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New Oscillation Results of Even-Order Emden–Fowler Neutral Differential Equations

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Abstract: The objective of this study is to establish new sufficient criteria for oscillation of solutions of even-order delay Emden-Fowler differential equations with neutral term $\left(r(\iota)\left((y(\iota)+m(\iota)y(g(\iota)))^{(n-1)}\right)^{\gamma}\right)' + \sum_{i=1}^{j} q_i(\iota)y^{\gamma}(\mu_i(\iota)) = 0$. We use Riccati transformation and the comparison with first-order differential inequalities to obtain theses criteria. Moreover, the presented oscillation conditions essentially simplify and extend known criteria in the literature. To show the importance of our results, we provide some examples. Symmetry plays an essential role in determining the correct methods for solutions to differential equations.

Keywords: even-order differential equations; neutral delay; oscillation

MSC: 34C10; 34K11

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1. Introduction

The aim of this work is to study the oscillation of solutions of the even-order neutral differential equation

$$\left(r(\iota)\left((y(\iota)+m(\iota)y(g(\iota)))^{(n-1)}\right)^{\gamma}\right)' + \sum_{i=1}^{j} q_i(\iota)y^{\gamma}(\mu_i(\iota)) = 0, \quad \iota \geq \iota_0, \tag{1}$$

where $j \geq 1$ and n is an even number. Throughout this work, we suppose that:

($\mathbf{P}_1$) $r \in C^1([\iota_0,\infty),(0,\infty)), r(\iota) > 0, r'(\iota) \geq 0, \kappa(\iota) = \int_{\iota_0}^{\infty} r^{-1/\gamma}(s)\mathrm{d}s = \infty$,
($\mathbf{P}_2$) $m \in C^n([\iota_0,\infty),(0,\infty)), q_i \in C([\iota_0,\infty),(0,\infty)), 0 \leq m(\iota) < 1,\ i = 1,2,\ldots,j$,
($\mathbf{P}_3$) $0 \leq m(\iota) \leq m_0 < \infty$,
($\mathbf{P}_4$) $m(\iota) > 1, \mu_i(\iota) \leq g(\iota)$,
($\mathbf{P}_5$) $g \in C^n([\iota_0,\infty),\mathbb{R}), \mu_i \in C([\iota_0,\infty),\mathbb{R}), g'(\iota) \geq g_0 > 0, \mu_i'(\iota) > 0, g(\iota) \leq \iota, \mu_i(\iota) \leq \iota$, $\lim_{\iota\to\infty} g(\iota) = \lim_{\iota\to\infty} \mu_i(\iota) = \infty$,
($\mathbf{P}_6$) $\gamma \in S = \{j : j = \frac{2m_1+1}{2m_2+1},\ m_1,\ m_2 \in \mathbb{N}^* = \{1,2,3,\ldots\}\}$.

Throughout this paper, we set

$$w(\iota) = y(\iota) + m(\iota)y(g(\iota)). \tag{2}$$

Definition 1. *A solution of Equation (1) is said to be non-oscillatory if it is positive or negative ultimately; otherwise, it is said to be oscillatory.*

Definition 2. *Equation (1) is said to be oscillatory if all its solutions are oscillatory.*

Definition 3. *A neutral delay differential equation, the highest order derivative of the unknown function, appears both with and without delay.*

In the past decades, the problem of establishing asymptotic behavior of solutions for differential equations with a delay term has been a very active research area. Due to the huge advantage of neutral differential equations in describing several neutral phenomena in engineering, biology, economics, medicine and physics that are of great academic and scientific values practically and theoretically for studying neutral differential equations. Furthermore, symmetrical properties contribute to the Euler equation in some variational problems. In other words, it contributes to determining the appropriate method for finding the correct solution to this equation [1–5].

2. Literature Review

In this section, we provide some auxiliary results of some published studies. A large amount of research attention has been focused on the oscillation problem of different kinds of differential equations. Zhang et al. [6] and Li and Rogovchenko [7] developed techniques for studying oscillation in order to improve the oscillation criteria of all solutions of even-order neutral differential equations. Agarwal et al. [8] and Moaaz et al. [9] gave new oscillation conditions for neutral differential equations. Therefore, there are many studies on the oscillation of different orders of some differential equations in canonical and noncanonical form, see [10–17]. The purpose of this paper is to continue the previous works [18,19].

In [20], the authors considered the oscillation of differential equation

$$\left(r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma}\right)' + q(\imath)f(y(\mu(\imath))) = 0,$$

where $f(y) = y^{\gamma}$, and they used the integral averaging technique to find the oscillation conditions. Xing et al. [18] discussed the following half-linear equation

$$\left(r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma_1}\right)' + q(\imath)y^{\gamma_2}(\mu(\imath)) = 0, \tag{3}$$

where n is even. They established some oscillation criteria for this equation by comparison principles. Baculikova et al. [19] presented oscillation results by comparison principles for the equation

$$\left(\left(w^{(n-1)}(\imath)\right)^{\gamma}\right)' + q(\imath)y^{\gamma}(\mu(\imath)) = 0. \tag{4}$$

The authors in [18,19] used the comparison technique that differs from the one we used in this article. Their approach is based on using comparison technique to reduce Equations (3) and (4) into a first-order equation, and they studied the qualitative properties of Equations (3) and (4) in the noncanonical case, that is $\int_{\imath_0}^{\infty} r^{-1/\gamma}(s)\mathrm{d}s < \infty$, while in our article, it is based on using the Riccati technique to reduce Equation (1) into a first-order inequality to find more effective some oscillation criteria for Equation (1) in the canonical case, that is $\int_{\imath_0}^{\infty} r^{-1/\gamma}(s)\mathrm{d}s = \infty$.

Motivated by these reasons mentioned above, in this work, we extend, generalize and improve the results for Equation (1) using the Riccati transformation and comparison technique. These oscillation conditions contribute to adding some important criteria that were previously studied in the papers.

3. Main Results

We need the following lemmas to prove our main results:

Lemma 1 ([21]). *Let $f \in C^n([\imath_0,\infty),(0,\infty))$. If $f^{(n)}(\imath)$ is eventually of one sign for all large $\imath$, then there exist a $\imath_y > \imath_1$ for some $\imath_1 > \imath_0$ and an integer m, $0 \le m \le n$ with $n+m$ even for $f^{(n)}(\imath) \ge 0$ or $n+m$ odd for $f^{(n)}(\imath) \le 0$ such that $m > 0$ implies that $f^{(k)}(\imath) > 0$ for $\imath > \imath_y$, $k = 0,1,\dots,m-1$ and $m \le n-1$ implies that $(-1)^{m+k} f^{(k)}(\imath) > 0$ for $\imath > \imath_y$, $k = m, m+1, \dots, n-1$.*

Lemma 2 ([22]). *Let $g(u) = ru - mu^{\beta+1/\beta}$, where r and m are positive constants, $\beta \in S = \{j : j = \frac{2m_1+1}{2m_2+1},\ m_1,\ m_2 \in \mathbb{N}^* = \{1,2,3,\dots\}\}$. Then, g attains its maximum value on $\mathbb{R}^+$ at $u^* = \left(\frac{\beta r}{(\beta+1)^m}\right)^\beta$ and*

$$\max_{u\in\mathbb{R}^+} g = g(u^*) = \frac{\beta^\beta r^{\beta+1}}{(\beta+1)^{\beta+1} m^\beta}.$$

Lemma 3 ([23]). *Let $y \in C^n([\imath_0,\infty),(0,\infty))$ such that $y^{(n-1)}(\imath) y^{(n)}(\imath) \le 0$ for all $\imath \ge \imath_1$. If $\lim_{\imath\to\infty} y(\imath) \ne 0$, then for every $\varepsilon \in (0,1)$, there exists $\imath_\varepsilon \ge \imath_1$ such that*

$$y(\imath) \ge \frac{\varepsilon}{(n-1)!} \imath^{n-1} \left| y^{(n-1)}(\imath) \right| \text{ for } \imath \ge \imath_\varepsilon.$$

Lemma 4 ([24]). *Let $\alpha \in (0,1)$ and $y_1, y_2 \ge 0$. Then,*

$$y_1^\alpha + y_2^\alpha \ge \frac{1}{2^{\alpha-1}} (y_1 + y_2)^\alpha \quad \text{if} \quad \alpha \ge 1,$$

and

$$y_1^\alpha + y_2^\alpha \ge (y_1 + y_2)^\alpha \quad \text{if} \quad 0 < \alpha < 1.$$

Lemma 5. *Let y be an eventually positive solution of Equation (1), then there exists $\imath_1 \ge \imath_0$, such that:*

$$w(\imath) > 0,\ w'(\imath) > 0,\ \ w^{(n-1)}(\imath) > 0,\ \ w^{(n)}(\imath) < 0. \tag{5}$$

More precisely, $w(\imath)$ has the following two cases for $\imath \ge \imath_1$:

$(\mathbf{I}_1)$ $w(\imath) > 0$, $w'(\imath) > 0$, $w''(\imath) > 0$, $w^{(n-1)}(\imath) > 0$, $w^{(n)}(\imath) < 0$;
$(\mathbf{I}_2)$ $w(\imath) > 0$, $w^{(j)}(\imath) > 0$, $w^{(j+1)}(\imath) < 0$ for all odd integer $j \in \{1,2,\dots,n-3\}$, $w^{(n-1)}(\imath) > 0$, $w^{(n)}(\imath) < 0$.

Proof. The proof of Equation (5) is similar to that of ([25], Lemma 2.3), and so we omit it. Furthermore, we can conclude that cases $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$ hold. □

Theorem 1. *Let $\gamma > 0$, $(\mathbf{P}_1)$, $(\mathbf{P}_3)$, $(\mathbf{P}_5)$ hold and*

$$\int_{\imath_1}^{\infty} C(\imath) \mathrm{d}\imath = \infty, \tag{6}$$

where $C(\imath) = \min\{q_i(\imath), q_i(g(\imath))\}$, then (1) is oscillatory.

Proof. Assume towards a contradiction that Equation (1) is not oscillatory. Then, we can clearly assume that $y > 0$ is eventually positive. By $\gamma > 0$, we need to divide into two situations to discuss—$\gamma \ge 1$ and $0 < \gamma < 1$.

When $\gamma \ge 1$ is satisfied, owing to Lemma 5, we find that Equation (5) holds. According to Equation (1), we see

$$\left(r(\imath) \left(w^{(n-1)}(\imath) \right)^\gamma \right)' = -\sum_{i=1}^{j} q_i(\imath) y^\gamma(\mu_i(\imath)) < 0,\ \imath \ge \imath_1. \tag{7}$$

Thus, $r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma}$ is not increasing for $\imath \geq \imath_1$.
Let
$$\varphi(\imath) = r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma}$$
and
$$\Psi(\imath) = y^{\gamma}(\mu_i(\imath)).$$

From Equations (2) and (7), we obtain

$$\varphi'(\imath) + \sum_{i=1}^{j} q_i(\imath)\Psi(\imath) + \frac{m_0^{\gamma}}{g'(\imath)}(\varphi(g(\imath)))' + m_0^{\gamma}\sum_{i=1}^{j} q_i(g(\imath))\Psi(g(\imath)) \leq 0,$$

which leads to

$$\varphi'(\imath) + \frac{m_0^{\gamma}}{g_0}(\varphi(g(\imath)))' + C(\imath)\left(\Psi(\imath) + +m_0^{\gamma}\Psi(g(\imath))\right) \leq 0. \tag{8}$$

According to Lemma 4 and $(\mathbf{P}_3)$, we have

$$\varphi'(\imath) + \frac{m_0^{\gamma}}{g_0}(\varphi(g(\imath)))' + \frac{1}{2^{\gamma-1}}C(\imath)w^{\gamma}(\mu_i(\imath)) \leq 0. \tag{9}$$

Integrating Equation (9) from $\imath_1$ to $\imath$, we obtain

$$\frac{1}{2^{\gamma-1}}\int_{\imath_1}^{\imath} C(s)w^{\gamma}(\mu_i(s)) \leq \varphi(\imath_1) - \varphi(\imath) + \frac{m_0^{\gamma}}{g_0}(\varphi(g(\imath_1)) - \varphi(g(\imath))). \tag{10}$$

By $w'(\imath) > 0$, we obtain $w(\mu_i(\imath)) > \alpha > 0$. By virtue of $(\mathbf{P}_1)$, Equations (5) and (7), we know that $\varphi(\imath) > 0$, $\varphi'(\imath) < 0$, and so $\varphi(\imath)$ is bounded. Thus, the right of Equation (10) is bounded, contrary to Equation (6).

If $0 < \gamma < 1$, the argument is analogous to that in the above discussion, so it is omitted. This completes the proof. □

Corollary 1. *Let $\gamma > 0$, $(\mathbf{P}_1)$, $(\mathbf{P}_3)$, $(\mathbf{P}_5)$ and Equation (6) hold. If the following inequality*

$$\left(\varphi(\imath) + \frac{m_0^{\gamma}}{g_0}(\varphi(g(\imath)))\right)' + \frac{1}{2^{\gamma-1}}\frac{C(\imath)}{r(\mu_i(\imath))}\Omega_1^{\gamma}(\mu_i(\imath))\varphi(\mu_i(\imath)) \leq 0, \tag{11}$$

has no eventually positive solution, where

$$\Omega_1(\imath) = \frac{\delta\imath^{n-1}}{(n-1)!},$$

for all $\delta \in (0,1)$, then Equation (1) is oscillatory.

Corollary 2. *Let $\gamma > 0$, $(\mathbf{P}_1)$, $(\mathbf{P}_3)$, $(\mathbf{P}_5)$ and Equation (6) hold, and*

$$\left(\varphi(\imath) + \frac{m_0^{\gamma}}{g_0}(\varphi(g(\imath)))\right)' + \frac{C(\imath)}{r(\mu_i(\imath))}\Omega_1^{\gamma}(\mu_i(\imath))\varphi(\mu_i(\imath)) \leq 0, \tag{12}$$

has no eventually positive solution, then Equation (1) is oscillatory.

Theorem 2. *Let $n \geq 4$ is an even and $(\mathbf{P}_1)$, $(\mathbf{P}_2)$, $(\mathbf{P}_5)$ hold. If $\xi, \sigma \in C^1([\imath_0, \infty), (0, \infty))$ such that*

$$\int_{\imath_1}^{\infty} (\xi(\imath)E(\imath) - K(\imath))\mathrm{d}\imath = \infty \tag{13}$$

and

$$\int_{\imath_1}^{\infty}\left(\sigma(\imath)\int_{\imath}^{\infty}(s-\imath)^{n-4}\frac{E^*(s)}{(n-4)!r^{1/\gamma}(s)}\mathrm{d}s-\frac{(\sigma'(\imath))^2}{4\sigma(\imath)\mu_i'(\imath)}\right)\mathrm{d}\imath=\infty, \tag{14}$$

where

$$E(\imath)=\sum_{i=1}^{j}q_i(\imath)(1-m(\mu_i(\imath)))^{\gamma},\quad E^*(\imath)=\left(\int_{\imath}^{\infty}E(s)ds\right)^{1/\gamma}$$

and

$$K(\imath)=\frac{r(\imath)(\xi'(\imath))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\xi(\imath)\mu_i'(\imath)\Omega_2(\mu_i(\imath)))^{\gamma}},\ \ \Omega_2(\imath)=\frac{\delta\imath^{n-2}}{(n-2)!},$$

for all $\delta\in(0,1)$*, then Equation (1) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1. By Lemma 5, w satisfies case $(\mathbf{I}_1)$ or case $(\mathbf{I}_2)$.

Assume that case $(\mathbf{I}_1)$ holds. Then, $\lim_{\imath\to\infty}w'(\imath)\neq\infty$. From that and Lemma 3, we achieve

$$w'(\imath)\geq\Omega_2(\imath)w^{(n-1)}(\imath).$$

By $\mu_i(\imath)\leq\imath$ and the fact that $w^{(n-1)}(\imath)$ is not increasing, we obtain

$$\frac{w'(\mu_i(\imath))}{w^{(n-1)}(\imath)}\geq\frac{w'(\mu_i(\imath))}{w^{(n-1)}(\mu_i(\imath))}\geq\Omega_2(\mu_i(\imath)). \tag{15}$$

Owing to $w'(\imath)>0$ and Equation (2), we obtain

$$y(\mu_i(\imath))\geq(1-m(\mu_i(\imath)))w(\mu_i(\imath)). \tag{16}$$

Let

$$\eta(\imath)=\xi(\imath)\frac{r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma}}{w^{\gamma}(\mu_i(\imath))},\imath\geq\imath_1, \tag{17}$$

Thus, $\eta(\imath)>0$ on $[\imath_1,\infty)$ and set

$$d(\imath)=\frac{\xi'(\imath)}{\xi(\imath)},\quad e(\imath)=\frac{\gamma\mu_i'(\imath)\Omega_2(\mu_i(\imath))}{(\xi(\imath)r(\imath))^{1/\gamma}}.$$

Then,

$$\eta'(\imath)\leq-\xi(\imath)E(\imath)+d(\imath)\eta(\imath)-e(\imath)\eta^{\gamma+1/\gamma}.$$

By Lemma 2, we obtain

$$d(\imath)\eta(\imath)-e(\imath)\eta^{\gamma+1/\gamma}\leq K(\imath).$$

Thus,

$$\eta'(\imath)\leq-\xi(\imath)E(\imath)+K(\imath).$$

This yields

$$\int_{\imath_1}^{\infty}(\xi(\imath)E(\imath)-K(\imath))\mathrm{d}\imath\leq\eta(\imath_1),$$

which contradicts Equation (13).

For the case $(\mathbf{I}_2)$, according to Equations (1) and (16), we achieve

$$\left(r(\imath)\left(w^{(n-1)}(\imath)\right)^{\gamma}\right)'+E(\imath)w^{\gamma}(\mu_i(\imath))\leq0. \tag{18}$$

Integrating Equation (18) from $\imath$ to ∞, from $w'(\imath) > 0$ and $(\mathbf{P}_5)$, we find

$$-w^{(n-1)}(\imath) + w(\mu_i(\imath)) \frac{E^*(\imath)}{r^{1/\gamma}(\imath)} \leq 0. \tag{19}$$

Integrating Equation (19) from $\imath$ to ∞, we see

$$w^{(n-2)}(\imath) + w(\mu_i(\imath)) \int_\imath^\infty \frac{E^*(s)}{r^{1/\gamma}(s)} ds \leq 0. \tag{20}$$

Continuously, if we integrate Equation (20) from $\imath$ to ∞ for all $(n-4)$ times, we find

$$w''(\imath) + w(\mu_i(\imath)) \int_\imath^\infty \frac{(s-\imath)^{n-4} E^*(s)}{(n-4)! r^{1/\gamma}(s)} ds \leq 0. \tag{21}$$

Let

$$\nu(\imath) = \sigma(\imath) \frac{w'(\imath)}{w(\mu_i(\imath))} > 0. \tag{22}$$

Since $w'(\imath)$ is decreasing and $\mu_i(\imath) \leq \imath$, according to Lemma 2, we find

$$\nu'(\imath) \leq -\sigma(\imath) \int_\imath^\infty \frac{(s-\imath)^{n-4} E^*(s)}{(n-4)! r^{1/\gamma}(s)} ds + \frac{(\sigma'(\imath))^2}{4\sigma(\imath)\mu_i'(\imath)}.$$

This implies that

$$\int_{\imath_1}^\infty \left(\sigma(\imath) \int_\imath^\infty \frac{(s-\imath)^{n-4} E^*(s)}{(n-4)! r^{1/\gamma}(s)} ds - \frac{(\sigma'(\imath))^2}{4\sigma(\imath)\mu_i'(\imath)} \right) \leq \nu(\imath_1).$$

This contradicts our assumption Equation (14), which completes the proof. $\square$

Corollary 3. *Let $n \geq 4$ be even and $(\mathbf{P}_1)$, $(\mathbf{P}_4)$, $(\mathbf{P}_5)$ hold. If $\xi, \sigma \in C^1([\imath_0, \infty), (0, \infty))$ and $\varepsilon \in (0,1)$ such that*

$$\int_{\imath_1}^\infty \left(\sum_{i=1}^j q_i(\imath) \xi(\imath) M(\mu_i(\imath)) - K^*(\imath) \right) \mathrm{d}\imath = \infty \tag{23}$$

and

$$\int_{\imath_1}^\infty \left(\sigma(\imath) \int_\imath^\infty (s-\imath)^{n-4} \frac{M^*(s)}{(n-4)! r^{1/\gamma}(s)} \mathrm{d}s - \frac{(\sigma'(\imath))^2}{4\sigma(\imath)\mu_i'(\imath) R(\imath)} \right) \mathrm{d}\imath = \infty, \tag{24}$$

where

$$\begin{aligned} M(\imath) &= \frac{1}{m(g^{-1}(\imath))} \left(1 - \frac{\left(g^{-1}(g^{-1}(\imath))\right)^{n-1/\varepsilon}}{(g^{-1}(\imath))^{n-1/\varepsilon} m(g^{-1}(g^{-1}(\imath)))} \right), \\ M^*(\imath) &= \left(\int_\imath^\infty \sum_{i=1}^j q_i(s) M^\gamma(\mu_i(s)) ds \right)^{1/\gamma} \end{aligned}$$

and

$$K^*(\imath) = \frac{r(\imath)(\xi'(\imath))^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left(\xi(\imath)\mu_i'(\imath) R(\imath) \Omega_2(g^{-1}(\mu_i(\imath)))\right)^\gamma}, \; R(\imath) = \left(g^{-1}(\mu_i(\imath))\right)',$$

then Equation (1) is oscillatory.

4. Examples

Example 1. *Consider the equation*

$$\left(e^{\imath}\left(\left(y(\imath)+\frac{1}{2}y\left(\frac{\imath}{3}\right)\right)'''\right)^{1/5}\right)'+e^{\imath}y^{1/5}\left(\frac{\imath}{4}\right)=0,\quad \imath\geq 1. \tag{25}$$

Let $n=4,\ \gamma=1/5,\ r(\imath)=q_i(\imath)=e^{\imath},\ m(\imath)=1/2,\ g(\imath)=\imath/3,\ \mu_i(\imath)=\imath/4$, *then it is easy to see that*

$$C(\imath)=q_i(g(\imath))=e^{\imath/3}$$

and

$$\int_1^\infty e^{s/3}\mathrm{d}s=\infty.$$

By Theorem 1, Equation (25) is oscillatory.

Example 2. *Let the equation*

$$\left(\imath\left(y(\imath)+\frac{1}{2}y\left(\frac{\imath}{2}\right)\right)'''\right)'+\frac{q_0}{\imath^3}y\left(\frac{9\imath}{10}\right)=0,\quad \imath\geq 1, \tag{26}$$

where $q_0\geq 1$. *Let* $n=4,\ \gamma=1,\ r(\imath)=\imath,\ q_i(\imath)=q_0/\imath^3,\ m(\imath)=1/2,\ g(\imath)=\imath/2,\ \mu_i(\imath)=9\imath/10$, *we set* $\xi(\imath)=\imath^2,\ \sigma(\imath)=\imath,\ \delta=\left(9^3-1\right)/9^3$, *then it is easy to see that*

$$E(\imath)=\frac{q_0}{2\imath^3},\ K(\imath)=\frac{2000}{9^3\delta},$$

then

$$\int_{\imath_1}^\infty\left(\frac{q_0}{2}-\frac{2000}{9^3\delta}\right)\imath^{-1}\mathrm{d}\imath\to\infty\ \text{as}\ \imath\to\infty\ \text{if}\ q_0\geq\frac{4000}{9^3-1}\approx 5.5,$$

and it is easy to see that

$$E(\imath)=\frac{q_0}{4\imath^2},$$

then

$$\int_{\imath_1}^\infty\left(\frac{q_0}{8}-\frac{5}{18}\right)\imath^{-1}\mathrm{d}\imath\to\infty\ \text{as}\ \imath\to\infty\ \text{if}\ q_0\geq\frac{20}{9}\approx 2.2$$

By Theorem 2, Equation (26) is oscillatory if $q_0\geq\frac{4000}{9^3-1}\approx 5.5$.

Example 3. *Consider the equation*

$$\left(\imath\left(y(\imath)+16y\left(\frac{\imath}{2}\right)\right)'''\right)'+\frac{q_0}{\imath^3}y\left(\frac{\imath}{3}\right)=0,\quad \imath\geq 1, \tag{27}$$

where $q_0\geq 1$. *Let* $n=4,\ \gamma=1,\ r(\imath)=\imath,\ q_i(\imath)=q_0/\imath^3,\ m(\imath)=16,\ g(\imath)=\imath/2,\ \mu_i(\imath)=\imath/3$, *we set* $\xi(\imath)=\imath^2,\ \sigma(\imath)=\imath,\ \delta=(27\times 8-1)/27\times 8,\ \varepsilon=\left(10^4-1\right)/10^4$, *then it is easy to see that*

$$M(\mu_i(\imath))=\frac{1}{16}\left(1-\frac{2^{3/\varepsilon}}{16}\right)\approx\frac{1}{32}$$

and

$$K^*(\imath)=\frac{27}{4\delta t},$$

then

$$\int_{\imath_1}^{\infty}\left(\sum_{i=1}^{j} q_i(\imath)\xi(\imath)M(\mu_i(\imath)) - K^*(\imath)\right) = \int_{\imath_1}^{\infty}\left(\frac{q_0\imath^2}{32\imath^3} - \frac{27}{4\delta t}\right)\mathrm{d}\imath$$

$$= \int_{\imath_1}^{\infty}\left(\frac{q_0}{32} - \frac{27}{4\delta}\right)\imath^{-1}\mathrm{d}\imath \to \infty \ \ as \ \ \imath \to \infty$$

$$if \ q_0 \geq \frac{27 \times 8}{\delta} \approx 217.1,$$

and it is also easy to see that

$$M^*(s) = \int_s^{\infty} \frac{q_0}{32s^3} ds = \frac{q_0}{64} s^{-2},$$

then

$$\int_{\imath_1}^{\infty}\left(\sigma(\imath)\int_{\imath}^{\infty}(s-\imath)^{n-4}\frac{M^*(s)}{(n-4)!r^{1/\gamma}(s)}\mathrm{d}s - \frac{(\sigma'(\imath))^2}{4\sigma(\imath)\mu_i'(\imath)R(\imath)}\right)\mathrm{d}\imath$$

$$\to \infty \ \ as \ \ \imath \to \infty \ \ if \ \ q_0 \geq \frac{3 \times 128}{8} = 48.$$

By Corollary 3, Equation (27) is oscillatory if $q_0 \geq 217.1$.

5. Conclusions

In this paper, we investigate oscillation conditions of Equation (1). New oscillation conditions are established by the comparison method and Riccati technique. These criteria simplify and extend many well-known results for oscillation of even-order delay Emden–Fowler differential equations with a neutral term. Continuing this work in the future, we can obtain the oscillation properties of the equation

$$\left(r(\imath)\left(w^{(n-1)}(\imath)\right)^{p-1}\right)' + \sum_{i=1}^{j} q_i(\imath)y^{p-1}(\mu_i(\imath)) = 0, \ \imath \geq \imath_0,$$

where $w(\imath) = y(\imath) + m(\imath)y(g(\imath))$.

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