -\frac{x}{|x|}.$$

It is known ([10], p. 30) that, if $(\nabla^2 + k^2)u = 0$ in Ω , $k = \text{const} > 0$ and $u = O(|x|^{-2})$ as $|x| \rightarrow \infty$, then $u = 0$ in Ω . From Formulas (31) and (33), it follows that $u = O(|x|^{-2})$ as $|x| \rightarrow \infty$. Lemma 2 is proved. $\square$

Proof. Let us continue with the proof of Theorem A. From Lemma 2, it follows that u solves the problem:

$$(\nabla^2 + k^2)u = -1 \text{ in } D, \quad u|_S = u_N|_S = 0. \quad (34)$$

Indeed, $u \in H_{loc}^2(\mathbb{R}^3)$, where H^2 is the Sobolev space. Therefore, if $u = 0$ in Ω , then $u|_S = u_N|_S = 0$, and $(\nabla^2 + k^2)u = -1$ in D . Thus, (34) holds.

Problem (34) is problem (1) with $c_0 = -1$, $c_1 = c_2 = 0$. By Theorem 1, it follows that D is a ball. In this case $C_1 = \frac{1}{k^2}$, $c_2 = 0$. Part (a) of Theorem A is proved.

(b) Let us prove part (b) of Theorem A.

Define $u := \int_S g(x, s) ds$. By the assumption $\int_S e^{ik\alpha \cdot s} ds = 0 \quad \forall \alpha \in S^2$, it follows, as above, that $u = 0$ in Ω . Consequently,

$$(\nabla^2 + k^2)u = 0 \text{ in } D, \quad u|_S = 0, \quad u_N|_S = 1, \quad (35)$$

where the equation $u_N|_S = 1$ follows from the known relation (see [10], p. 18):

$$u_N^+ - u_N^- = 1.$$

In our case, $u_N^- = 0$ since $u = 0$ in Ω , so $u_N^+ := u_N|_S = 1$. By u_N^- (u_N^+), the limiting values of the normal derivative on S from Ω (D) are denoted.

Problem (35) is a particular case of problem (1). It follows that D is a ball and S is a sphere. In this case, $C_1 = 0$, $c_2 = 1$. Part (b) of Theorem A is proved.

Theorem A is proved. $\square$

5. Proof of the Schiffer's Conjecture

Let us prove the **Schiffer's conjecture**.

Theorem B. Assume that the problem

$$(\nabla^2 + k^2)u = 0 \text{ in } D, \quad u|_S = 0, \quad u_N|_S = 1, \quad (36)$$

is solvable. Then, D is a ball.

Proof of Theorem B. Problem (36) is a particular case of problem (1). The conclusion of Theorem B follows from Theorem 1. Theorem B is proved. $\square$

6. Solution to the Pompeiu Problem

Let us solve the **Pompeiu problem**. This problem is discussed in [10–12]. In modern formulation, the problem can be stated as follows: Assume that

$$\int_{\sigma(D)} f(x) dx = 0, \quad (37)$$

where σ is the set of all rigid motions, that is, translations and rotations, $f \in L^1_{loc}(\mathbb{R}^m)$ and $D \subset \mathbb{R}^m$ is a bounded connected domain with a Lipschitz boundary S .

The modern formulation of the Pompeiu problem is:

Can (37) imply that $f = 0$ if D is not a ball?

It is known (see [1]) that, if D is a ball, then (37) may hold for $f \neq 0$.

Let us rewrite Equation (37) as

$$\int_D f(gx + y) dx = 0 \quad \forall y \in \mathbb{R}^m, \forall g \in G, \quad (38)$$

where G is the group of rotations and y describes the translations.

Define the Fourier transform by the formula $\tilde{f} := \int_{\mathbb{R}^m} f(x) e^{ix \cdot \xi} dx$. Take the Fourier transform of (38) and get

$$\tilde{f}(\xi) \overline{\tilde{\chi}(g^{-1}\xi)} = 0, \quad (39)$$

where $\chi(x)$ is the characteristic function of D and the overline stands for a complex conjugate. For the supp $\tilde{f}$ to be non-void, that is, for $f \neq 0$, it is necessary and sufficient that the set of zeros of $\tilde{\chi}(g^{-1}\xi)$ be non-void. Since D is bounded, the function $\tilde{\chi}(\xi)$ is an entire function of ξ . Since $g \in G$ is arbitrary, the set of zeros of $\tilde{\chi}(\xi)$ must be spherically symmetric. Let C_k be the spherical surface of zeros of $\tilde{\chi}(\xi)$, where $k > 0$ is the radius of the sphere C_k . Therefore, if and only if $f \neq 0$, there exists a $k > 0$ such that C_k is the spherical surface of zeros of $\tilde{\chi}(\xi)$. Then, by Theorem A, it follows that D is a ball. We have proved the following:

Theorem C. Equation (37) can hold for $f \neq 0$ if and only if D is a ball.

7. Possible Values of the Radii of the Balls in Theorem 1

Can the radius a of the ball in Theorem 1 be arbitrary? Let us consider the case $m = 3$, which is the main interest physically. The spherically symmetric solution of Equation (3) in the spherical coordinates, regular at the origin, is $cj_0(kr)$, where $j_0(r) := \frac{J_{1/2}(kr)}{r^{1/2}}$, $c = \text{const}$, substitute this function into the boundary conditions (3) for a sphere of radius a and get: $cj_0(ka) = C_1$, $ckj'_0(ka) = c_2$. Eliminating c yields

$$kC_1j'_0(ka) = c_2j_0(ka). \quad (40)$$

Any possible radius a of the ball in Theorem 1 should solve Equation (40).

In the Schiffer's conjecture $C_1 = 0$, $c_2 = 1$, so Equation (40) yields equation $j_0(ka) = 0$ for possible values of the radius a .

In the Pompeiu problem and Theorem A, part (a), $C_1 = \frac{1}{k^2}$, $c_2 = 0$, and Equation (40) yields equation $j'_0(ka) = 0$ for all possible values of the radius a .

8. A Symmetry Problem for the Laplace Equation

Assume that

$$\nabla^2 u = c_0 \text{ in } D, \quad u|_S = 0, \quad u_N = c_2, \quad c_j = \text{const}, \quad j = 0, 2, \quad c_0|D| = c_2|S|, \quad |c_0| > 0. \quad (41)$$

The assumptions on D and S are the same as in the previous sections. Our result is the following.

Theorem 2. If problem (41) is solvable, then D is a ball.

Proof of Theorem 2. First, note that the condition $|c_0| > 0$ is necessary: if $c_0 = 0$, then, for any shape of D , problem (41) has only the trivial solution $u = 0$ and $c_2 = 0$.

Secondly, integrating Equation (41) over D and using the formula

$$c_0|D| = \int_D \nabla^2 v dx = \int_S v_N ds = c_2|S|,$$

one concludes that the condition $c_0|D| = c_2|S|$ is necessary for the solvability of problem (41).

To prove that the solvability of problem (41) implies that D is a ball, we argue as in the proof of Theorem 1 and obtain the equation similar to (26):

$$\sum_{j=1}^m v_{x_j x_j} = c_0, \quad (42)$$

and the relation

$$v_{x_j x_j}(s_0) + v_{x_p x_p}(s_0) = 2c_2 k_j(s_0), \quad j \neq p.$$

As in the proof of Theorem 1, we conclude that $\kappa(s) = \text{const} > 0$ on S , where κ is the mean curvature of S . If the mean curvature of a closed smooth surface S is a positive constant, then, by Lemma 1, S is a sphere, and D is a ball.

Theorem 2 is proved. $\square$

9. Conclusions

In this paper, new results on symmetry problems for PDE are established. These results are formulated in Theorems 1 and 2 and Theorems A, B, and C. The symmetry results for PDE are significant and interesting.

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