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On Ulam Stability of Functional Equations in 2-Normed Spaces—A Survey

Anna Bahyrycz [1,†], Janusz Brzdęk [1,*,†], El-sayed El-hady [2,3,†] and Zbigniew Leśniak [4,†]

1 Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland; bahyrycz@agh.edu.pl
2 Mathematics Department, College of Science, Jouf University, P.O. Box 2014, Sakaka 72388, Saudi Arabia; eaelhady@ju.edu.sa or elsayed_elhady@ci.suez.edu.eg
3 Basic Science Department, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt
4 Department of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland; zbigniew.lesniak@up.krakow.pl
* Correspondence: brzdek@agh.edu.pl
† These authors contributed equally to this work.

Abstract: The theory of Ulam stability was initiated by a problem raised in 1940 by S. Ulam and concerning approximate solutions to the equation of homomorphism in groups. It is somehow connected to various other areas of investigation such as, e.g., optimization and approximation theory. Its main issue is the error that we make when replacing functions satisfying the equation approximately with exact solutions of the equation. This article is a survey of the published so far results on Ulam stability for functional equations in 2-normed spaces. We present and discuss them, pointing to the various pitfalls they contain and showing possible simple generalizations. In this way, in particular, we demonstrate that the easily noticeable symmetry between them and the analogous results obtained for the classical metric or normed spaces is in fact only apparent.

Keywords: Ulam stability; functional equation; 2-norm; n-norm

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1. Introduction

The theory of Ulam stability has drawn the attention of many researchers because of various possible applications. It is also somehow related, e.g., to some issues in optimization and to the notion of shadowing (see, e.g., in [1]). It addresses the errors we encounter when replacing functions that satisfy equations approximately by the exact solutions to them.

Every year numerous papers are published in this field, and many of them contain various gaps and mistakes. In this paper, we are reviewing the results on Ulam stability proved for function taking values in n-normed spaces. We present and discuss them, pointing to the various pitfalls they may contain and showing possible simple generalizations.

As the number of such results is very big, in this paper we focus only on the case of n-normed spaces with $n = 2$, that is, on the case of 2-normed spaces. Moreover, also in this area, we discuss in some detail only some part of such outcomes. The remaining results for n-normed spaces and 2-normed spaces will be considered in future publications. However, even from the limited number of examples that we consider in this paper, it follows that the easily noticeable and commonly expected symmetry between stability outcomes for 2-normed spaces and the analogous results obtained for classical normed spaces is in fact only apparent.

Roughly speaking, the Ulam stability concerns the following issue: how much an approximate solution to an equation differs from the exact solutions. For the first time, such a problem was formulated by Ulam in 1940 for the equation of group homomorphism, and now it is commonly known in the following form (see in [2]):

Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(x+y), f(x)+f(y)) < \delta, \qquad x, y \in G_1,$$

then a homomorphism $T : G_1 \to G_2$ exists with

$$d(f(x), T(x)) < \varepsilon, \qquad x \in G_1?$$

D.H. Hyers [2] published the subsequent partial affirmative answer to the question:

Let E, Y be Banach spaces and $\varepsilon > 0$. Then, for every $f : E \to Y$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \qquad x, y \in E,$$

the limit

$$\varphi(x) = \lim_{n \to +\infty} \frac{f(2^n x)}{2^n}$$

exists for each $x \in E$ and the function φ is the unique additive function such that

$$\|f(x) - \varphi(x)\| \leq \varepsilon, \qquad x \in E.$$

The method that was used by D.H. Hyers is called now a *direct method*. Nowadays, we often describe Hyers' result by saying simply that the Cauchy functional equation

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

is Hyers–Ulam (or Ulam–Hyers)-stable in the class of functions mapping a Banach space into a Banach space.

At present, much more sophisticated results are known in this area and we refer to the work in [3] for more details.

We should mention here that a result somewhat similar to that of Hyers was proved much earlier (in 1924) by G. Pólya and G. Szegö [4], and it reads as follows:

For every real sequence $(b_n)_{n \in N}$ with

$$\sup_{n,m \in N} |b_{n+m} - b_n - b_m| \leq 1\,,$$

there is a real number c such that

$$\sup_{n \in N} |b_n - cn| \leq 1\,.$$

Moreover,

$$c = \lim_{n \to \infty} \frac{b_n}{n}\,.$$

The result of Hyers inspired several other papers in the next years, and we refer to the work in [5] for more information on this subject. However, a new direction in this field was opened by a result, which was first proved by T. Aoki [6] and passed unnoticed by a wider audience. It became widely known only due to a paper of Th.M. Rassias [7], who independently rediscovered it to a large extent. The result reads as follows.

Theorem 1. *Assume that X is a normed space, Y is a Banach space, $\eta \geq 0$, and $0 < p < 1$. Let $f\colon X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \eta(\|x\|^p + \|y\|^p), \qquad x, y \in X\,. \tag{1}$$

Then, there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\eta\|x\|^p}{1 - 2^{p-1}}, \qquad x \in X. \tag{2}$$

Z. Gajda [8] extended the result contained in Theorem 1 for $p > 1$ and provided an example showing that for $p = 1$ it is not valid. Moreover, estimate (2) is the best possible. For $p < 0$, the analogous result is also valid, which was noticed in [9], but by now we know that actually for $p < 0$ each function $f : X \to Y$ satisfying (1) (of course only for $x \neq 0$ and $y \neq 0$) must be additive and completeness of Y is not necessary in such situation (see [3] for more details). Very recently, much more precise results, but only for functions taking real values, have been obtained in [10] by applying the Banach limit technique (see also in [11] for the application of that technique in the stability of functional equations in a single variable). For instance, the following has been proved in ([10], Theorem 8) (see also ([10], Remark 7)).

Theorem 2. *Let X be a normed space, $X_0 := X \setminus \{0\}$, $\varepsilon \geq 0$, $p, \chi, \rho \in \mathbb{R}$ (the set of real numbers), $p \neq 1$, and $\chi \leq \rho$. Assume that $f\colon X \to \mathbb{R}$ satisfies the inequality*

$$\chi(\|x\|^p + \|y\|^p) \leq f(x+y) - f(x) - f(y) \leq \rho(\|x\|^p + \|y\|^p), \qquad x, y \in X_0. \tag{3}$$

Then, there exists a unique additive $T\colon X \to \mathbb{R}$ such that, in the case $p < 1$,

$$\frac{\chi}{1 - 2^{p-1}}\|x\|^p \leq T(x) - f(x) \leq \frac{\rho}{1 - 2^{p-1}}\|x\|^p, \qquad x \in X_0, \tag{4}$$

and, in the case $p > 1$,

$$\frac{\chi}{2^{p-1} - 1}\|x\|^p \leq f(x) - T(x) \leq \frac{\rho}{2^{p-1} - 1}\|x\|^p, \qquad x \in X_0. \tag{5}$$

Moreover, if f is continuous at a point, then T is continuous.

The situation of the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \eta\|x\|^p\|y\|^q, \qquad x, y \in X \setminus \{0\}, \tag{6}$$

with $p, q \in \mathbb{R}$, was considered in [12,13] (see also in [14]). Later, more general approaches have appeared with the stability inequalities (1) and (6) replaced by a more general condition:

$$\|f(x+y) - f(x) - f(y)\| \leq \phi(x, y), \qquad x, y \in X. \tag{7}$$

Furthermore, stability for various other functional equations have also been considered with many possible kinds of the stability inequalities (see in [3,5,15,16] for more information and further references). Roughly speaking (see, e.g., in [17]), we can say that a functional equation is stable in the sense of Ulam, in some class of functions, if any function from that class, satisfying the equation approximately (in some sense), is near (in some way) to an exact solution of the equation.

The direct method used by Hyers in [2] has been successfully applied for study of the stability of large variety of equations, but unfortunately (see [18]) it does not work in numerous significant cases. The second most popular technique of proving the stability of functional equations is the fixed point method (see in [19,20] for more details). The other methods are shortly presented in [3].

The concepts of an approximate solution and nearness of two functions can be understood in various (also nonstandard) ways. This depends on the needs of a particular situation and tools that are available for us. One of such non-classical ways of measuring a distance can be provided by the notions of 2-norms and n-norms (we provide necessary details on this subject in the next section). In this paper, we discuss the Ulam stability results that are somehow related to those notions.

For more information on functional equations and their applications we refer to the works in [21–25].

Through the article, we use $\mathbb{R}$ to denote the set of reals, $\mathbb{R}_+$ the set of nonnegative reals, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $\mathbb{Q}$ to denote the set of rational numbers, $\mathbb{N}$ to denote the set of positive integers, $\mathbb{N}_0$ to denote $\mathbb{N} \cup \{0\}$, and A^B denotes the family of all functions mapping a set B into a set A.

The article is organized as follows. In the next section, we provide auxiliary information on n-normed spaces (which includes also the 2-normed spaces), and in Section 3 we proof some simple, but general results on stability of some functional equations, which are very useful in further parts of the paper. The main survey of different stability results of functional equations in 2-Banach spaces is included in Section 4. In Section 5, we give some final remarks.

2. Auxiliary Information on n-Normed Spaces

The notion of n-normed spaces was introduced by A. Misiak (see [26]). In this way, he generalized an earlier concept of 2-normed spaces (i.e., n-normed spaces with $n = 2$) defined by S. Gähler [27,28].

Below we recall some basic definitions and facts concerning such spaces (for more details we refer to the works in [26,29–32]).

Let $n \in \mathbb{N}$ and $\mathfrak{L}$ be a real linear space, which is at least $n-$dimensional. If $\|\cdot, \dots, \cdot\|$ is a function mapping $\mathfrak{L}^n$ into $\mathbb{R}$ such that, for every $\beta \in \mathbb{R}$ and $k_1, k_2, l_1, \dots, l_n \in \mathfrak{L}$, the following four conditions are valid:

(N1) $\|l_1, \dots, l_n\| = 0$ if and only if vectors $l_1, \dots, l_n$ are linearly dependent;

(N2) $\|l_1, \dots, l_n\|$ is invariant under permutation of $l_1, \dots, l_n$;

(N3) $\|\beta l_1, \dots, l_n\| = |\beta| \|l_1, \dots, l_n\|$;

(N4) $\|k_1 + k_2, l_2, \dots, l_n\| \leq \|k_1, l_2, \dots, l_n\| + \|k_2, l_2, \dots, l_n\|$,

then $\|\cdot, \dots, \cdot\|$ is called an n-norm on $\mathfrak{L}$, and the pair $(\mathfrak{L}, \|\cdot, \dots, \cdot\|)$ is said to be an n-normed space.

If $n \in \mathbb{N}$, $n > 1$, and $(\mathfrak{L}, < \cdot, \cdot >)$ is a real inner product space (and, as before, $\mathfrak{L}$ is at least n-dimensional), then an n-norm on $\mathfrak{L}$ can be defined by the formula

$$\|l_1, \dots, l_n\|_S = \mathrm{abs}\left(\begin{vmatrix} < l_1, l_1 > & < l_1, l_2 > & \dots & < l_1, l_n > \\ \vdots & \vdots & \ddots & \vdots \\ < l_n, l_1 > & < l_n, l_2 > & \dots & < l_n, l_n > \end{vmatrix} \right)^{1/2}$$

for $l_1, \dots, l_n \in \mathfrak{L}$. Here, for every real number a, $\mathrm{abs}(a)$ denotes the module (absolute value) of a.

In the case $\mathfrak{L} = \mathbb{R}^n$ (with the usual inner product), this n-norm also can be expressed by

$$\|l_1, \dots, l_n\|_S = |\det(l_{ij})|, \qquad l_i = (l_{i1}, \dots, l_{in}) \in \mathbb{R}^n, i \in 1, \dots, n,$$

where

$$\det(l_{ij}) = \begin{vmatrix} l_{11} & l_{12} & \dots & l_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{vmatrix}.$$

Then it is called the Euclidean n-norm on $\mathbb{R}^n$ and often denoted by $\|\cdot, \dots, \cdot\|_E$.

Remark 1. *H. Gunawan and M. Mashadi [30] demonstrated that from every n-norm we can obtain an $(n-1)$-norm and finally also a norm. For a somewhat analogous observation, we refer to ([33], Remark 2).*

In what follows, to simplify some formulas, we write

$$\|x,z\| := \|x,z_1,\dots,z_{n-1}\|, \qquad x \in \mathfrak{L}, z = (z_1,\dots,z_{n-1}) \in \mathfrak{L}^{n-1}.$$

We also have the following definitions and properties.

Definition 1. *Let $(l_k)_{k\in\mathbb{N}}$ be a sequence of elements of an n-normed space $(\mathfrak{L}, \|\cdot,\dots,\cdot\|)$. Then we say that $(l_k)_{k\in\mathbb{N}}$ is a* Cauchy sequence *if*

$$\lim_{m,k\to\infty} \|l_m - l_k, y\| = 0, \qquad y \in \mathfrak{L}^{n-1}.$$

The sequence $(l_k)_{k\in\mathbb{N}}$ is convergent *if there is $l \in \mathfrak{L}$ (called the limit of $(l_k)_{k\in\mathbb{N}}$) with*

$$\lim_{k\to\infty} \|l_k - l, y\| = 0, \qquad y \in \mathfrak{L}^{n-1}.$$

Such a limit is unique; we denote it by $\lim_{k\to\infty} l_k$ and so we write $l = \lim_{k\to\infty} l_k$.

An n-normed space is an n-Banach space if every Cauchy sequence in it is convergent. Moreover, we have the following property stated in [32] (see also in [29]).

Lemma 1. *Let $(\mathfrak{L}, \|\cdot,\dots,\cdot\|)$ be an n-normed space. Then, the following four conditions hold:*

(i) *if $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence of elements of $\mathfrak{L}$, then*

$$\lim_{k\to\infty} \|x_k, y\| = \Big\|\lim_{k\to\infty} x_k, y\Big\|, \qquad y \in \mathfrak{L}^{n-1};$$

(ii) *if $x, z \in \mathfrak{L}$ and $y \in \mathfrak{L}^{n-1}$, then*

$$\big|\|x,y\| - \|z,y\|\big| \le \|x - z, y\|;$$

(iii) *if $x \in \mathfrak{L}$ and*

$$\|x,y\| = 0, \qquad y \in \mathfrak{L}^{n-1},$$

then $x = 0$.

The notion of Ulam stability in n-normed spaces can be understood as in the subsequent definition (A^B stands for the family of all mappings from a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 2. *Let $(\mathfrak{L}, \|\cdot,\dots,\cdot\|)$ be an n-normed space, D be a nonempty set, $\mathcal{U}_0 \subset \mathcal{U} \subset \mathfrak{L}^D$ and $\mathcal{E} \subset \mathbb{R}_+{}^{D\times\mathfrak{L}^{n-1}}$ be nonempty, $\mathcal{S} : \mathcal{E} \to \mathbb{R}_+{}^{D\times\mathfrak{L}^{n-1}}$ and $\mathcal{F} : \mathcal{U} \to \mathfrak{L}^D$. The equation*

$$\mathcal{F}(\phi) = \phi$$

is $\mathcal{F}$-stable in $\mathcal{U}_0$ provided, for any $\phi \in \mathcal{U}_0$ and $\delta \in \mathcal{E}$ with

$$\big\|(\mathcal{F}\phi)(s) - \phi(s), z\big\| \le \delta(s,z), \qquad s \in D, z \in \mathfrak{L}^{n-1},$$

there is a solution $\psi \in \mathcal{U}$ of the equation with

$$\big\|\phi(s) - \psi(s), z\big\| \le (\mathcal{S}\delta)(s,z), \qquad s \in D, z \in \mathfrak{L}^{n-1}.$$

For further information on n-normed spaces refer to the works in [29,31,32,34–38].

3. Preliminary Stability Results

In this section, we show two very simple stability results. We assume that n is a positive integer and $(\mathcal{N}, \|\cdot,\ldots,\cdot\|_{\mathcal{N}})$ is an n-normed space.

Proposition 1. *Let $m \in \mathbb{N}$, T, and $W_1,\ldots,W_m, X_1,\ldots,X_m$ be nonempty sets. If $F : W_1 \times \ldots \times W_m \to \mathcal{N}$, $\xi_i : T \to X_i$, and $g_i : X_i \to W_i$, for $i = 1,\ldots,m$, are such that*

$$\big\|F\big(g_1(\xi_1(t)),\ldots,g_m(\xi_m(t))\big), z\big\|_{\mathcal{N}} \le \varphi(t), \qquad t \in T, z \in \mathcal{N}^{n-1}, \tag{8}$$

with some function $\varphi\colon T \to [0,\infty)$, then

$$F\big(g_1(\xi_1(t)),\ldots,g_m(\xi_m(t))\big) = 0, \qquad t \in T. \tag{9}$$

Proof. Fix $t \in T$ and $z \in \mathcal{N}^{n-1}$. Then,

$$\big\|F\big(g_1(\xi_1(t)),\ldots,g_m(\xi_m(t))\big), lz\big\|_{\mathcal{N}} \le \varphi(t), \qquad l \in \mathbb{N},$$

and consequently

$$\big\|F\big(g_1(\xi_1(t)),\ldots,g_m(\xi_m(t))\big), z\big\|_{\mathcal{N}} \le \frac{1}{l^{n-1}}\varphi(t), \qquad l \in \mathbb{N}.$$

Therefore, letting $l \to \infty$, we get $\big\|F\big(g_1(\xi_1(t)),\ldots,g_m(\xi_m(t))\big), z\big\|_{\mathcal{N}} = 0$ for every $t \in T$ and $z \in \mathcal{N}^{n-1}$, which means that (9) holds (see Lemma 1 (iii)). □

The proposition yields at once the following corollary.

Corollary 1. *Let E be a linear space over a field $\mathbb{K}$, $k, m \in \mathbb{N}$, $a_{ij} \in \mathbb{K}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,k$, $A_1,\ldots,A_m \in \mathbb{R}$, $\varphi\colon E^k \to [0,\infty)$, and let $g_1,\ldots,g_m : E \to \mathcal{N}$ be mappings satisfying*

$$\Big\|\sum_{i=1}^{m} A_i g_i\Big(\sum_{j=1}^{k} a_{ij}x_j\Big), z\Big\|_{\mathcal{N}} \le \varphi(x_1,\ldots,x_k), \qquad x_1,\ldots,x_k \in E, z \in \mathcal{N}^{n-1}. \tag{10}$$

Then,

$$\sum_{i=1}^{m} A_i g_i\Big(\sum_{j=1}^{k} a_{ij}x_j\Big) = 0, \qquad x_1,\ldots,x_k \in E. \tag{11}$$

Proof. It is enough to apply Proposition 1 with $T = E^k$, $X_i = E$, $W_i = \mathcal{N}$,

$$\xi_i(x_1,\ldots,x_k) = \sum_{j=1}^{k} a_{ij}x_j, \qquad x_1,\ldots,x_k \in E, i = 1,\ldots,m,$$

and $F(w_1,\ldots,w_m) = \sum_{i=1}^{m} A_i w_i$ for $w_1,\ldots,w_m \in \mathcal{N}$. □

Note that Equation (11) is a generalized (the so-called pexiderized) version of the functional equation

$$\sum_{i=1}^{m} A_i g\Big(\sum_{j=1}^{k} a_{ij}x_j\Big) = 0, \qquad x_1,\ldots,x_k \in E, \tag{12}$$

the stability of which was studied, e.g., in [39–42]. Particular cases of (12) are the Cauchy functional equation

$$f(x+y) = f(x) + f(y),$$

the Jensen functional equation

$$2f(x+y) = f(2x) + f(2y),$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

For more information on those (and many other) functional equations, we refer to the works in [21,22,24]. In the next sections, we will discuss stability of them as well as of some other particular cases of (11).

Finally, note that in a way similar as in Proposition 1, we also can easily extend (to the case of n-normed spaces) the stability results presented, e.g., in [43–47].

4. Ulam Stability in 2-Normed Spaces

In this section, we survey results on stability in 2-normed spaces. To shorten the statements of some theorems, we always assume that $(G,+)$ is a groupoid (i.e., G is a nonempty set endowed with a binary operation $+ : G^2 \to G$, which is not necessarily commutative), $(X, \|\cdot,\cdot\|)$ and $(Y, \|\cdot,\cdot\|_Y)$ are 2-normed spaces, $(B, \|\cdot,\cdot\|_B)$ is a 2-Banach space, $n \in \mathbb{N}$, $n > 1$, and $(\mathcal{N}, \|\cdot,\ldots,\cdot\|_{\mathcal{N}})$ is an n-normed space.

To the best of our knowledge, the first stability results involving the notion of 2-normed spaces have been given in [48]. We present them in the form of the subsequent two theorems. The first one (Theorem 2.1 in [48]) reads as follows.

Theorem 3. *Let the dimensions of X and B be greater than 2. If $f : X \to B$ is such that*

$$\|f(x+y) - f(x) - f(y), z\|_B \le c, \qquad x, y \in X, z \in B, \tag{13}$$

with some $c > 0$, then there is a unique additive mapping $h : X \to B$ such that

$$\|f(x) - h(x), z\|_B \le c, \qquad x \in X, z \in B. \tag{14}$$

Moreover, if the mapping $\mathbb{R} \ni t \to f(tx)$ is continuous for each $x \in X$, then h is linear.

In this theorem (and in the rest of this section) a mapping $h : X \to B$ is said to be additive if fulfills the Cauchy functional equation

$$h(x+y) = h(x) + h(y), \qquad x, y \in X. \tag{15}$$

(Analogously for mappings $f : G \to \mathcal{N}$.) A mapping $h : X \to B$ is linear if (as usual) it is additive and $h(ax) = ah(x)$ for every $x \in X$ and $a \in \mathbb{R}$.

Note that our Corollary 1 yields the following generalization of Theorem 3.

Proposition 2. *If $f : G \to \mathcal{N}$ is such that*

$$\|f(x+y) - f(x) - f(y), z\|_{\mathcal{N}} \le c, \qquad x, y \in G, z \in \mathcal{N}^{n-1}, \tag{16}$$

with some $c > 0$, then f is additive.

Note that at first glance, Theorem 3 seems to show that the situation in 2-normed spaces is analogous as in the result of Hyers [2], but from Proposition 2 we see that it is not, because for functions that take values in the 2-normed space, we actually get that functions satisfying (16) must be additive, i.e., we obtain hyperstability (see in [49] for more information on this notion). Later in this article, we show that a similar lack of symmetry between the results on Ulam stability in 2-normed spaces and classical normed spaces occurs for several other equations.

If G is a real linear space, then we can add in Proposition 2 the following statement: *Moreover, if the mapping* $\mathbb{R} \ni t \to f(tx)$ *is continuous for each* $x \in G$*, then* f *is linear*. The next remark shows how we can derive it from the properties of additive mappings, if we understand the notion of continuity as usual in n-normed spaces (in [48] it has not been explained).

Remark 2. Let E be a real linear space and $f : E \to \mathcal{N}$ be additive. Assume that the mapping $\mathbb{R} \ni t \to f(tx)$ is continuous for each $x \in E$, i.e.,

$$\lim_{n\to\infty} f(t_n x) = f(tx) \quad (\text{in } \mathcal{N}) \tag{17}$$

for every $x \in E$, every $t \in \mathbb{R}$, and every sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $t = \lim_{n\to\infty} t_n$ (with regard to the usual topology in $\mathbb{R}$).

We show that then $f(ax) = af(x)$ for every $a \in \mathbb{R}$ and $x \in E$. Therefore, fix $x \in E$, $x \neq 0$, and $a \in \mathbb{R}$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers with $a = \lim_{n\to\infty} q_n$ (in $\mathbb{R}$). Then,

$$\lim_{n\to\infty} \|q_n f(x) - af(x), z\|_{\mathcal{N}} = \lim_{n\to\infty} \big(|q_n - a| \, \|f(x), z\|_{\mathcal{N}}\big) = 0, \qquad z \in \mathcal{N}^{n-1},$$

which means that $\lim_{n\to\infty}(q_n f(x)) = af(x)$ (in $\mathcal{N}$). Further, in view of the additivity of f, we have $f(q_n x) = q_n f(x)$ for every $n \in \mathbb{N}$. Consequently, by (17) (with $t = a$ and $t_n = q_n$) and Lemma 1 (i),

$$\begin{aligned}
\Big\| f(ax) - \lim_{n\to\infty} q_n f(x), z \Big\|_{\mathcal{N}} &= \Big\| \lim_{n\to\infty} f(q_n x) - \lim_{n\to\infty} q_n f(x)), z \Big\|_{\mathcal{N}} \\
&= \Big\| \lim_{n\to\infty} \big(f(q_n x) - q_n f(x)\big), z \Big\|_{\mathcal{N}} \\
&= \lim_{n\to\infty} \Big\| f(q_n x) - q_n f(x), z \Big\|_{\mathcal{N}} = 0, \qquad z \in \mathcal{N}^{n-1},
\end{aligned}$$

whence

$$f(ax) = \lim_{n\to\infty} q_n f(x) = af(x).$$

The second main result in [48] (Theorem 2.2) can be formulated as follows.

Theorem 4. *Let* $f : X \to B$ *be a surjective mapping and the dimensions of* X *and* B *be greater than 2. If there exist* $c > 0$ *and* $r \in \mathbb{R}$*,* $r \neq 1$*, such that*

$$\begin{aligned}\|f(x+y) - f(x) - f(y), f(z)\|_B \le c\big(\|x,z\|^r + \|y,z\|^r\big),& \\ x,y,z \in X,&\end{aligned} \tag{18}$$

then there exists a unique additive mapping $h : X \to B$ *such that*

$$\|f(x) - h(x), f(z)\|_B \le \frac{2c}{|2 - 2^r|} \, \|x,z\|^r, \qquad x,z \in X. \tag{19}$$

Unfortunately, the author does not explain how to understand $\|x,z\|^r$ when $r < 0$ and $\|x,z\| = 0$ (which happens for linearly-dependent x and z). Therefore, it is natural to presume that actually (18) should be in the case $r < 0$ formally rewritten as

$$\begin{aligned}\|f(x+y) - f(x) - f(y), f(z)\|_B \le c\big(\|x,z\|^r + \|y,z\|^r\big),& \\ x,y,z \in X, \|x,z\| \, \|y,z\| \neq 0.&\end{aligned} \tag{20}$$

For the same reason, for $r < 0$, (19) should be

$$\|f(x) - h(x), f(z)\|_B \leq \frac{2c}{|2 - 2^r|} \|x, z\|^r, \qquad x, z \in X, \|x, z\| \neq 0. \tag{21}$$

The proof of Theorem 2.2 in [48] should be modified accordingly.

The main results in [50] can be stated as in the following theorem.

Theorem 5. *Let U be a normed space, $\eta, \theta, p, q \in \mathbb{R}_+$, and $f : U \to B$. The following three statements are valid.*

(i) If $p \neq 1 \neq q$ and

$$\|f(x + y) - f(x) - f(y), z\|_B \leq \eta\|x\|^p + \theta\|y\|^q, \qquad x, y \in U, z \in B, \tag{22}$$

then there is a unique additive mapping $A : U \to B$ such that

$$\|f(x) - A(x), z\|_B \leq \frac{\eta\|x\|^p}{|2 - 2^p|} + \frac{\theta\|x\|^q}{|2 - 2^q|}, \qquad x \in U, z \in B. \tag{23}$$

(ii) If $p \neq 1 \neq q$ and

$$\left\|2f\left(\frac{x + y}{2}\right) - f(x) - f(y), z\right\|_B \leq \eta\|x\|^p + \theta\|y\|^q, \qquad x, y \in U, z \in B, \tag{24}$$

then there is a unique mapping $J : U \to B$ such that

$$2J\left(\frac{x + y}{2}\right) = J(x) + J(y), \qquad x, y \in U, \tag{25}$$

$$\|f(x) - J(x), z\|_B \leq \frac{2\eta}{|3 - 3^p|}\|x\|^p + \theta\frac{1 + 3^q}{|3 - 3^q|}\|x\|^q, \qquad x \in U, z \in B. \tag{26}$$

(iii) If $p \neq 2 \neq q$ and

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), z\|_B \leq \eta\|x\|^p + \theta\|y\|^q, \qquad x, y \in U, z \in B, \tag{27}$$

then there is a unique mapping $Q : U \to B$ such that

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \qquad x, y \in U, \tag{28}$$

$$\|f(x) - Q(x), z\|_B \leq \frac{\eta\|x\|^p}{|4 - 2^p|} + \frac{\theta\|x\|^q}{|4 - 2^q|} + \frac{1}{3}\|f(0), z\|, \qquad x \in U, z \in B. \tag{29}$$

However, from Corollary 1 we can obtain the following generalizations of those results.

Proposition 3. *Let U be a normed space, $\eta, \theta, p, q \in \mathbb{R}_+$, and $f : U \to Y$. The following three statements are valid.*

(i) If (22) holds with B replaced by Y, then f is additive.

(ii) If (24) holds with B replaced by Y, then f is a solution to Equation (25).

(iii) If (27) holds with B replaced by Y, then f is a solution to Equation (28).

In [51], the inequalities (22), (24), and (27) have also been considered, but with the function $\eta\|x\|^p + \theta\|y\|^q$ replaced by $\theta\|x\|^p\|y\|^q$. Clearly, from Corollary 1 we can deduce analogous (as in Proposition 3) generalizations of those results. Here, we also see the lack of symmetry between the results on Ulam stability in 2-normed spaces and classical normed spaces (cf., e.g., Theorem 1 and comments after it), contrary to what Theorem 5 suggests.

In [52], the stability of the quartic functional equation

$$q(2x+y) + q(2x-y) = 4q(x+y) + 4q(x-y) + 24q(x) - 6q(y) \tag{30}$$

has been investigated in 2-Banach spaces. The main result provided there can be written as follows.

Theorem 6. *Let U be a normed space and $\varphi : U^2 \to \mathbb{R}_+$ be such that*

$$\tilde{\varphi}(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^{4k}} \varphi(2^k x, 2^k y) < \infty, \qquad x, y \in U, \tag{31}$$

$$\lim_{n\to\infty} \frac{1}{2^{4n}} \varphi(2^n x, 2^n y) = 0, \qquad x, y \in U. \tag{32}$$

Suppose that $f : U \to B$ is such that

$$\begin{aligned} \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y), z\|_B \\ \le \varphi(x,y), \qquad x, y \in U, z \in B. \end{aligned} \tag{33}$$

Then, there exists a unique solution $q : U \to B$ of (30) such that

$$\|f(x) - q(x), z\|_B \le \frac{1}{32}\tilde{\varphi}(x,0), \qquad x \in U, z \in B.$$

Note that actually (32) results from (31), so (32) is superfluous, but this does not matter, as again, with Corollary 1 we get the following generalization of Theorem 6.

Proposition 4. *Let $(U,+)$ be a group, $\varphi : U^2 \to \mathbb{R}_+$ and $f : U \to Y$ be such that (33) holds. Then, f is a solution to (30).*

The stability of the functional equation

$$f\Big(lx_0 + \sum_{i=1}^{l} x_i\Big) = \sum_{i=1}^{l} f(x_0 + x_i) \tag{34}$$

with a fixed $l \in \mathbb{N}$, $l > 1$, has been studied in [53]. We guess (because of many confusing misprints in the paper) that the main result can be rewritten as follows.

Theorem 7. *Let $j \in \{-1, 1\}$ and $\alpha : X^{l+1} \to \mathbb{R}_+$ be such that*

$$\sum_{m=0}^{\infty} \frac{\alpha\Big(l^{mj}x_0, l^{mj}x_1, \ldots, l^{mj}x_l\Big)}{l^{mj}} < \infty, \qquad x_0, x_1, \ldots, x_l \in X.$$

Suppose that a function $f : X \to B$ satisfies

$$\begin{aligned} \Big\|f\Big(lx_0 + \sum_{i=1}^{l} x_i\Big) - \sum_{i=1}^{l} f(x_0 + x_i), u\Big\|_B \le \alpha(x_0, x_1, \ldots, x_l), \\ x_0, x_1, \ldots, x_l \in X, u \in B. \end{aligned} \tag{35}$$

Then, there exists a unique additive function $A : X \to B$ such that

$$\|f(x) - A(x), u\|_B \le \frac{1}{l} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(l^{kj}x, 0, \ldots, 0\right)}{l^{kj}}, \qquad x \in X, u \in B.$$

The mapping $A(x)$ is defined by

$$A(x) := \lim_{m\to\infty} \frac{f(l^{mj}x)}{l^{mj}}, \qquad x \in X.$$

The author also considers the equation

$$g\left(lx_0 - \sum_{i=1}^{l} x_i\right) = \sum_{i=1}^{l} g(x_0 - x_i), \tag{36}$$

but this is not necessary, as (36) with x_i replaced by $-x_i$ actually becomes (34).

Note that Theorem 3.1 in [53] (the proof of the theorem has some drawbacks, but it is true) and our Corollary 1 imply the following result.

Proposition 5. *Let $\alpha : X^{l+1} \to \mathbb{R}_+$ and $f : X \to Y$ satisfy*

$$\left\| f\left(lx_0 + \sum_{i=1}^{l} x_i\right) - \sum_{i=1}^{l} f(x_0 + x_i), u\right\|_Y \le \alpha(x_0, x_1, \ldots, x_l), \tag{37}$$
$$x_0, x_1, \ldots, x_l \in X, u \in Y.$$

Then, f is a solution of Equation (34), whence (in view of ([53], Theorem 3.1)) it is additive.

From Proposition 5, it follows that Examples 5.1 and 5.2 in [53] are not correct (the main problem is probably that they are constructed for $X = \mathbb{R}$, which has dimension 1 and therefore does not admit any nontrivial 2-norm).

A very similar situation, as in [53], is with all stability results and examples in [54], where the authors have investigated the stability of the functional equation

$$\begin{aligned} &11\ [g(u + 2v + 2w) + g(u - 2v - 2w)] + 66g(u) + 48g(2v + 2w) \\ &\quad = 44[g(u + v + w) + g(u - v - w)] + 12g(3v + 3w) + 60g(v + w). \end{aligned} \tag{38}$$

From Corollary 1, we obtain the subsequent corollary, which generalizes the main stability outcomes contained in Theorems 3.1.1–3.1.5 in [54] in the following way.

Proposition 6. *Let $(G, +)$ be a group, $\alpha : G^3 \to \mathbb{R}_+$, $f : G \to \mathcal{N}$, and*

$$\begin{aligned} Df(u, v, w) := {}& 11[g(u + 2v + 2w) + g(u - 2v - 2w)] + 66g(u) \\ &+ 48g(2v + 2w) - 44[g(u + v + w) + g(u - v - w)] \\ &+ 12g(3v + 3w) + 60g(v + w), \qquad u, v, w \in G. \end{aligned}$$

If

$$\left\|Df(u, v, w), z\right\|_{\mathcal{N}} \le \alpha(u, v, w), \qquad u, v, w \in G, z \in \mathcal{N}^{n-1}, \tag{39}$$

then f is a solution to Equation (38).

A description of solutions to Equation (38) is given in Theorem 3.1 in [54].

In [55], a fixed point method has been used to investigate in 2-Banach spaces the stability of the functional equation

$$\begin{aligned} f(2x+y\,,2z+w)&+f(2x-y,2z+w)+f(2x+y,2z-w) \\ &= -f(2x-y,2z-w)+4f(x+y,z+w)+4f(x+y,z-w) \\ &\quad +24f(x+y,z)+4f(x-y,z+w)+4f(x-y,z-w) \\ &\quad +24f(x-y,z)+24f(x,z+w)+24f(x,z-w)+144f(x,z). \end{aligned} \tag{40}$$

Corollary 1 yields the following generalization of the main result in [55].

Theorem 8. *Let $(G,+)$ be a group and $\varphi : G^4 \to [0,\infty)$. If $f : G^2 \to \mathcal{N}$ satisfies*

$$\|D_f(x,y,z,w),t\|_{\mathcal{N}} \le \varphi(x,y,z,w), \qquad x,y,z,w \in G, t \in \mathcal{N}^{n-1},$$

where

$$\begin{aligned} D_f(x,y,z,w) := {} & f(2x+y,2z+w)+f(2x-y,2z+w) \\ & +f(2x+y,2z-w)+f(2x-y,2z-w) \\ & -4f(x+y,z+w)-4f(x+y,z-w)-24f(x+y,z) \\ & -4f(x-y,z+w)-4f(x-y,z-w)-24f(x-y,z) \\ & -24f(x,z+w)-24f(x,z-w)-144f(x,z), \end{aligned}$$

then f is a solution to Equation (40).

In [56], the author studied in 2-Banach spaces the stability of the functional equation

$$f(x_{11}+x_{12},\cdots,x_{n1}+x_{n2}) = \sum_{i_1,\cdots,i_n\in\{1,2\}} f(x_{1i_1},\cdots,x_{ni_n}). \tag{41}$$

The main stability result in [56] can be written as follows.

Theorem 9. *Let $(V,+)$ be a commutative semigroup with an identity element. Assume also that $\varphi : V^{2n} \to \mathbb{R}_+$ is a function such that for any $(x_{11},x_{12},\cdots,x_{n1},x_{n2}) \in V^{2n}$ we have*

$$\begin{aligned} &\tilde{\varphi}(x_{11}\,,x_{12},\cdots,x_{n1},x_{n2}) \\ &\qquad := \sum_{j=0}^{\infty} \frac{1}{2^{n(j+1)}} \varphi(2^j x_{11},2^j x_{12},\cdots,2^j x_{n1},2^j x_{n2}) < \infty. \end{aligned} \tag{42}$$

If $f : V^n \to B$ is a mapping satisfying

$$\begin{aligned} \Big\| f(x_{11}+x_{12},\cdots,x_{n1}+x_{n2}) - \sum_{i_1,\cdots,i_n\in\{1,2\}} f(x_{1i_1},\cdots,x_{ni_n}),y \Big\|_B & \\ \le \varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2}),& \end{aligned} \tag{43}$$

for $(x_{11},x_{12},\cdots,x_{n1},x_{n2}) \in V^{2n}$, $y \in Y$, then there exists a unique mapping $F : V^n \to B$ satisfying Equation (41) such that

$$\|f(x_{11},\cdots,x_{n1}) - F(x_{11},\cdots,x_{n1}),y\|_B \le \tilde{\varphi}(x_{11},x_{11},\cdots,x_{n1},x_{n1}),$$

for $(x_{11},\cdots,x_{n1}) \in V^n$ and $y \in B$. The mapping F is given by

$$F(x_{11},\cdots,x_{n1}) :\, = \lim_{j\to\infty} \frac{1}{2^{nj}} f(2^j x_{11},\cdots,2^j x_{n1}), \tag{44}$$

for $(x_{11},\cdots,x_{n1}) \in V^n$.

In view of Proposition 1, the theorem can be easily generalized in the following way.

Proposition 7. *Let $(V, +)$ be a semigroup, $\varphi : V^{2n} \to \mathbb{R}_+$, and $f : V^n \to Y$ be a mapping satisfying the inequality*

$$\Big\| f(x_{11} + x_{12}, \cdots, x_{n1} + x_{n2}) - \sum_{i_1,\cdots,i_n \in \{1,2\}} f(x_{1i_1}, \cdots, x_{ni_n}), y \Big\|_Y \leq \varphi(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}),$$

for all $(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) \in V^{2n}$ and all $y \in Y$. Then, f is a solution to Equation (41).

In the analogous way, we can generalized the theorems from [57], where the stability of equations somewhat similar to (41) has been investigated in 2-Banach spaces.

In [58], the author studied (in 2-Banach spaces) the stability of Cauchy, Jensen, and quadratic functional equations, with the control function $\varphi(x, y) = c\|x\|^p\|y\|^q\|z\|^r$ for $x, y, z \in U$, where $(U, \|\cdot\|)$ is a normed space, $c \in \mathbb{R}_+$, and $p, q, r \in (0, \infty)$. Unfortunately, there is a confusion there, because the function φ should rather have the form $\varphi(x, y) = c\|x\|^p\|y\|^q\|z\|_0^r$ for $x, y \in U$ and $z \in Y$, where $\|\cdot\|_0$ is a norm in Y (see Remark 1). These results with such modified form of φ can be collected in the form of the subsequent theorem.

Theorem 10. *Let U be a normed space, $\|\cdot\|_0$ be a norm in Y, $f : U \to Y$, $p, q, r \in (0, \infty)$ and $c \in \mathbb{R}_+$. Then, the following three statements are valid.*

(i) If $p + q \neq 1$ and

$$\|f(x + y) - f(x) - f(y), z\|_Y \leq c\|x\|^p\|y\|^q\|z\|_0^r, \qquad x, y \in U, z \in Y,$$

then there is a unique additive mapping $A : U \to Y$ such that

$$\|f(x) - A(x), z\|_Y \leq \frac{c\|x\|^{p+q}\|z\|_0^r}{|2 - 2^{p+q}|}, \qquad x \in U, z \in Y.$$

(ii) If $p + q \neq 1$ and

$$\Big\| f\Big(\frac{x+y}{2}\Big) - \frac{f(x) + f(y)}{2}, z \Big\|_Y \leq c\|x\|^p\|y\|^q\|z\|_0^r, \qquad x, y \in U, z \in Y,$$

then there is a unique solution $J : U \to Y$ to the Jensen Equation (25) such that

$$\|f(x) - J(x), z\|_Y \leq \frac{(1 + 3^q)\|x\|^{p+q}\|z\|_0^r}{|3 - 3^{p+q}|}, \qquad x \in U, z \in Y.$$

(iii) If $p + q \neq 2$ and

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), z\|_Y \leq c\|x\|^p\|y\|^q\|z\|_0^r, \qquad x, y \in U, z \in Y,$$

then there is a unique solution $Q : U \to Y$ to the quadratic equation such that

$$\|f(x) - Q(x), z\|_Y \leq \frac{c\|x\|^{p+q}\|z\|_0^r}{|4 - 2^{p+q}|} + \frac{1}{3}\|f(0), z\|_Y, \qquad x \in U, z \in Y.$$

In the case $r \neq 1$, generalizations of the above results can be easily derived from the following modification of Corollary 1.

Proposition 8. *Let $k, m \in \mathbb{N}$, T and $E_1, \ldots, E_m$ be nonempty sets, $A_1, \ldots, A_m \in \mathbb{R}$, $\varphi : T^k \to \mathbb{R}_+$, $r \in \mathbb{R}_+ \setminus \{1\}$, $\|\cdot\|_0$ be a norm in Y, and $\xi_i : T^k \to E_i$ and $g_i : E_i \to Y$ for $i = 1, \ldots, m$ be such that*

$$\Big\| \sum_{i=1}^{m} A_i g_i\Big(\xi_i(x_1 \ldots, x_k)\Big), z\Big\|_Y \leq \varphi(x_1, \ldots, x_k)\, \|z\|_0^r, \qquad x_1, \ldots, x_k \in T, z \in Y. \tag{45}$$

Then, $g_1, \ldots, g_m$ satisfy the equation

$$\sum_{i=1}^{m} A_i g_i\Big(\xi_i(x_1 \ldots, x_k)\Big) = 0, \qquad x_1, \ldots, x_k \in T. \tag{46}$$

Proof. Fix $x_1, \ldots, x_k \in T$ and $z \in Y$, and put

$$s: = \begin{cases} 1 & \text{for} \quad 0 \leq r < 1; \\ -1 & \text{for} \quad r > 1\,. \end{cases}$$

Then,

$$\Big\| \sum_{i=1}^{m} A_i g_i\Big(\sum_{j=1}^{k} a_{ij} x_j\Big) + A, l^s z\Big\|_Y \leq \varphi(x_1, \cdots, x_k) \cdot \big\|l^s z\big\|_0^r, \qquad l \in \mathbb{N},$$

and consequently

$$\Big\| \sum_{i=1}^{m} A_i g_i\Big(\sum_{j=1}^{k} a_{ij} x_j\Big) + A, z\Big\|_Y \leq l^{s(r-1)} \varphi(x_1, \cdots, x_k) \cdot \|z\|_0^r, \qquad l \in \mathbb{N}.$$

Therefore, letting $l \to \infty$, for every $x_1, \cdots, x_k \in T$ and $z \in Y$, we get

$$\Big\| \sum_{i=1}^{m} A_i g_i\Big(\sum_{j=1}^{k} a_{ij} x_j\Big) + A, z\Big\|_Y = 0,$$

which means that (11) holds. □

Clearly, Proposition 8 implies the following generalization of Theorem 10.

Proposition 9. *Let $(G, +)$ be a group, $\|\cdot\|_0$ be a norm in Y, $\varphi\colon G^2 \to \mathbb{R}_+$, $r \in \mathbb{R}_+ \setminus \{1\}$, and $f : G \to Y$. Then, the following three statements are valid.*

(i) If, for every $x, y \in G$ and $z \in Y$,

$$\|f(x+y) - f(x) - f(y), z\|_Y \leq \varphi(x,y)\|z\|_0^r,$$

then f is additive.

(ii) If, for every $x, y \in G$ and $z \in Y$,

$$\Big\|f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2}, z\Big\|_Y \leq \varphi(x,y)\|z\|_0^r,$$

then f is a solution to the Jensen Equation (25).

(iii) If, for every $x, y \in G$ and $z \in Y$,

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\|_Y \leq \varphi(x,y)\|z\|_0^r,$$

then f is a solution to the quadratic Equation (28).

In [59], the stability of the following functional equation:

$$f(2x+y) - f(x+2y) = 3f(x) - 3f(y) \tag{47}$$

has been investigated in 2-Banach spaces and the main result can be stated as follows.

Theorem 11. *Let $\|\cdot\|_0$ be a norm in X, $\varepsilon \geq 0$, $p, q \in \mathbb{R}_+ \setminus \{2\}$ and $r > 0$. If $f : X \to X$ is a function such that*

$$\|D_f(x,y), z\| \leq \varepsilon(\|x\|_0^p + \|y\|_0^q)\|z\|_0^r, \qquad x, y, z \in X,$$

where

$$D_f(x,y) := f(2x+y) - f(x+2y) - 3f(x) + 3f(y), \tag{48}$$

then there exists a unique mapping $Q : X \to X$ satisfying (47) such that

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon\|x\|_0^p\|z\|_0^r}{|4-2^p|}, \qquad x, z \in X.$$

Clearly, in the case $r \neq 1$, from Proposition 8 we can easily derive the following generalization of Theorem 11.

Proposition 10. *Let $(G,+)$ be a group, $\|\cdot\|_0$ be a norm in Y, $\varphi \colon G^2 \to \mathbb{R}_+$, and $r \in \mathbb{R}_+ \setminus \{1\}$. If $f : G \to Y$ is such that*

$$\|D_f(x,y), z\|_Y \leq \varphi(x,y)\|z\|_0^r, \qquad x, y \in G, z \in Y,$$

where D_f is given by (48) for every $x, y \in G$, then f is a solution to functional Equation (47).

The stability of the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+2y) - 4f(x+y) + 18f(x) - 12f(y) \tag{49}$$

in 2-normed spaces has been studied in [60] and one of the two main results can be written as follows.

Theorem 12. *Let $(U, \|\cdot\|)$ be a normed space and $\varepsilon, r, p, q \in (0,\infty)$. Assume that $p+q < 3$ or $p, q \in (3,\infty)$. If $f : U \to B$ is such that*

$$\|D_f(x,y), z\|_B \leq \varepsilon(\|x\|^p\|y\|^q + \|x\|^p + \|y\|^q)\|z\|^r, \qquad x, y \in U, z \in B,$$

where $D_f(x,y) := f(2x+y) + f(2x-y) - 2f(x+2y) + 4f(x+y) - 18f(x) + 12f(y)$, then there is a unique solution $C : U \to B$ of (49) with

$$\|f(x) - C(x), z\|_B \leq \frac{\varepsilon\|x\|^p\|z\|^r}{2|8-2^p|}, \qquad x \in U, z \in B.$$

The authors have not specified the meaning of $\|z\|$ for $z \in B$, but if we assume that $\|\cdot\|$ also means a classic norm in B, then the whole reasoning is still correct. Moreover, if $r \neq 1$ in Theorem 12, then by Proposition 8, f must be a solution of (49). Solutions to (49) have been characterized in Theorem 2.1 in [60].

The other main result presented in [60] can be formulated as follows.

Theorem 13. *Let $\varepsilon, p, q \in (0,\infty)$. Assume that $p+q < 3$ or $p, q \in (3,\infty)$. If $f : X \to B$ is such that*

$$\|D_f(x,y), z\|_B \leq \varepsilon(\|x,z\|^p\|y,z\|^q + \|x,z\|^p + \|y,z\|^q), \qquad x, y \in X, z \in B,$$

where $D_f(x,y) := f(2x+y) + f(2x-y) - 2f(x+2y) + 4f(x+y) - 18f(x) + 12f(y)$*, then there is a unique solution* $C : X \to B$ *of* (49) *with*

$$\|f(x) - C(x), z\|_B \leq \frac{\varepsilon \|x,z\|^p}{2|8-2^p|}, \qquad x \in X, z \in B. \tag{50}$$

Unfortunately, it appears that the authors confused the 2-norms in X and B and the only reasonable possibility to keep the theorem valid is to assume that $X = B$. In such a case, if $p \neq 1$, then from (50) (replacing z by lz or by $\frac{1}{l}z$, with $l \in \mathbb{N}$) we easily obtain that $f = C$.

In [61], the authors have investigated the stability of the equation

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 2f(2x) - 2f(x), \tag{51}$$

in 2-Banach spaces. The main results in [61] can be written as follows.

Theorem 14. *Let* $(U, \|\cdot\|)$ *be a normed space,* $\theta \in [0,\infty)$*,* $p,q,r \in (0,\infty)$*, and* $p+q \neq 2$*. Let* $f : U \to B$ *be such that* $f(0) = 0$ *and*

$$\begin{aligned} \|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 2f(2x) + 2f(x), z\|_B \\ \leq \theta \|x\|^p \|y\|^q \|z\|^r \end{aligned} \tag{52}$$

for all $x,y,z \in U$*. Then, there is a unique solution* $Q : U \to B$ *of* (51) *with*

$$\|f(x) - Q(x), z\|_B \leq \frac{4+3^q}{3^{p+q}|2^{p+q}-4|}\theta \|x\|^{p+q}\|z\|^r, \qquad x,z \in U.$$

Unfortunately, already at the very beginning of the proofs (in Formula (2.3)) the authors commit simple but meaningful mistakes, which makes the remaining reasoning doubtful. Moreover, to avoid evident confusion between the norm and 2-norm in Theorem 14, it should be assumed that $U = B$ (and B is endowed with a norm). However, from Proposition 8 we can easily derive the following.

Proposition 11. *Let* $(G,+)$ *be a group,* $\|\cdot\|_0$ *be a norm in* Y*,* $r \in \mathbb{R}_+$*,* $r \neq 1$*, and* $\phi : G^2 \to \mathbb{R}_+$*. If* $f : G \to Y$ *is such that*

$$\begin{aligned} \|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 2f(2x) + 2f(x), z\|_Y \\ \leq \phi(x,y)\|z\|_0^r, \qquad x,y \in G, z \in Y, \end{aligned}$$

then f *is a solution of* (51)*.*

In [62] (Theorems 4.5 and 4.8), the authors claim that have used the direct and fixed point methods to investigate the stability of a functional equation (called the decic functional equation), being a particular case of (11), in the 2-normed spaces. However, because the results are stated in very unclear (for us) manner and without proofs, we are not recalling them here.

A very similar situation is in [63] (Theorems 4.1 and 5.1), where the stability of a functional equation of type (11) (called the n-dimensional quartic functional equation) has been studied in the 2-normed spaces.

In [64] (Theorems 3.1 and 4.1), the authors apply two different methods to prove the stability results for a functional equation of type (11) (called the $n-$dimensional quartic functional equation, with $n \geq 5$) in 2-normed spaces. The main results are presented with proofs, but unfortunately (analogously as in [62,63]) in a manner that is unclear for us. So we have decided not to present them here.

5. Some Other Results

Several other stability results in 2-normed spaces (also non-Archimedean and random) have been presented in [65] (for the Pexiderized Cauchy functional equation), in [17,66] (for the Cauchy equation), in [67] (for a generalized radical cubic functional equation related to quadratic functional equation), in [68] (for the radical quartic functional equation), in [69] (for the quadratic functional equation), in [70] (for the generalized Cauchy functional equation), in [71] (for a functional equation called the Cauchy–Jensen functional equation), in [72] (for a general p-radical functional equation) [73] (for radical sextic functional equation), in [74] (for several functional equations of quadratic-type), in [75,76] (for the functional equation of p-Wright affine functions), in [77] (for a system of additive-cubic-quartic functional equations with constant coefficients in non-Archimedean 2-normed spaces), in [78] (for a functional inequality in non-Archimedean 2-normed spaces), in [79] (for a cubic functional equation in random 2-normed spaces), in [80] (for the Pexiderized quadratic functional equation in the random 2-normed spaces) and in [81] (for radical functional equations in 2-normed spaces and p-2-normed spaces). However, as these results are more involved and of a different character than those presented so far, we will discuss them in more details in another publication.

Furthermore, the stability results in n-normed spaces, which can be found in [29,31–33,35–37,53,82–86], will be discussed in a future publication.

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References

1. Hayes, W.; Jackson, K.R. A survey of shadowing methods for numerical solutions of ordinary differential equations. *Appl. Numer. Math.* **2005**, *53*, 299–321. [CrossRef]
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Brzdęk, J.; Fechner, W.; Moslehian, M.S.; Sikorska, J. Recent developments of the conditional stability of the homomorphism equation. *Banach J. Math. Anal.* **2015**, *9*, 278–326. [CrossRef]
4. Pólya, G.; Szegö, G. *Aufgaben und Lehrsätze aus der Analysis I*; Springer: Berlin, Germany, 1925.
5. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhäuser: Boston, MA, USA, 1998.
6. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [CrossRef]
7. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
8. Gajda, Z. On stability of additive mappings. *Int. J. Math. Math. Sci.* **1991**, *14*, 431–434. [CrossRef]
9. Rassias, T.M. On a modified Hyers-Ulam sequence. *J. Math. Anal. Appl.* **1991**, *158*, 106–113. [CrossRef]
10. Badora, R.; Brzdęk, J.; Ciepliński, K. Applications of Banach limit in Ulam stability. *Symmetry* **2021**, *13*, 841. [CrossRef]
11. Badora, R.; Brzdęk, J. Banach limit in the stability problem of a linear functional equation. *Results Math.* **2021**, *76*, 51. [CrossRef]
12. Rassias, J.M. On approximation of approximately linear mappings by linear mappings. *J. Funct. Anal.* **1982**, *46*, 126–130. [CrossRef]
13. Rassias, J.M. On a new approximation of approximately linear mappings by linear mappings. *Discuss. Math.* **1985**, *7*, 193–196.
14. Brzdęk, J. A hyperstability result for the Cauchy equation. *Bull. Australian Math. Soc.* **2014**, *89*, 33–40. [CrossRef]
15. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA, 2011,
16. Brzdęk, J.; Popa, D.; Raşa, I.; Xu, B. *Ulam Stability of Operators, Mathematical Analysis and its Applications v. 1*; Academic Press: Oxford, UK, 2018.
17. Brzdęk, J.; Ciepliński, K. On a fixed point theorem in 2-Banach spaces and some of its applications. *Acta Math. Sci.* **2018**, *38*, 377–390. [CrossRef]
18. Kaiser, Z.; Páles, Z. An example of a stable functional equation when the Hyers method does not work. *J. Inequal. Pure Appl. Math.* **2005**, *6*, 11.
19. Brzdęk, J.; Cadariu, L.; Ciepliński, K. Fixed point theory and the Ulam stability. *J. Funct. Spaces* **2014**, *2014*, 829419. [CrossRef]

20. Ciepliński, K. Applications of fixed point theorems to the Hyers-Ulam stability of functional equations—A survey. *Ann. Funct. Anal.* **2012**, *3*, 151–164. [CrossRef]
21. Aczél, J. *Lectures on Functional Equations and Their Applications*; Academic Press: New York, NY, USA, 1966.
22. Aczél, J.; Dhombres, J. *Functional Equations in Several Variables. Volume 31 of Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1989.
23. Castillo, E.; Ruiz-Cobo, M.R. *Functional Equations and Modelling in Science and Engineering*; Marcel Dekker: New York, NY, USA, 1992.
24. Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, 2nd ed.; Birkhäuser: Basel, Switzerland, 2009.
25. Kuczma, M.; Choczewski, B.; Ger, R. *Iterative Functional Equations*; Cambridge University Press: Cambridge, UK, 1990.
26. Misiak, A. n-inner product spaces. *Math. Nachr.* **1989**, *140*, 299–319. [CrossRef]
27. Gähler, S. 2-metrisch Räume und ihre topologische Struktur. *Math. Nachr.* **1963**, *26*, 115–148. [CrossRef]
28. Gähler, S. Lineare 2-normierte Räumen. *Math. Nachr.* **1964**, *28*, 1–43. [CrossRef]
29. Chu, H.-Y.; Kim, A.; Park, J. On the Hyers-Ulam stabilities of functional equations on $n-$Banach spaces. *Math. Nachr.* **2016**, *289*, 1177–1188. [CrossRef]
30. Gunawan, H.; Mashadi, M. On n-normed spaces. *Int. J. Math. Math. Sci.* **2001**, *27*, 631–639. [CrossRef]
31. Xu, T.Z. Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces. *Abstr. Appl. Anal.* **2013**, *2013*, 648709. [CrossRef]
32. Xu, T.Z.; Rassias, J.M. On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 926390. [CrossRef]
33. Brzdęk, J.; El-hady, E.-s. On hyperstability of the Cauchy functional equation in n-Banach spaces. *Mathematics* **2020**, *8*, 1886. [CrossRef]
34. Chen, X.Y.; Song, M.M. Characterizations on isometries in linear n-normed spaces. *Nonlinear Anal.* **2010**, *72*, 1895–1901. [CrossRef]
35. Chu, H.-Y.; Lee, K.; Park, C.-G. On the Aleksandrov problem in linear n-normed spaces. *Nonlinear Anal.* **2004**, *59*, 1001–1011.
36. Dutta, H. On some n-normed linear space valued difference sequences. *J. Franklin Inst.* **2011**, *348*, 2876–2883. [CrossRef]
37. Eghbali, N.; Rassias, J.M.; Taheri, M. On the stability of a k-cubic functional equation in intuitionistic fuzzy n-normed spaces. *Res. Math.* **2016**, *70*, 233–248. [CrossRef]
38. Gunawan, H.; Neswan, O.; Sukaesih, E. Fixed point theorems in bounded sets in an n-normed space. *J. Math. Anal.* **2015**, *6*, 51–58.
39. Bahyrycz, A.; Olko, J. On stability of the general linear equation. *Aequ. Math.* **2015**, *89*, 1461–1474. [CrossRef]
40. Bahyrycz, A.; Olko, J. Hyperstability of general linear functional equation. *Aequ. Math.* **2016**, *90*, 527–540. [CrossRef]
41. Phochai, T.; Saejung, S. Hyperstability of generalised linear functional equations in several variables. *Bull. Australian Math. Soc.* **2020**, *102*, 293–302. [CrossRef]
42. Zhang, D. On hyperstability of generalised linear functional equations in several variables. *Bull. Australian Math. Soc.* **2015**, *92*, 259–267. [CrossRef]
43. Brzdęk, J.; Ciepliński, K.; Leśniak, Z. On Ulam's type stability of the linear equation and related issues. *Discret. Dyn. Nat. Soc.* **2014**, *2014*, 536791. [CrossRef]
44. Brzdęk, J.; Fošner, A. Remarks on the stability of Lie homomorphisms. *J. Math. Anal. Appl.* **2013**, *400*, 585–596. [CrossRef]
45. Agarwal, R.P.; Xu, B.; Zhang, W. Stability of functional equations in single variable. *J. Math. Anal. Appl.* **2003**, *288*, 852–869. [CrossRef]
46. Brzdęk, J.; Popa, D.; Xu, B. Selections of set-valued maps satisfying a linear inclusion in a single variable. *Nonlinear Anal.* **2011**, *74*, 324–330. [CrossRef]
47. Xu, B.; Brzdęk, J.; Zhang, W. Fixed point results and the Hyers-Ulam stability of linear equations of higher orders. *Pac. J. Math.* **2015**, *273*, 483–498. [CrossRef]
48. Gao, J. On the stability of the linear mapping in 2-normed spaces. *Nonlinear Funct. Anal. Appl.* **2009**, *14*, 801–807.
49. Brzdęk, J.; Ciepliński, K. Hyperstability and superstability. *Abstr. Appl. Anal.* **2013**, *2013*, 401756. [CrossRef]
50. Chung, S.C.; Park, W.G. Hyers-Ulam stability of functional equations in 2-Banach spaces. *Int. J. Math. Anal. (Ruse)* **2012**, *6*, 951–961.
51. Cho, Y.J.; Park, C.; Eshaghi Gordji, M. Approximate additive and quadratic mappings in 2-Banach spaces and related topics. *Int. J. Nonlin. Anal. Appl.* **2012**, *3*, 75–81.
52. Almahalebi, M.; Kabbaj, S. A fixed point approach to stability of the quartic equation in 2-Banach spaces. *J. Math. Comput. Sci.* **2013**, *3*, 972–984.
53. Arunkumar, M. Stability of $n-$dimensional additive functional equation in generalized 2-normed space. *Demonstr. Math.* **2016**, *49*, 319–330. [CrossRef]
54. Rassias, J.M.; Arunkumar, M.; Sathya, E.; Kumar, N.M. Solution and stability of a acq functional equation in generalized 2-normed spaces. *Int. J. Fuzzy Math. Arch.* **2015**, *7*, 213–224.
55. Almahalebi, M.; Chahbi, A.; Kabbaj, S. A fixed point approach to the stability of a bi-cubic functional equations in 2-Banach spaces. *Palest. J. Math.* **2016**, *5*, 220–227.
56. Ciepliński, K. Approximate multi-additive mappings in 2-Banach spaces. *Bull. Iranian Math. Soc.* **2015**, *41*, 785–792.

57. Ciepliński, K.; Xu, T.Z. Approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces. *Carpathian J. Math.* **2013**, *29*, 159–166. [CrossRef]
58. Park, W.G. Approximate additive mappings in 2-Banach spaces and related topics. *J. Math. Anal. Appl.* **2011**, *376*, 193–202. [CrossRef]
59. Patel, B.M.; Patel, A.B. Stability of quadratic functional equations in 2-Banach space. *Gen. Math. Notes* **2013**, *15*, 1–13.
60. Kim, C.I.; Jung, K.H. Stability of a cubic functional equation in 2-normed spaces. *J. Appl. Math. Inf.* **2014**, *32*, 817–825. [CrossRef]
61. Eivani, S.; Ostadbashi, S. Approximate mixed additive and quadratic functional in 2-Banach spaces. *Casp. J. Math. Sci. (CJMS)* **2015**, *4*, 167–173.
62. Arunkumar, M.; Bodaghi, A.; Rassias, J.M.; Sathya, E. The general solution and approximations of a decic type functional equation in various normed spaces. *J. Chungcheong Math. Soc.* **2016**, *29*, 287–328. [CrossRef]
63. Murthy, S.; Govindhan, V.; Velan, M.S.S. Solution and stability of two types of n-dimensional quartic functional equation in generalized 2-normed spaces. *Int. J. Pure Appl. Math.* **2016**, *111*, 249–272. [CrossRef]
64. Arunkumar, M.; Murthy, S.; Namachivayam, T.; Ganapathy, G. Stability of $n-$dimensional quartic functional equation in generalized $2-$normed spaces using two different methods. *Malaya J. Math.* **2015**, *1*, 242–251.
65. Eskandani, G.Z.; Gavruta, P. Hyers-Ulam-Rassias stability of Pexiderized Cauchy functional equation in 2-Banach spaces. *J. Nonlinear Sci. Appl.* **2012**, *5*, 459–465. [CrossRef]
66. Brzdęk, J.; El-hady, E.-s. On approximately additive mappings in 2-Banach spaces. *Bull. Australian Math. Soc.* **2020**, *101*, 299–310. [CrossRef]
67. Ghali, R.E.; Kabbaj, S. 2-Banach stability results for the radical cubic functional equation related to quadratic mapping. *J. Linear Topol. Algebra (JLTA)* **2020**, *9*, 35–51.
68. EL-Fassi, I.-I. Approximate solution of radical quartic functional equation related to additive mapping in 2-Banach spaces. *J. Math. Anal. Appl.* **2017**, *455*, 2001–2013. [CrossRef]
69. Sayar, K.Y.N.; Bergam, A. Approximate solutions of a quadratic functional equation in 2-Banach spaces using fixed point theorem. *J. Fixed Point Theory Appl.* **2020**, *22*, 1–16. [CrossRef]
70. Aiemsomboon, L.; Sintunavarat, W. On new approximations for generalized Cauchy functional equations using Brzdęk and Ciepliński's fixed point theorems in 2-Banach spaces. *Acta Math. Sci.* **2020**, *40*, 824–834. [CrossRef]
71. Sayar, K.Y.N.; Bergam, A. Some hyperstability results for a Cauchy-Jensen type functional equation in 2-Banach spaces. *Proyecciones (Antofagasta)* **2020**, *39*, 73–89. [CrossRef]
72. AL-Ali, S.A.; Almahalebi, M.; Elkettani, Y. Stability of a general p-radical functional equation related to additive mappings in 2-Banach spaces. *Proyecciones (Antofagasta)* **2021**, *40*, 49–71. [CrossRef]
73. EL-Fassi, I.-I. New stability results for the radical sextic functional equation related to quadratic mappings in $(2,\beta)$-Banach spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 1–17. [CrossRef]
74. Park, W.G.; Bae, J.H. Hyers-Ulam stability of quadratic forms in 2-normed spaces. *Demonstr. Math.* **2019**, *52*, 496–502. [CrossRef]
75. El-hady, E.-s. On stability of the functional equation of p-Wright affine functions in 2-Banach spaces. In *Ulam Type Stability*; Springer: Cham, Switzerland, 2019; pp. 131–141.
76. El-hady, E.-s. On stability of the functional equation of p-Wright affine functions in $(2,\alpha)$-Banach spaces. *J. Egypt. Math. Soc.* **2019**, *27*, 1–9. [CrossRef]
77. Park, C.; Gordji, M.E.; Ghaemi, M.B.; Majani, H. Fixed points and approximately octic mappings in non-Archimedean 2-normed spaces. *J. Inequal. Appl.* **2012**, *2012*, 1–12. [CrossRef]
78. Park, S.; Kim, C. The generalized Hyers-Ulam stability of additive functional inequalities in non-Archimedean 2-normed space. *Korean J. Math.* **2014**, *22*, 339–348.
79. Alotaibi, A.; Mohiuddine, S.A. On the stability of a cubic functional equation in random 2-normed spaces. *Adv. Differ. Equ.* **2012**, *2012*, 39. [CrossRef]
80. Alghamdi, M.A. Stability of Pexiderized quadratic functional equation in random 2-normed spaces. *J. Funct. Spaces* **2015**, *2015*, 828967. [CrossRef]
81. Khodaei, H.; Gordji, M.E.; Kim, S.S.; Cho, Y.J. Approximation of radical functional equations related to quadratic and quartic mappings. *J. Math. Anal. Appl.* **2012**, *395*, 284–297. [CrossRef]
82. Ciepliński, K. On Ulam stability of a functional equation. *Results Math.* **2020**, *75*, 1–11. [CrossRef]
83. Brzdęk, J.; Ciepliński, K. A fixed point theorem in n-Banach spaces and Ulam stability. *J. Math. Anal. Appl.* **2019**, *470*, 632–646. [CrossRef]
84. Choy, J.; Chu, H.Y.; Kim, A. A Remark for the Hyers-Ulam stabilities on n-Banach spaces. *Axioms* **2021**, *10*, 2. [CrossRef]
85. EL-Fassi, I.-I.; Elqorachi, E.; Khodaei, H. A fixed point approach to stability of k-th radical functional equation in non-archimedean (n,β)-Banach spaces. *Bull. Iran. Math. Soc.* **2021**, *47*, 487–504. [CrossRef]
86. Yang, X.; Chang, L.; Liu, G.; Shen, G. Stability of functional equations in (n,β)-normed spaces. *J. Inequal. Appl.* **2015**, *2015*, 112. [CrossRef]