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Completeness of $b-$Metric Spaces and Best Proximity Points of Nonself Quasi-Contractions

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Abstract: The aims of this article are twofold. One is to prove some results regarding the existence of best proximity points of multivalued non-self quasi-contractions of $b-$metric spaces (which are symmetric spaces) and the other is to obtain a characterization of completeness of $b-$metric spaces via the existence of best proximity points of non-self quasi-contractions. Further, we pose some questions related to the findings in the paper. An example is provided to illustrate the main result. The results obtained herein improve some well known results in the literature.

Keywords: completeness; quasi-contraction; best proximity points; $b-$metric; non-self mappings

MSC: 47H10; 47H04; 47H09; 90C26

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1. Introduction and Preliminaries

In 1922, Banach [1] presented one of the important and basic results known as the Banach contraction principle (shortly as BCP) in metric fixed point theory. Since then, fixed point theory has been used frequently to prove the existence of solutions of functional equations (compare [2–6]). Due to its usefulness and applicability, BCP has been generalized in one too many directions. In 1974, Ciric [7] introduced quasi-contractions and generalized BCP for self quasi-contractions of orbitally complete metric spaces. On the other hand, in 1969, Nadler [8] extended BCP from self mappings to multivalued mappings of complete metric spaces. Amini Harandi [9] introduced multivalued quasi-contractions and generalized Nadler's result.

Due to its significance, the concept of distance has been generalized in many directions (compare [10]). For instance, $b-$metric space was introduced as a proper generalization of metric space (see [11,12]). Since then, there have been a lot of developments in the context of fixed point theory of $b-$metric spaces; for more details one can see the reference [13]. Czerwik [12,14,15] obtained BCP in the context of $b-$metric spaces for single valued and multivalued mappings and also discussed some results concerning stability. Afshari [16] developed some fixed points results in the context of quasi$-b-$metric and $b-$metric$-$like spaces and also provided the solution of some fractional differential equations. Aydi et al. [17] obtained results for multivalued quasi-contractions of $b-$metric spaces. Ciric et al. [18] obtained Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two $b-$metrics. Alo et al. [19] and Ali et al. [20] obtained the existence of fixed points of multivalued quasi-contractions along with a completeness characterization of underlying $b-$metric spaces.

On the other hand, if A and B are two non-empty subsets of a metric space $(\mathcal{W}, p)$, and $\mathcal{F} : A \to B$ a non-self mapping then a point $x \in A$ such that

$$p(x, \mathcal{F}x) = p(A, B),$$

(if it exists) is called a best proximity point (shortly as BPP) of $\mathcal{F}$ in A, where

$$p(A,B) = \inf_{a\in A, b\in B} p(a,b).$$

Note that if $A = B$, then x becomes the fixed point of $\mathcal{F}$. Fan [21] presented a result that guarantees the existence of best proximity points (shortly as BPPs) of a continuous mapping of a non-empty compact convex subset of a Hausdorff locally convex topological vector space. Hussain et al. [22] obtained Fan type result in ordered Banach spaces. Sehgal and Singh [23] generalized Fan's result for multivalued mappings (also compare [24,25]). Basha and Naseer [26] explored the existence of BPP theorems for generalized proximal contractions of metric spaces (see also [27]). Mishra et al. [28] developed some best proximity points results in the context of $b-$metric spaces. Abkar and Gabeleh [29] and Hussain et al. [30] obtained BPP results for Suzuki type contractions of metric spaces. George et al. [31] studied BPP results for cyclic contractions of $b-$metric spaces. Gabeleh and Plebaniak [32] obtained BPPs of multivalued contractions of $b-$metric spaces.

The "Completeness Problem (CP)" is an important problem in mathematics which is equivalent to the "End Problem (EP)" in behavioral sciences. The end problem is to determine where and when a human dynamics defined as a succession of positions that starts from an initial position and follows transitions ends. For details on the completeness problem and the end problem, we refer to [33,34] and references therein. In 1959, Connel presented an example ([35], (Example 3)) (also compare [20]) which shows that BCP does not characterize metric ($b-$metric) completeness. That is, there exists an incomplete metric ($b-$metric) space $\mathcal{W}$ such that every Banach contraction on $\mathcal{W}$ has a fixed point. Suzuki [36] presented a fixed point theorem that generalized BCP and characterized metric completeness as well. Recently, Ali et al. [20] (compare with [19]) obtained completeness characterizations of $b-$metric spaces via the fixed point of Suzuki type contractions.

In this paper, first we study the existence of BPPs of generalized multivalued non-self quasi-contractions of $b-$metric spaces and then we obtain a characterization of the completeness of $b-$metric spaces which are symmetric spaces. For more on the connection between completeness and symmetry we refer the interested reader to [37,38].

Throughout this article, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_1$, denote the set of nonnegative reals, reals, positive integers, and nonnegative integers, respectively.

Definition 1 ([11,12])**.** *Let $\mathcal{W}$ be a non-empty set. A mapping $p : \mathcal{W} \times \mathcal{W} \to [0,\infty)$ is a $b-$metric and $(\mathcal{W}, p)$ is called $b-$metric space if there exists a real number $k \geq 1$ such that p satisfies the following:*

(a_1) $p(x,y) = 0$ *if and only if* $x = y$ *for all* $x, y \in \mathcal{W}$;
(a_2) $p(x,y) = p(y,x)$ *for all* $x, y \in \mathcal{W}$;
(a_3) $p(x,y) \leq k[p(x,z) + p(z,y)]$ *for all* $x, y, z \in \mathcal{W}$.

Note that, throughout this article, $k \geq 1$, will be used as $b-$metric constant.

Definition 2. *A sequence $\{x_n\}$ in a $b-$metric space $(\mathcal{W}, p)$ is:*

(i) *convergent if there is an $x \in \mathcal{W}$, such that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $p(x_n, x) < \varepsilon$ for all $n > n_0$, that is, $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;*
(ii) *Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$ and $p \in \mathbb{N}_1$, that is, $\lim_{n\to\infty} p(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}_1$.*

Remark 1 (compare [39])**.** *A $b-$metric p is not necessarily continuous but if it is continuous in one variable then it is continuous in the second variable as well and the subset:*

$$B_\epsilon(u_0) = \{u \in \mathcal{W} : p(u_0, u) < \epsilon\},$$

of $b-$metric space $(\mathcal{W}, p)$ is not an open set (in general) but if p is continuous in one variable then $B_\epsilon(u_0)$ is open in $\mathcal{W}$. Moreover, throughout in this article, assume that the b-metric p is continuous in one variable.

The following lemma has been used as sufficient condition for a contractive sequence to be a Cauchy sequence.

Lemma 1 ([40])**.** *If a sequence $\{x_n\}$ in a $b-$metric space $(\mathcal{W}, p)$ satisfies*

$$p(x_{n+1}, x_{n+2}) \leq rp(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}_1$ and for some

$$0 \leq r < \frac{1}{k},$$

then it is a Cauchy sequence in $\mathcal{W}$.

Recently, Suzuki [41] improved the previous lemma as follows.

Lemma 2. *If a sequence $\{x_n\}$ in a $b-$metric space $(\mathcal{W}, p)$ satisfies*

$$p(x_{n+1}, x_{n+2}) \leq rp(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}_1$ and for some $r \in [0, 1)$, then it is a Cauchy sequence in $\mathcal{W}$.

Let $(\mathcal{W}, p)$ be a $b-$metric space then $C(\mathcal{W}), C_B(\mathcal{W}), P(\mathcal{W})$ represent the set of non-empty closed, non-empty closed and bounded subsets, non-empty subsets of $\mathcal{W}$. For $A, B \in C_B(\mathcal{W})$, the mapping H_p defined as:

$$H_p(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

is called Hausdorff metric on $C_B(\mathcal{W})$ induced by p, where

$$\delta(A, B) = \sup_{a \in A} p(a, B) \text{ and } p(a, B) = \inf_{b \in B} p(a, b).$$

The following lemma lists some important properties of $b-$metric spaces that will be used in the sequel to prove the main results.

Lemma 3 ([12,15,40])**.** *For a $b-$metric space $(\mathcal{W}, p)$, $x, y \in \mathcal{W}$ and $A, B \in C_B(\mathcal{W})$, the following assertions hold:*

(b_1) *$(C_B(\mathcal{W}), H_p)$ is a $b-$metric space.*
(b_2) *For all $a \in A$, $p(a, B) \leq H_p(A, B)$.*
(b_3) *For all $x, y \in \mathcal{W}$, $p(x, A) \leq kp(x, y) + kp(y, A)$.*
(b_4) *For $t > 1$ and $a \in A$, there is a $b \in B$ so that $p(a, b) \leq tH_p(A, B)$.*
(b_5) *For $t > 0$ and $a \in A$, there is a $b \in B$ so that $p(a, b) \leq H_p(A, B) + t$.*
(b_6) *$a \in \overline{A} = A$, if and only if $p(a, A) = 0$, where $\overline{A}$ is the closure of A in $(\mathcal{W}, p)$.*
(b_7) *For any sequence $\{x_n\}$ in $\mathcal{W}$*

$$p(x_0, x_n) \leq kp(x_0, x_1) + k^2 p(x_1, x_2) + \dots + k^{n-1} p(x_{n-2}, x_{n-1}) + k^{n-1} p(x_{n-1}, x_n).$$

Ciric [7] introduced quasi-contractions of metric space $(\mathcal{W}, p)$. A self-mapping $f : \mathcal{W} \to \mathcal{W}$ is a quasi-contraction of $\mathcal{W}$ if:

$$p(fu, fy) \leq r \max\{p(u, y), p(u, fu), p(y, fy), p(u, fy), p(y, fu)\},$$

for some $0 \leq r < 1$. Further, they obtained fixed point results for quasi-contractions in orbitally complete metric spaces. Nadler [8] extended the BCP as follows.

Theorem 1. *Let $(\mathcal{W}, p)$ be a complete metric space and $\mathcal{F} : \mathcal{W} \longrightarrow C_B(\mathcal{W})$ such that*

$$H_p(\mathcal{F}u, \mathcal{F}y) \leq rp(u, y),$$

for all $u, y \in \mathcal{W}$ and some $r \in [0, 1)$, then $F_{ix}(\mathcal{F})$ (set of fixed points of $\mathcal{F}$) is non-empty.

Amini-Harandi [9] generalized Theorem 1 for multivalued quasi-contractions.

Theorem 2 ([9])**.** *Let $(\mathcal{W}, p)$ be a complete metric space and $\mathcal{F} : \mathcal{W} \to C_B(\mathcal{W})$. If*

$$H_p(\mathcal{F}u, \mathcal{F}y) \leq r \max\{p(u, y), p(u, \mathcal{F}u), p(y, \mathcal{F}y), p(u, \mathcal{F}y), p(y, \mathcal{F}u)\},$$

for all $u, y \in \mathcal{W}$ and some $r \in \left[0, \frac{1}{2}\right)$. Then $F_{ix}(\mathcal{F})$ is non-empty.

On the other hand, Aydi et al. [17] obtained a $b-$metric version of Theorem 2.

Theorem 3 ([17])**.** *Let $(\mathcal{W}, p)$ be a complete $b-$metric space and $\mathcal{F} : \mathcal{W} \to C_B(\mathcal{W})$. If $\mathcal{F}$ satisfies*

$$H_p(\mathcal{F}u, \mathcal{F}y) \leq r \max\{p(u, y), p(u, \mathcal{F}u), p(y, \mathcal{F}y), p(u, \mathcal{F}y), p(y, \mathcal{F}u)\},$$

for all $u, y \in \mathcal{W}$ and for some $r \in [0, 1)$ with $r < \dfrac{1}{k^2 + k}$, then $F_{ix}(\mathcal{F})$ is non-empty.

Let $(\mathcal{W}, p)$ be a $b-$metric space, and fix $A, B \in P(\mathcal{W})$. Define

$$A_0 = \{a \in A : p(a, b) = p(A, B) \text{ for some } b \in B\} \text{ and}$$
$$B_0 = \{b \in B : p(a, b) = p(A, B) \text{ for some } a \in A\}.$$

If A_0 is non-empty then the pair (A, B) has the weak $P-$property if:

$$\begin{cases} p(x_1, y_1) = p(A, B) \\ p(x_2, y_2) = p(A, B) \end{cases} \text{implies } p(x_1, x_2) \leq p(y_1, y_2),$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$. Further, define

$$H_p^*(C, D) = H_p(C, D) + p(A, B) \text{ for all } C, D \in P(\mathcal{W}),$$

$$p^*(x, y) = p(x, y) + p(A, B) \text{ for all } x, y \in \mathcal{W},$$

$$\Theta = \left\{ \zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} : \zeta(s, t) \leq \frac{s}{k} - t \right\}, \text{ and}$$

$$\zeta^*(s, t) = \zeta(s, t) - p(A, B).$$

Moreover, for a non-self mapping $f : A \to B$ and a multivalued non-self mapping $\mathcal{F} : A \to C_B(B)$, consider the following notations that we use in the sequel.

$$L_{\mathcal{F}}(x, y) = \max \left\{ \begin{array}{c} p(x, y), p(x, \mathcal{F}x), p(y, \mathcal{F}y), p(x, \mathcal{F}y), p(y, \mathcal{F}x), \\ \left(\dfrac{p(x, \mathcal{F}x) p(y, \mathcal{F}y)}{p(x, y)} \right) \end{array} \right\},$$

$$M_S(x, y) = \max \left\{ \begin{array}{c} p(x, y), p(x, Sx) - kp(A, B), p(y, Sy) - kp(A, B), \\ p(x, Sy) - k^2 p(A, B), p(y, Sx) - kp(A, B) \end{array} \right\},$$

for $S \in \{f, \mathcal{F}\}$. Further, we denote the set of BPPs of the mapping S by $B_{PP}(S)$.

Definition 3 ([20]). *Let $(\mathcal{W}, p)$ be a $b-$metric space. A mapping $\mathcal{F} : \mathcal{W} \to C_B(\mathcal{W})$ is a generalized multivalued Ciric-Suzuki type (shortly $CS-$type) quasi-contraction if there exists an $r \in [0, 1)$ with $r < \dfrac{1}{2k}$ such that*

$$\zeta(p(x, \mathcal{F}x), p(x, y)) \leq 0 \text{ implies } H_p(\mathcal{F}x, \mathcal{F}y) \leq rL_{\mathcal{F}}(x, y),$$

for all $x, y \in A$, with $x \neq y$ and for some $\zeta \in \Theta$.

Theorem 4 ([20]). *Let $(\mathcal{W}, p)$ be a complete $b-$metric space and $\mathcal{F} : \mathcal{W} \to C_B(\mathcal{W})$ a generalized multivalued $CS-$type quasi-contraction. Then $F_{ix}(\mathcal{F})$ is non-empty.*

Definition 4. *Let $(\mathcal{W}, p)$ be a $b-$metric space and A, B non-empty subsets of $\mathcal{W}$.*

1- *A mapping $\mathcal{F} : A \to C_B(B)$ is a generalized multivalued Ciric Suzuki type (shortly $CS-$type) non-self quasi-contraction if there exists an $r \in [0, 1)$ with $r < \dfrac{1}{k^4 + k^3}$ such that*

$$\zeta^*(p(x, \mathcal{F}x), p(x, y)) \leq 0 \text{ implies } H_p^*(\mathcal{F}x, \mathcal{F}y) \leq rM_{\mathcal{F}}(x, y), \tag{1}$$

for all $x, y \in A$ and for some $\zeta \in \Theta$.

2- *A mapping $f : A \to B$ is a generalized Ciric Suzuki type (shortly $CS-$type) non-self quasi-contraction if there exists an $r \in [0, 1)$ with $r < \dfrac{1}{k^4 + k^3}$ such that*

$$\zeta^*(p(x, fx), p(x, y)) \leq 0, \text{ implies } p^*(fx, fy) \leq rM_f(x, y), \tag{2}$$

for all $x, y \in A$ and for some $\zeta \in \Theta$.

In this article, we provide the existence of BPPs for generalized multivalued $CS-$type non-self quasi-contractions of $b-$metric space and establish some results for the completeness of the underlying $b-$metric space.

2. Existence of BPPs of Generalized Multivalued Nonself Quasi-Contractions

Following is the first main result about the existence of BPPs of generalized multivalued $CS-$type non-self quasi-contractions of $b-$metric space.

Theorem 5. *Let $(\mathcal{W}, p)$ be a complete $b-$metric space, $A, B \in C(\mathcal{W})$ and $\mathcal{F} : A \to C_B(B)$ a generalized multivalued $CS-$type non-self quasi-contraction. Assume that A_0 is non-empty such that for each $x \in A_0$, $\mathcal{F}x \subseteq B_0$ and the pair (A, B) satisfies the weak $P-$property. Then, $B_{PP}(\mathcal{F})$ is non-empty.*

Proof. Let r_1 be a real number such that $0 \leq r < r_1 < \dfrac{1}{k^4 + k^3}$. We can choose a positive real α such that

$$\frac{r_1}{2} + \alpha = \frac{1}{2(k^4 + k^3)} \text{ implies } r_1 + \alpha = \frac{1}{2}\left(\frac{1}{k^4 + k^3} + r_1\right).$$

If $\beta = r_1 + \alpha$, then

$$0 < \beta < \frac{1}{k^4 + k^3}.$$

As A_0 is non-empty, so we pick an $x_0 \in A_0$. By the given assumption, $\mathcal{F}x_0 \subseteq B_0$. Choose $y_1 \in \mathcal{F}x_0$. That is $y_1 \in B_0$ implies that there is an $x_1 \in A_0$ such that

$$p(x_1, y_1) = p(A, B). \tag{3}$$

If $y_1 \in \mathcal{F}x_1$, then x_1 is the BPP. Assume $y_1 \notin \mathcal{F}x_1$. As:

$$\begin{aligned} \zeta^*(p(x_0, \mathcal{F}x_0), p(x_0, x_1)) &= \zeta(p(x_0, \mathcal{F}x_0), p(x_0, x_1)) - p(A, B) \\ &\leq \frac{1}{k}p(x_0, \mathcal{F}x_0) - p(x_0, x_1) - p(A, B) \\ &\leq \frac{1}{k}p(x_0, y_1) - p(x_0, x_1) - p(A, B) \\ &\leq 0, \end{aligned}$$

so by (1), we have

$$\begin{aligned} &H_p(\mathcal{F}x_0, \mathcal{F}x_1) \leq H_p(\mathcal{F}x_0, \mathcal{F}x_1) + p(A, B) \\ &= H_p^*(\mathcal{F}x_0, \mathcal{F}x_1) \leq rM_{\mathcal{F}}(x_0, x_1) \leq r_1 M_{\mathcal{F}}(x_0, x_1). \end{aligned} \tag{4}$$

If $h = \dfrac{\beta}{r_1} > 1$, then by Lemma 3, there is $y_2 \in \mathcal{F}x_1$ such that

$$p(y_1, y_2) \leq hH_p(\mathcal{F}x_0, \mathcal{F}x_1) = \beta r_1^{-1} H_p(\mathcal{F}x_0, \mathcal{F}x_1). \tag{5}$$

As $\mathcal{F}x_1 \subseteq B_0$, so there exists $x_2 \in A_0$ such that

$$p(x_2, y_2) = p(A, B). \tag{6}$$

From (4) and (5), we get

$$\begin{aligned} &p(y_1, y_2) \leq \beta r_1^{-1} H_p(\mathcal{F}x_0, \mathcal{F}x_1) \leq \beta M_{\mathcal{F}}(x_0, x_1) \\ &= \beta \max\left\{ \begin{array}{l} p(x_0, x_1), p(x_0, \mathcal{F}x_0) - kp(A, B), p(x_1, \mathcal{F}x_1) - kp(A, B), \\ p(x_0, \mathcal{F}x_1) - k^2 p(A, B), p(x_1, \mathcal{F}x_0) - kp(A, B) \end{array} \right\} \\ &\leq \beta \max\left\{ \begin{array}{l} p(x_0, x_1), p(x_0, y_1) - kp(A, B), p(x_1, y_2) - kp(A, B), \\ p(x_0, y_2) - k^2 p(A, B), p(x_1, y_1) - kp(A, B) \end{array} \right\} \\ &\leq \beta \max\left\{ \begin{array}{l} p(x_0, x_1), kp(x_0, x_1) + kp(x_1, y_1) - kp(A, B), \\ kp(x_1, y_1) + kp(y_1, y_2) - kp(A, B), \\ kp(x_0, y_1) + kp(y_1, y_2) - k^2 p(A, B), (1-k)p(A, B) \end{array} \right\} \\ &\leq \beta \max\{kp(x_0, x_1), kp(y_1, y_2), k^2(p(x_0, x_1) + p(y_1, y_2))\}. \end{aligned}$$

Hence,

$$p(y_1, y_2) \leq k^2 \beta (p(x_0, x_1) + p(y_1, y_2)).$$

That is,

$$p(y_1, y_2) \leq \frac{k^2 \beta}{1 - k^2 \beta} p(x_0, x_1). \tag{7}$$

As the pair (A, B) satisfies the weak $P-$property, so from (3) and (6), we get

$$p(x_1, x_2) \leq p(y_1, y_2). \tag{8}$$

Combining (7) and (8)

$$p(x_1, x_2) \leq p(y_1, y_2) \leq \frac{k^2 \beta}{1 - k^2 \beta} p(x_0, x_1).$$

Continuing like this, we obtain sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0 such that

$$y_{n+1} \in \mathcal{F}x_n,\ y_{n+1} \notin \mathcal{F}x_{n+1} \text{ and } p(x_n, y_n) = p(A, B),$$

$$p(x_n, x_{n+1}) \leq p(y_n, y_{n+1}) \leq \frac{k^2\beta}{1-k^2\beta} p(x_n, x_{n-1}).$$

Set $p(x_n, x_{n+1}) = \alpha_n$, and $\gamma = \dfrac{k^2\beta}{1-k^2\beta}$ in the above, we obtain

$$\alpha_n \leq \gamma\alpha_{n-1}.$$

As $k \geq 1$ and $r_1 < \dfrac{1}{k^4+k^3}$, so we have

$$k^2\beta = \frac{k^2}{2}\left(\frac{1}{k^4+k^3} + r_1\right) < \frac{1}{2} \text{ implies } \gamma < 1.$$

By Lemma 2, $\{x_n\}$ is a Cauchy sequence in A, similarly, we can prove $\{y_n\}$ is a Cauchy sequence in B. That is, there exist $x \in A$ and $y \in B$ such that

$$\lim_{n\to\infty} p(x_n, x) = 0 \tag{9}$$

and

$$\lim_{n\to\infty} p(y_n, y) = 0. \tag{10}$$

As

$$p(x_n, y_n) = p(A, B),$$

so on taking limit as n tends to ∞, implies

$$p(x, y) = p(A, B). \tag{11}$$

From (9), we can choose an $n_0 \in \mathbb{N}$ so that

$$p(x_n, x) < \frac{1}{3k^2} p(x, w),$$

for all $n \geq n_0$ and $x \neq w$. Hence,

$$\begin{aligned}
\zeta^*(p(x_n,\mathcal{F}x_n),p(x_n,w)) &= \zeta(p(x_n,\mathcal{F}x_n),p(x_n,w)) - p(A,B) \\
&\leq \frac{1}{k}p(x_n,\mathcal{F}x_n) - p(x_n,w) - p(A,B) \\
&\leq \frac{1}{k}p(x_n,y_{n+1}) - p(x_n,w) - p(A,B) \\
&= p(x_n,x_{n+1}) - p(x_n,w) \\
&\leq kp(x_n,x) + kp(x,x_{n+1}) - p(x_n,w) \\
&\leq \frac{k}{3k^2}p(x,w) + \frac{k}{3k^2}p(x,w) - p(x_n,w) \\
&\leq \frac{2}{3k}p(x,w) - p(x_n,w) \\
&= \frac{1}{k}\left(p(x,w) - \frac{1}{3}p(x,w)\right) - p(x_n,w) \\
&\leq \frac{1}{k}(p(x,w) - k^2p(x_n,x)) - p(x_n,w) \\
&\leq \frac{1}{k}(p(x,w) - kp(x_n,x)) - p(x_n,w) \\
&\leq \frac{1}{k}(kp(x_n,w)) - p(x_n,w) = 0.
\end{aligned}$$

Consequently by (1), we get

$$\begin{aligned}
&p(x_{n+1},\mathcal{F}w) \leq kp(x_{n+1},y_{n+1}) + kp(y_{n+1},\mathcal{F}w) \\
&\leq kp(A,B) + kH_p(\mathcal{F}x_n,\mathcal{F}w) \\
&= kH_p^*(\mathcal{F}x_n,\mathcal{F}w) \leq krM_{\mathcal{F}}(x_n,w) \\
&= kr\max\left\{\begin{array}{l} p(x_n,w), p(x_n,\mathcal{F}x_n) - kp(A,B), \\ p(w,\mathcal{F}w) - kp(A,B), \\ p(x_n,\mathcal{F}w) - k^2p(A,B), p(w,\mathcal{F}x_n) - kp(A,B) \end{array}\right\} \\
&\leq kr\max\left\{\begin{array}{l} p(x_n,w), kp(x_n,x_{n+1}) + k(x_{n+1},y_{n+1}) - kp(A,B), \\ p(w,\mathcal{F}w) - kp(A,B), p(x_n,\mathcal{F}w) - k^2p(A,B), \\ kp(w,x_{n+1}) + kp(x_{n+1},y_{n+1}) - kp(A,B) \end{array}\right\} \\
&\leq kr\max\left\{\begin{array}{l} p(x_n,w), kp(x_n,x_{n+1}), p(w,\mathcal{F}w) - kp(A,B), \\ p(x_n,\mathcal{F}w) - k^2p(A,B), kp(w,x_{n+1}) \end{array}\right\}.
\end{aligned}$$

On taking limit as n tends to infinity in the above inequality, we get

$$p(x,\mathcal{F}w) \leq kr\max\Big\{kp(x,w), p(w,\mathcal{F}w) - kp(A,B), p(x,\mathcal{F}w) - k^2p(A,B)\Big\}, \tag{12}$$

for all $w \neq x$. If $p(x,\mathcal{F}w) = 0$, then

$$p(x,\mathcal{F}w) \leq kr\max\{kp(x,w), p(w,\mathcal{F}w) - kp(A,B)\}. \tag{13}$$

If $p(x,\mathcal{F}w) > 0$ and

$$\begin{aligned}
&\max\big\{kp(x,w), p(w,\mathcal{F}w) - p(A,B), p(x,\mathcal{F}w) - k^2p(A,B)\big\} \\
&= p(x,\mathcal{F}w) - k^2p(A,B),
\end{aligned}$$

in (12), then we obtain

$$p(x,\mathcal{F}w) \leq krp(x,\mathcal{F}w) - rk^3p(A,B) < p(x,\mathcal{F}w),$$

a contradiction. Consequently (13) holds for all $w \neq x$.

Now, we show that x is the BPP of $\mathcal{F}$. Assume on the contrary that

$$p(x, \mathcal{F}x) - p(A, B) \neq 0.$$

That is $p(x, \mathcal{F}x) - p(A, B) > 0$. As

$$r < \frac{1}{k^4 + k^3},$$

so choose an s such that $r < s < \frac{1}{k^4 + k^3}$. That is,

$$\frac{1}{s(k^4 + k^3)} - 1 > 0 \text{ and } s\left(k^4 + k^3\right) - 1 < 0.$$

Hence for

$$\varepsilon = \left(\frac{1}{s(k^4 + k^3)} - 1\right)(p(x, \mathcal{F}x) - p(A, B)) > 0,$$

there exists $a \in \mathcal{F}x$ such that

$$\begin{aligned}
&p(x, a) < p(x, \mathcal{F}x) + \varepsilon \\
&= p(x, \mathcal{F}x) + \left(\tfrac{1}{s\left(k^4+k^3\right)} - 1\right)(p(x, \mathcal{F}x) - p(A, B)) \\
&= p(x, \mathcal{F}x) + \left(\tfrac{1}{s\left(k^4+k^3\right)} - 1\right)p(x, \mathcal{F}x) - \left(\tfrac{1}{s\left(k^4+k^3\right)} - 1\right)p(A, B) \\
&= p(x, \mathcal{F}x) + \tfrac{1}{s\left(k^4+k^3\right)}p(x, \mathcal{F}x) - p(x, \mathcal{F}x) - \left(\tfrac{1}{s\left(k^4+k^3\right)} - 1\right)p(A, B) \\
&= \tfrac{1}{s\left(k^4+k^3\right)}p(x, \mathcal{F}x) - \left(\tfrac{1}{s\left(k^4+k^3\right)} - 1\right)p(A, B).
\end{aligned}$$

Hence

$$s\left(k^4 + k^3\right)p(x, a) < p(x, \mathcal{F}x) + s\left(k^4 + k^3\right)p(A, B) - p(A, B). \tag{14}$$

As from (11) $x \in A_0$, so by given assumption $\mathcal{F}x \subseteq B_0$. Hence $a \in B_0$. This implies that there exists $z \in A_0$ such that

$$p(a, z) = p(A, B).$$

Since

$$\begin{aligned}
\zeta^*(p(x, \mathcal{F}x), p(x, z)) &= \zeta(p(x, \mathcal{F}x), p(x, z)) - p(A, B) \\
&\leq \frac{1}{k}p(x, \mathcal{F}x) - p(x, z) - p(A, B) \\
&\leq \frac{1}{k}(p(x, a)) - p(x, z) - p(A, B) \\
&\leq \frac{1}{k}(kp(x, z) + kp(z, a)) - p(x, z) - p(A, B) \\
&= p(x, z) + p(z, a)) - p(x, z) - p(A, B) = 0.
\end{aligned}$$

Consequently by (1), we obtain

$$
\begin{aligned}
&H_p(\mathcal{F}x,\mathcal{F}z)+p(A,B)=H_p^*(\mathcal{F}x,\mathcal{F}z)\le rM_{\mathcal{F}}(x,z)\\
&=r\max\left\{\begin{array}{l} p(x,z),p(x,\mathcal{F}x)-kp(A,B),p(z,\mathcal{F}z)-kp(A,B),\\ p(x,\mathcal{F}z)-k^2p(A,B),p(z,\mathcal{F}x)-kp(A,B)\end{array}\right\}\\
&\le r\max\left\{\begin{array}{l} p(x,z),p(x,a)-kp(A,B),p(z,\mathcal{F}z)-kp(A,B),\\ p(x,\mathcal{F}z)-k^2p(A,B),p(z,a)-kp(A,B)\end{array}\right\}\\
&\le r\max\left\{\begin{array}{l} p(x,z),kp(x,z),p(z,\mathcal{F}z)-kp(A,B),\\ p(x,\mathcal{F}z)-k^2p(A,B),(1-k)p(A,B)\end{array}\right\}\\
&\le r\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B),p(x,\mathcal{F}z)-k^2p(A,B)\}.
\end{aligned}
$$

Using (13), we obtain

$$
\begin{aligned}
&H_p(\mathcal{F}x,\mathcal{F}z)+p(A,B)\\
&\le r\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B),p(x,\mathcal{F}z)-k^2p(A,B)\}\\
&\le r\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B),p(x,\mathcal{F}z)\}\\
&\le r\max\left\{\begin{array}{l} kp(x,z),p(z,\mathcal{F}z)-kp(A,B),\\ kr\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B)\}\end{array}\right\}\\
&=r\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B)\}.
\end{aligned}
$$

Further, if

$$\max\{p(x,z),p(z,\mathcal{F}z)-p(A,B)\}=p(z,\mathcal{F}z)-kp(A,B),$$

then

$$
\begin{aligned}
H_p(\mathcal{F}x,\mathcal{F}z) &\le H_p(\mathcal{F}x,\mathcal{F}z)+p(A,B)\\
&\le r(p(z,\mathcal{F}z)-kp(A,B))\\
&\le r(kp(z,a)+kp(a,\mathcal{F}z)-kp(A,B))\\
&\le rkp(a,\mathcal{F}z)\le rkH_p(\mathcal{F}x,\mathcal{F}z)\\
&< H_p(\mathcal{F}x,\mathcal{F}z),
\end{aligned}
$$

a contradiction. Consequently we have

$$H_p(\mathcal{F}x,\mathcal{F}z)\le krp(x,z)-p(A,B). \tag{15}$$

From (13) and (15), we get

$$
\begin{aligned}
&p(x,\mathcal{F}z)\le kr\max\{kp(x,z),p(z,\mathcal{F}z)-kp(A,B)\}\\
&\le kr\max\{kp(x,z),kp(z,a)+kp(a,\mathcal{F}z)-kp(A,B)\}\\
&\le kr\max\{kp(x,z),kp(a,\mathcal{F}z)\}\\
&\le kr\max\{kp(x,z),kH_p(\mathcal{F}x,\mathcal{F}z)\}\\
&\le kr\max\{kp(x,z),k^2rp(x,z)-kp(A,B)\}\le k^2rp(x,z).
\end{aligned}
\tag{16}
$$

Now from (14)–(16), we get

$$\begin{aligned}
p(x,\mathcal{F}x) &\leq kp(x,\mathcal{F}z) + kH_p(\mathcal{F}x,\mathcal{F}z)) \\
&\leq k^3 r(p(x,z)) + k^2 r(p(x,z)) - kp(A,B) \\
&\leq \left(k^3+k^2\right) r(p(x,z)) - kp(A,B) \\
&\leq r\left(k^3+k^2\right)(kp(x,a)+kp(a,z)) - kp(A,B) \\
&< s\left(k^3+k^2\right)(kp(x,a)+kp(A,B)) - kp(A,B) \\
&= s\left(k^4+k^3\right)p(x,a) + s\left(k^4+k^3\right)p(A,B) - kp(A,B) \\
&< \begin{array}{l} p(x,\mathcal{F}x) + s\left(k^4+k^3\right)p(A,B) - p(A,B) \\ +s\left(k^4+k^3\right)p(A,B) - kp(A,B) \end{array} \\
&= p(x,\mathcal{F}x) + 2s\left(k^4+k^3\right)p(A,B) - p(A,B) - kp(A,B) \\
&< p(x,\mathcal{F}x) + 2p(A,B) - p(A,B) - kp(A,B) \leq p(x,\mathcal{F}x),
\end{aligned}$$

a contradiction. Hence, x is the BPP of $\mathcal{F}$. This completes the proof. □

Remark 2. *As best proximity point theory is a natural generalization of fixed point theory, so Theorem 5 is a natural generalization of Theorems 1–3 (compare corollaries below). Some questions arise naturally out of this work which have been mentioned in the conclusion.*

Now we give an example to explain the above result.

Example 1. *Let* $\mathcal{W} = \mathbb{R}^2$,

$$p(P,Q) = |x_1 - x_2|^2 + |y_1 - y_2|^2,$$

where $P = (x_1, y_1), Q = (x_2, y_2) \in \mathcal{W}$. *Note that* p *is the* $b-$*metric with* $k = 2$ *as* p *is the square of usual metric on* $\mathcal{W}$ *(compare [42]). Let*

$$\begin{aligned}
A &= \{(1,9^n) : n \in \mathbb{N}_1\}, \\
B &= \left\{\left(0,\frac{1}{9^n}\right) : n \in \mathbb{N}_1\right\} \cup \{(0,0)\}.
\end{aligned}$$

Note that $p(A,B) = 1$. *Define a mapping* $\mathcal{F} : A \to B$ *as*

$$\mathcal{F}(1,9^n) = \left\{\left(0,\frac{1}{9^a}\right) : 0 \leq a \leq n\right\}.$$

As

$$A_0 = \{(1,1)\} \text{ and } B_0 = \{(0,1)\},$$

so

$$\mathcal{F}(x) \subseteq B_0 \text{ for all } x \text{ in } A_0.$$

Let

$$r = \frac{1}{25} < \frac{1}{k^4+k^3},$$

and $P_1 = (1,9^{n_1})$, $P_2 = (1,9^{n_2})$ *be any two points in* A, *where* $n_2 > n_1$. *Now*

$$\mathcal{F}(P_1) = \left\{\left(0,\frac{1}{9^{n_1}}\right),\ldots,(0,1)\right\}$$

and

$$\mathcal{F}(P_2) = \left\{\left(0,\frac{1}{9^{n_2}}\right),\ldots,(0,1)\right\}.$$

It implies

$$\begin{aligned} H_p^*(\mathcal{F}(P_1),\mathcal{F}(P_2)) &= H_p(\mathcal{F}(P_1),\mathcal{F}(P_2)) + p(A,B) \\ &= \left(\frac{1}{9^{n_1}} - \frac{1}{9^{n_2}}\right)^2 + 1 \\ &= \frac{(9^{n_2-n_1}-1)^2 + (9^{n_2})^2}{(9^{n_2})^2}, \end{aligned}$$

as $n_2 - n_1 \leq n_2$, *it implies*

$$\frac{(9^{n_2-n_1}-1)^2 + (9^{n_2})^2}{(9^{n_2})^2} < 2,$$

therefore

$$H_p^*(\mathcal{F}(P_1),\mathcal{F}(P_2)) < 2. \tag{17}$$

Now, consider

$$\begin{aligned} p(P_2,\mathcal{F}(P_2)) - kp(A,B) &= p\left((1,9^{n_2}),\left\{\left(0,\frac{1}{9^{n_2}}\right),\ldots,(0,1)\right\}\right) - 2 \\ &= (1)^2 + (9^{n_2}-1)^2 - 2 \\ &= 9^{2n_2} - 2(9^{n_2}) = 9^{n_2}(9^{n_2}-2). \end{aligned}$$

It implies

$$r(p(P_2,\mathcal{F}(P_2)) - kp(A,B)) = \frac{9^{n_2}(9^{n_2}-2)}{25} > 2.$$

Consequently,

$$r(p(P_2,\mathcal{F}(P_2)) - kp(A,B)) < rM_{\mathcal{F}}(P_1,P_2),$$

so

$$rM_{\mathcal{F}}(P_1,P_2) > 2. \tag{18}$$

From (17) and (18), we get

$$H_p^*(\mathcal{F}(P_1),\mathcal{F}(P_2)) < rM_{\mathcal{F}}(P_1,P_2).$$

Hence,

$$\zeta^*(p(x,\mathcal{F}x),p(x,y)) \leq 0 \text{ implies } H_p^*(\mathcal{F}x,\mathcal{F}y) \leq rM_{\mathcal{F}}(x,y),$$

for all $x,y \in A$ *and for some* $\zeta \in \Theta$, *where* $\zeta(s,t) = \frac{s}{k} - t$. *That is,* $\mathcal{F}$ *is a generalized multivalued* $CS-$*type non-self quasi-contraction. All axioms of Theorem 5 are satisfied. There exist* $(1,1) \in A$ *which is the BPP of* $\mathcal{F}$.

Corollary 1. *Let* $(\mathcal{W},p)$ *be a complete* $b-$*metric space and* $\mathcal{F}: A \to C_B(B)$. *If*

$$p(x,\mathcal{F}x) \leq k(p(x,y) + p(A,B)) \text{ implies } H_p(\mathcal{F}x,\mathcal{F}y) \leq rM_{\mathcal{F}}(x,y) - p(A,B),$$

for all $x,y \in A$ *and for some* $r \in \left[0,\frac{1}{k^4+k^3}\right)$, *where* $A,B \in C(\mathcal{W})$. *Assume that* A_0 *is non-empty such that for each* $x \in A_0$, $\mathcal{F}x \subseteq B_0$ *and the pair* (A,B) *satisfies the weak* $P-$*property. Then* $B_{PP}(\mathcal{F})$ *is non-empty.*

Proof. Put $\zeta(s,t) = \frac{s}{k} - t$, in Theorem 5. □

Corollary 2. *Let $(\mathcal{W}, p)$ be a complete $b-$metric space and $\mathcal{F} : A \to C_B(B)$. If*

$$H_p(\mathcal{F}x, \mathcal{F}y) \leq rM_{\mathcal{F}}(x, y) - p(A, B),$$

for all $x, y \in A$ and for some $r \in \left[0, \frac{1}{k^4 + k^3}\right)$, where $A, B \in C(\mathcal{W})$. Assume that A_0 is non-empty such that for each $x \in A_0$, $\mathcal{F}x \subseteq B_0$ and the pair (A, B) satisfies the weak $P-$property. Then $B_{PP}(\mathcal{F})$ is non-empty.

If we replace multivalued mappings $\mathcal{F}$ by a single valued non-self mapping $\mathcal{F} : A \to B$ in Theorem 5, we get the following result.

Corollary 3. *Let $(\mathcal{W}, p)$ be a complete $b-$metric space, $A, B \in C(\mathcal{W})$ and $\mathcal{F} : A \to B$ a generalized $CS-$type non-self quasi-contraction. Assume that A_0 is non-empty such that for each $x \in A_0$, $\mathcal{F}x \in B_0$ and the pair (A, B) satisfies the weak $P-$property. Then, $B_{PP}(\mathcal{F})$ is singleton.*

Proof. By Theorem 5, $\mathcal{F}$ has a BPP. In order to prove the uniqueness, suppose on the contrary that u_0 and u_1 are two BPPs. Then,

$$p(u_0, \mathcal{F}u_0) = p(A, B) \text{ and } p(u_1, \mathcal{F}u_1) = p(A, B). \tag{19}$$

Now,

$$\begin{aligned} \zeta^*(p(u_0, \mathcal{F}u_0), p(u_0, u_1)) &= \zeta(p(u_0, \mathcal{F}u_0), p(u_0, u_1)) - p(A, B) \\ &\leq \frac{p(u_0, \mathcal{F}u_0)}{k} - p(u_0, u_1) - p(A, B) \\ &= \frac{p(A, B)}{k} - p(u_0, u_1) - p(A, B) \\ &\leq -p(u_0, u_1) < 0. \end{aligned}$$

Since $\mathcal{F}$ satisfies the weak $P-$property, from (19) and using the fact that $\mathcal{F}$ is a generalized $CS-$type quasi-contraction, we have

$$\begin{aligned} & p(u_0, u_1) \leq p(\mathcal{F}u_0, \mathcal{F}u_1) \leq p^*(\mathcal{F}u_0, \mathcal{F}u_1) \leq rM_{\mathcal{F}}(u_0, u_1) \\ & = r \max \left\{ \begin{array}{l} p(u_0, u_1), p(u_0, \mathcal{F}u_0) - kp(A, B), \\ p(u_1, \mathcal{F}u_1) - kp(A, B), p(u_0, \mathcal{F}u_1) - k^2 p(A, B), \\ p(u_1, \mathcal{F}u_0) - kp(A, B) \end{array} \right\} \\ & = r \max \left\{ \begin{array}{l} p(u_0, u_1), (1 - k)p(A, B), (1 - k)p(A, B), \\ kp(u_0, u_1) + kp(u_1, \mathcal{F}u_1) - k^2 p(A, B), \\ kp(u_1, u_0) + kp(u_0, \mathcal{F}u_0) - kp(A, B) \end{array} \right\} \\ & = r \max \left\{ \begin{array}{l} p(u_0, u_1), (1 - k)p(A, B), (1 - k)p(A, B), \\ kp(u_0, u_1) + (k - k^2)p(A, B), kp(u_1, u_0) \end{array} \right\} \\ & \leq rkp(u_0, u_1) < \frac{1}{k^3 + k^2} p(u_0, u_1) < p(u_0, u_1), \end{aligned}$$

a contradiction. Hence, $\mathcal{F}$ has a unique BPP. □

Corollary 4. *Let $(\mathcal{W}, p)$ be a complete $b-$metric space, $A, B \in C(\mathcal{W})$ and $\mathcal{F} : A \to B$. If*

$$\vartheta(r)p(x, \mathcal{F}x) \leq k(p(x, y) + p(A, B)) \text{ implies } p(\mathcal{F}x, \mathcal{F}y) \leq rp(x, y) - p(A, B),$$

for all $x, y \in A$ and for some $r \in \left[0, \frac{1}{k^4 + k^3}\right)$ and $\vartheta : [0, 1) \to (0, 1]$. Assume that A_0 is non-empty such that for each $x \in A_0$, $\mathcal{F}x \in B_0$ and the pair (A, B) satisfies the weak $P-$property. Then $B_{PP}(\mathcal{F})$ is singleton.

Proof. If
$$\zeta(s,t) = \frac{\vartheta(r)}{k}s - t,$$
then
$$\zeta^*(p(x,\mathcal{F}x), p(x,y)) \leq 0,$$
implies
$$\frac{\vartheta(r)}{k}p(x,\mathcal{F}x) - p(x,y) - p(A,B) \leq 0,$$
that is
$$\vartheta(r)p(x,\mathcal{F}x) \leq k(p(x,y) + p(A,B)),$$
which further implies
$$p(\mathcal{F}x,\mathcal{F}y) \leq rp(x,y) - p(A,B).$$
Consequently the result follows by Corollary 3. □

Now we derive some important results in $b-$metric fixed point theory.

Corollary 5. *Let $(\mathcal{W},p)$ be a complete $b-$metric space and $\mathcal{F}:\mathcal{W}\to C_B(\mathcal{W})$. If*
$$\zeta(p(x,\mathcal{F}x), p(x,y)) \leq 0 \text{ implies } H_p(\mathcal{F}x,\mathcal{F}y) \leq rM_{\mathcal{F}}(x,y),$$
for all $x,y\in\mathcal{W}$ and for some $r \in \left[0, \frac{1}{k^4+k^3}\right)$, then $\mathcal{F}$ has a fixed point.

Proof. Put $A = B = \mathcal{W}$ in Theorem 5. □

The following result is the generalization of Theorems 1 and 2.

Corollary 6. *Let $(\mathcal{W},p)$ be a complete $b-$metric space and $\mathcal{F}:\mathcal{W}\to C_B(\mathcal{W})$. If*
$$H_p(\mathcal{F}x,\mathcal{F}y) \leq rM_{\mathcal{F}}(x,y),$$
for all $x,y\in\mathcal{W}$ and for some $r \in \left[0, \frac{1}{k^4+k^3}\right)$, then $\mathcal{F}$ has a fixed point.

Corollary 7. *Let $(\mathcal{W},p)$ be a complete $b-$metric space and $\mathcal{F}:\mathcal{W}\to\mathcal{W}$. If*
$$\vartheta(r)p(x,\mathcal{F}x) \leq kp(x,y) \text{ implies } p(\mathcal{F}x,\mathcal{F}y) \leq rp(x,y),$$
for all $x,y\in\mathcal{W}$ and for some $r \in \left[0, \frac{1}{k^4+k^3}\right)$ and $\vartheta:[0,1)\to(0,1]$. Then $\mathcal{F}$ has a unique fixed point.

Proof. Put $A = B = \mathcal{W}$ in Corollary 4. □

Remark 3.

1. *Corollary 5 is a generalization of Theorem 3 for $0 \leq r < \frac{1}{k^4+k^3}$, which is a generalization of Theorem 2.*
2. *If in Corollary 6, we set $k = 1$, we get Theorems 2 which is a partial generalization of Theorem 1, ([43], (Corollary 3.3)) and ([44], (Theorem 3.3)).*

3. Completeness of $b-$Metric Spaces

In the following theorem, we obtain completeness of $b-$metric spaces via the BPP theorem.

Theorem 6. *Let $(\mathcal{W},p)$ be a $b-$metric space, $\vartheta:[0,1)\to(0,1]$ and $A,B\in C(\mathcal{W})$. Let $A_{r,\vartheta}$ be a class of mappings $\mathcal{F}:A\to B$ that satisfies (a)–(b)* *(a)* *for* $x, y \in A$,

$$\vartheta(r)p(x, \mathcal{F}x) \leq k(p(x,y) + p(A,B)) \text{ implies } p(\mathcal{F}x, \mathcal{F}y) \leq rp(x,y) - p(A,B), \quad (20)$$

where $r \in \left[0, \frac{1}{k^4 + k^3}\right)$.

(b) A_0 *is non-empty and for each* $x \in A_0$, $\mathcal{F}x \in B_0$ *and the pair* (A, B) *satisfies the weak* $P-$*property.*

Let $A^*_{r,\vartheta}$ *be a class of mappings* $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ *that satisfies:*

(c) *for* $x, y \in \mathcal{W}$,

$$\vartheta(r)p(x, \mathcal{F}x) \leq kp(x,y) \text{ implies } p(\mathcal{F}x, \mathcal{F}y) \leq rp(x,y), \quad (21)$$

where $r \in \left[0, \frac{1}{k^4 + k^3}\right)$.

Let $B_{r,\vartheta}$ *be a class of mappings* $\mathcal{F}$ *that satisfies (d) and*

(d) $\mathcal{F}(\mathcal{W})$ *is denumerable,*
(e) *every* $M \subseteq \mathcal{F}(\mathcal{W})$ *is closed.*

Then the statements (1)–(4) are equivalent:

1. *The* $b-$*metric space* $(\mathcal{W}, p)$ *is complete.*
2. $B_{PP}(\mathcal{F})$ *is non-empty for every mapping* $\mathcal{F} \in A_{r,\vartheta}$ *and for all* $r \in [0,1)$ *with* $r < \frac{1}{k^4 + k^3}$.
3. $F_{ix}(\mathcal{F})$ *is non-empty for every mapping* $\mathcal{F} \in A^*_{r,\vartheta}$ *and for all* $r \in [0,1)$ *with* $r < \frac{1}{k^4 + k^3}$.
4. $F_{ix}(\mathcal{F})$ *is non-empty for every mapping* $\mathcal{F} \in B_{r,\vartheta}$ *and some* $r \in [0,1)$ *with* $r < \frac{1}{k^4 + k^3}$.

Proof. By Corollary 4, (1) implies (2). For $A = B = \mathcal{W}$, $A^*_{r,\vartheta} \subseteq A_{r,\vartheta}$. Hence (2) implies (3). Since $B_{r,\vartheta} \subseteq A^*_{r,\vartheta}$, therefore, (3) implies (4). For (4) implies (1), assume on the contrary that (4) holds but $(\mathcal{W}, p)$ is incomplete. That is, there is a sequence $\{u_n\}$ which is Cauchy but does not converge. Define $g : \mathcal{W} \to [0, \infty)$ as:

$$g(x) = \limsup_{n\to\infty} p(x, u_n),$$

for $x \in \mathcal{W}$. As $\{u_n\}$ is Cauchy, so for $\epsilon = \frac{1}{k} > 0$, we can choose $m_\epsilon \in \mathbb{N}$, so that for all $n \in \mathbb{N}$,

$$p(u_{m_\epsilon}, u_{m_\epsilon+n}) \leq \frac{1}{k}.$$

Hence for all $n \in \mathbb{N}$, we get

$$p(x, u_{m_\epsilon+n}) \leq kp(x, u_{m_\epsilon}) + kp(u_{m_\epsilon}, u_{m_\epsilon+n}) \leq kp(x, u_{m_\epsilon}) + 1,$$

implies that the sequence $\{p(x, u_n)\}$ is bounded in $\mathbb{R}$ for every $x \in \mathcal{W}$. This further implies that the function g is well defined. Further, $g(x) > 0$ for all x in $\mathcal{W}$. For $\epsilon > 0$, there exists $K_\epsilon \in \mathbb{N}$ such that for all $p \in \mathbb{N}$,

$$p(u_n, u_{n+p}) < \epsilon,$$

for all $n \geq K_\epsilon$. Hence, we get for all $k \in \mathbb{N}$,

$$0 \leq g(u_m) = \limsup_{n\to\infty} p(u_m, u_{m+n}) < \epsilon,$$

for all $m \geq K_\epsilon$. That is,

$$\lim_{m\to\infty} g(u_m) = 0. \quad (22)$$

From (22), for every $x \in \mathcal{W}$, there exists a $v \in \mathbb{N}$ such that

$$g(u_v) \leq \left(\frac{r\vartheta(r)}{3k^4 + rk\vartheta(r)}\right)g(x). \tag{23}$$

If $\mathcal{F}(x) = u_v$, then

$$g(\mathcal{F}x) \leq \left(\frac{r\vartheta(r)}{3k^4 + rk\vartheta(r)}\right)g(x) \text{ and } \mathcal{F}x \in \{u_n : n \in \mathbb{N}\}, \tag{24}$$

for all $x \in \mathcal{W}$. From (24), we have $g(\mathcal{F}x) < g(x)$, hence $\mathcal{F}x \neq x$ for all $x \in \mathcal{W}$. That is, $F_{ix}(\mathcal{F})$ is empty. As $\mathcal{F}(\mathcal{W}) \subset \{u_n : n \in \mathbb{N}\}$, so (e) holds. Note that (f) holds as well. Further, g satisfies

$$\begin{array}{l} g(x) - kg(y) \leq kp(x,y) \text{ for all } x,y \in \mathcal{W}, \\ g(y) - kg(x) \leq kp(x,y) \text{ for all } x,y \in \mathcal{W}, \\ g(x) - kg(\mathcal{F}x) \leq kp(x,\mathcal{F}x) \text{ for all } x \in \mathcal{W}, \\ p(\mathcal{F}x,\mathcal{F}y) \leq kg(\mathcal{F}x) + kg(\mathcal{F}y) \text{ for all } x,y \in \mathcal{W}. \end{array}$$

Now fix $x, y \in \mathcal{W}$ such that

$$\vartheta(r)p(x,\mathcal{F}x) \leq kp(x,y).$$

We need to show that (21) holds. Observe that

$$\left\{ \begin{array}{l} p(x,y) \geq \dfrac{\vartheta(r)}{k}p(x,\mathcal{F}x) \geq \dfrac{\vartheta(r)}{k^2}(g(x) - kg(\mathcal{F}x)) \\ \geq \dfrac{\vartheta(r)}{k^2}\left(1 - \dfrac{r\vartheta(r)}{3k^3 + r\vartheta(r)}\right)g(x) = \dfrac{3k\vartheta(r)}{3k^3 + r\vartheta(r)}g(x). \end{array} \right. \tag{25}$$

We have two cases. Case (1) Suppose $g(y) \geq 2kg(x)$, then,

$$\begin{array}{rcl} p(\mathcal{F}x,\mathcal{F}y) & \leq & kg(\mathcal{F}x) + kg(\mathcal{F}y) \\ & \leq & \dfrac{r\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) + \dfrac{r\vartheta(r)}{3k^3 + r\vartheta(r)}g(y) \\ & \leq & \dfrac{r}{3k^2}(g(x) + g(y)) + \dfrac{2r}{3k^2}(g(y) - 2kg(x)) \\ & = & \dfrac{r}{3k}\left(\dfrac{1}{k}g(x) + \dfrac{1}{k}g(y) + \dfrac{2}{k}g(y) - 4g(x)\right) \\ & \leq & \dfrac{r}{3k}\left(\dfrac{3}{k}g(y) - 3g(x)\right) \\ & \leq & \dfrac{r}{k}\left(\dfrac{1}{k}g(y) - g(x)\right) \leq rp(x,y). \end{array}$$

Case (2) whenever $g(y) < 2kg(x)$, from (25)

$$\begin{array}{rcl} p(\mathcal{F}x,\mathcal{F}y) & \leq & kg(\mathcal{F}x) + kg(\mathcal{F}y) \\ & \leq & \dfrac{r\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) + \dfrac{r\vartheta(r)}{3k^3 + r\vartheta(r)}g(y) \\ & \leq & \dfrac{kr\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) + \dfrac{2kr\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) \\ & = & \dfrac{3kr\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) = r\dfrac{3k\vartheta(r)}{3k^3 + r\vartheta(r)}g(x) \leq rp(x,y). \end{array}$$

Hence,

$$\vartheta(r)p(x,\mathcal{F}x) \leq kp(x,y) \text{ implies } p(\mathcal{F}x,\mathcal{F}y) \leq rp(x,y),$$

for all $x, y \in \mathcal{W}$. From (4) $F_{ix}(\mathcal{F})$ is non-empty, a contradiction. Hence $\mathcal{W}$ is complete. $\square$

4. Conclusions

Quasi-contractions are of the utmost importance in applications as these contractions are not necessarily continuous. Such contractions have been discussed and studied in the context of fixed points but, to the best of our knowledge, these contractions have not been considered in the context of best proximity points. The best proximity point results for quasi-contractions we have proved in this article generalize fixed point results for quasi-contractions of metric and b-metric spaces. To obtain the best proximity point results for quasi-contractions of $b-$metric space, we had to employ some restrictions on the $b-$metric constant. Based on our findings, we pose some questions for future considerations as follows:

Question 01: Does the conclusion of Theorem 5 remain true for $\frac{1}{k^4+k^3} \leq r < 1$?

Question 02: Does the conclusion of Theorem 5 remain true if we replace H_p^* by H_p?

Question 03: Is it possible to extend Theorem 4 for best proximity points? Note that in Theorem 4 they used a contraction condition which is more general than the quasi-contractions used in [7,9,17].

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