Article

Use of the Symmetries in the Study of Vibration Response of a Hollow Cylinder

Călin Itu ¹, Sorin Vlase ^{1,2,*} , Marin Marin ³ and Ana Toderiță ²

¹ Department of Mechanical Engineering, Transilvania University of Braşov, B-dul Eroilor 20, 500036 Braşov, Romania; calinitu@unitbv.ro

² Romanian Academy of Technical Sciences, B-dul Dacia 26, 030167 Bucharest, Romania; ana.toderita@unitbv.ro

³ Department of Mathematics, Transilvania University of Braşov, B-dul Eroilor 20, 500036 Braşov, Romania; m.marin@unitbv.ro

* Correspondence: svlase@unitbv.ro; Tel.: +40-722-643-020

Abstract: The paper studies the vibration response of an elastic solid that has geometric symmetries. These determine special properties of the equations of motion of such a system, presented in the case of a cylindrical body (hollow cylinder). The properties of the eigenvalues and eigenmodes of these systems are theoretically established. A validation of these results is made using the finite element method. The use of the obtained results can lead to an easing of the vibration analysis of such a system and, consequently, to the decrease of the cost related to the design and manufacture of such a structure. The properties presented and demonstrated in the paper can simplify the numerical calculation and experimental verifications of such a structure. Serving these symmetries, the computation cost decrease substantially and will depend not in the number of the identical parts.

Keywords: symmetry; hollow cylinder; FEM; eigenvalue; eigenmode; vibration

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1. Introduction

A problem that often arises in the field of civil engineering, thus gaining importance, is the use of symmetries that can be solved by obtaining the ease of calculation and design time of a structure with identical parts. In engineering practice, most buildings, works of art, halls, etc., are composed of identical parts and have symmetries. This has been happening since antiquity and the reasons are many. First, a lighter, faster design, then a cheaper realization and, less important for engineers but important for beneficiaries, for aesthetic reasons. In engineering, civil engineering and other fields, such as the machine or equipment manufacturing industry, the automotive industry, or the aerospace industry, there are products, elements, machines and components that contain identical, repetitive parts, which have in their composition parts that have symmetries of different types. Within the constructions, most of the buildings, works of art, halls, etc., are composed of identical parts and have symmetries. These properties can be used successfully to facilitate the static and dynamic analysis of some structures. Symmetries of different types that provide specific properties have been observed for a long time and are used especially in the static case.

Structures that present different symmetries or structures with repetitive elements are common both in the field of engineering applications, architecture and construction, but also in nature. These mechanical systems have identical geometry, properties, boundary conditions and connection with the main structure.

As a general result the eigenvalues and eigenmodes for a mechanical system are defined by the composition, geometry and properties of the whole system. For structures with symmetries or/and repetitive parts, some eigenvalues and eigenmodes are identical to those of the whole system and this can facilitate the calculation of such a system. In this

way the necessary calculation time and costs for the design of such a structure decreases considerably.

From the multitude of aspects that can be studied we focused on the analysis of the dynamic response of these structures, especially on the study of vibration behavior. Although for the static case the problem has been studied and properties have been deduced that facilitate the design and calculation work, in the field of dynamic analysis the use of these properties is still reduced. In the paper, we propose to identify some vibration properties of cylindrical bodies, properties that can make the process of design, calculation and manufacturing of such structures easier.

In engineering practice (but also in the natural world) there are many structures presenting some type of symmetry [1]. A structure assembled by a group of identical substructures can be defined as repetitive. These identical substructures must be identical in all the parameters defining the mechanical behavior (geometrical shape, physical properties, boundary conditions and connections). To make a vibrational analysis of such types of structure, using finite element method (FEM), is necessary to mesh the whole structure, a very costly step. The use of the information about the symmetry properties of such structures could lead to a more economical way to obtain the results. In this way, for example, the eigenvalues of the whole structure can be determined from those of its single substructure. The properties of the repetitive structure were studied in [2–8]. Another way to study this type of problem is to analyze the wave propagations in linear periodic systems [9–13].

Thus far, the work presented previously has focused on the study of discrete systems. In the paper, we extend the method to continuous mechanical systems. The developed models are solved using the finite element method, which possesses, the well-developed methods for determining the eigenfrequencies and the eigenmodes. This can become a major advantage using the classical finite elements for the study of vibrations of large structures. Conventional FEM is used in the paper to form the classical mass and stiffness matrices.

There are many kinds of symmetry. In this paper, we will study a hollow cylinder that presents a mirror-symmetry and an axis-symmetry. A mirror-symmetric structure is if its geometric shape, physical properties and boundary conditions are all symmetric with respect to a plane (or a straight line). An axis-symmetric structure is if its geometry, physical properties and boundary conditions are all unaltered after rotating it by an arbitrary angle with respect to an axis (a straight line).

The existence of symmetries can be used to facilitate the analysis of vibration of bodies or structures. Although properties concerning the vibrations behavior of the mechanical systems with certain symmetries were set by many researchers [14,15] and some are known intuitively, after practice, by the users of finite element software, a systematic study of the effects of these symmetries is not yet done. The problem is complex, it is necessary to study several types of symmetries and their effect on the behavior of bodies or mechanical vibration systems. Meirovitch [6] specified, allusively, the importance of considering the symmetries. A first systematic approach to the problem is made in [10] and developed in [16]. A rigorous mathematical demonstration of mechanical properties of the equations of motion written for symmetrical systems with symmetries is made in [17]. New researches in the domain are presented in [18–27]. For the problems with complex symmetries a systematic study is not yet done.

2. Materials and Methods

In the following, we will present the motion equations for a cylinder considered as a continuous solid [28–30]. Motion equations can be written, using the cylindrical coordinates (r, θ, z) , in terms of stress components $\sigma_r, \sigma_\theta, \sigma_z, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z}$ are expressed by:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_r - \sigma_\theta) + b_r = \mu \ddot{u}_r, \quad (1)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r\theta} + b_{\theta} = \mu \ddot{u}_{\theta}, \quad (2)$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r} \sigma_{rz} + b_z = \mu \ddot{u}_z, \quad (3)$$

where μ is the mass density of the material, b_r, b_{θ}, b_z are the body forces per unit volume, $\ddot{u}_r, \ddot{u}_{\theta}, \ddot{u}_z$ the acceleration. The strains can be written, in cylindrical coordinate, as:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta} = \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \quad (4)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right), \quad \varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \quad \varepsilon_{\theta z} = \frac{1}{2} \left(\frac{\partial u_{\theta}}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \quad (5)$$

where $\varepsilon_r, \varepsilon_{\theta}, \varepsilon_z, \varepsilon_{r\theta}, \varepsilon_{rz}, \varepsilon_{\theta z}$ are the strains components, λ and G are the Lamé constants.

The stresses can be obtained using the generalized Hooke law:

$$\sigma_r = \lambda \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2G \frac{\partial u_r}{\partial r} \quad (6)$$

$$\sigma_{\theta} = \lambda \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2G \left(\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{\partial u_r}{\partial r} \right) \quad (7)$$

$$\sigma_z = \lambda \left(\frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2G \frac{\partial u_z}{\partial z} \quad (8)$$

$$\sigma_{r\theta} = G \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) \quad (9)$$

$$\sigma_{rz} = G \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (10)$$

$$\sigma_{\theta z} = G \left(\frac{\partial u_{\theta}}{\partial z} + \frac{1}{2r} \frac{\partial u_z}{\partial \theta} \right) \quad (11)$$

If we denote the cubic dilatation with:

$$I_{1\varepsilon} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \quad (12)$$

the motion Equation (1) can be written:

$$G \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta}}{\partial \theta} \right) + (\lambda + G) \frac{\partial I_{1\varepsilon}}{\partial \varepsilon} + b_r = \mu \ddot{u}_r, \quad (13)$$

$$G \left(\nabla^2 u_{\theta} - \frac{u_{\theta}}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) + (\lambda + G) \frac{1}{r} \frac{\partial I_{1\varepsilon}}{\partial \theta} + b_{\theta} = \mu \ddot{u}_{\theta}, \quad (14)$$

$$G \nabla^2 u_z + (\lambda + G) \frac{\partial I_{1\varepsilon}}{\partial z} + b_z = \mu \ddot{u}_z, \quad (15)$$

or:

$$G \nabla^2 u + (\lambda + G) \nabla(\nabla u) = \mu \frac{\partial u}{\partial t^2}. \quad (16)$$

The equations represents a system of three, second-order partial differential equations. The equations are, generally, complex, with many terms. A verified and widely used method of solving is FEM.

3. Use of FEM for the Study of a Hollow Cylinder

Consider a hollow cylinder as a linear elastic solid. The hollow cylinder is made by two identical parts, separated with a liaison plan (Figure 1). At the same time, if we divide

the cylinder in two, using a transversal plan xy , these two parts are identical. Here, we have multiple symmetries.

Figure 1. The hollow cylinder considered composed of different identical parts. (a) Full cylinder; (b,d) half cylinder; (c,e) quarter cylinder; (f) eight parts of a cylinder.

The motion equations of the free undamped vibrations, using FEM, can be written under the form [31–34]:

$$[m]\{\ddot{x}\} + [k]\{x\} = 0 \quad (17)$$

where:

- $[m]$ is the mass matrix;
- $[k]$ the stiffness matrix.

Element type used in analyzes: HEX8—first order hexahedral element with 8 corner nodes, each node having 3 degrees of nodal freedom.

Let us consider now the problem of determining the eigenvalues for a hollow cylinder represented in Figure 1. This system, denoted with (S) , is composed of two identical subsystems (S_1) —half cylinder. Using the classical methods of mechanics [35–38] is possible to write the equations of the free non-damped vibrations for the whole structure as:

$$\begin{bmatrix} m_a & 0 & m_{ab} & m_{ab} \\ 0 & m_a & m_{ab} & m_{ab} \\ m_{ab} & m_{ab} & m_b & 0 \\ m_{ab} & m_{ab} & 0 & m_b \end{bmatrix} \begin{Bmatrix} \ddot{x}_s \\ \ddot{x}_d \\ \ddot{x}_{sd} \\ \ddot{x}_{ds} \end{Bmatrix} + \begin{bmatrix} k_a & 0 & k_{ab} & k_{ab} \\ 0 & k_a & k_{ab} & k_{ab} \\ k_{ab} & k_{ab} & k_b & 0 \\ k_{ab} & k_{ab} & 0 & k_b \end{bmatrix} \begin{Bmatrix} x_s \\ x_d \\ x_{sd} \\ x_{ds} \end{Bmatrix} = 0 \quad (18)$$

m_a , m_b , k_a , k_b are symmetrical square matrices; m_a is the mass matrix for a half cylinder, m_b the mass matrix for the nodes of the common part (the nodes being in the common plan of the identical parts), k_a the stiffness matrix of the half-cylinder, k_b the stiffness matrix of the system common part (assuring the elastic liaison between the nodes being in the common plan and the other nodes of the identical parts), m_{ab} and k_{ab} being the mass, respectively, stiffness matrix that describes the liaison between the two identical parts via the common plan.

For half a structure, the equations become:

$$[m_a]\{x_s\} + [k_a]\{x_s\} = 0 \quad (19)$$

for the left part and

$$[m_a]\{x_d\} + [k_a]\{x_d\} = 0 \quad (20)$$

for the right part. These two equations are identical and have the same eigenvalues and eigenvectors. By numerical calculation, using the method of finite elements [11,12,14], the following property has been established:

Property 1: The eigenvalues for system (9) are eigenvalues for system (8) as well.

That means the solutions of algebraic equations

$$\det([k_a] - \omega^2[m_a]) = 0 \quad (\text{or } |[k_a] - \omega^2[m_a]| = 0) \quad (21)$$

are also solutions of the algebraic equation

$$\det \left(\begin{bmatrix} m_a & 0 & m_{ab} & m_{ab} \\ 0 & m_a & m_{ab} & m_{ab} \\ m_{ab} & m_{ab} & m_b & 0 \\ m_{ab} & m_{ab} & 0 & m_b \end{bmatrix} - \omega^2 \begin{bmatrix} k_a & 0 & k_{ab} & k_{ab} \\ 0 & k_a & k_{ab} & k_{ab} \\ k_{ab} & k_{ab} & k_b & 0 \\ k_{ab} & k_{ab} & 0 & k_b \end{bmatrix} \right) = 0 \quad (22)$$

or

$$\begin{vmatrix} k_a - \omega^2 m_a & 0 & k_{ab} - \omega^2 m_{ab} & k_{ab} - \omega^2 m_{ab} \\ 0 & k_a - \omega^2 m_a & k_{ab} - \omega^2 m_{ab} & k_{ab} - \omega^2 m_{ab} \\ k_{ab} - \omega^2 m_{ab} & k_{ab} - \omega^2 m_{ab} & k_b - \omega^2 m_b & 0 \\ k_{ab} - \omega^2 m_{ab} & k_{ab} - \omega^2 m_{ab} & 0 & k_b - \omega^2 m_b \end{vmatrix} = 0 \quad (23)$$

We could write:

$$|[k_a] - \omega^2[m_a]| = 0 \Rightarrow (23) \quad (24)$$

that implies the polynomial in ω^2 from Equation (11) is divided by the polynomial in ω^2 from Equation (10). In following is presented the Proposition, proved in [39], that assures the validity of the Property 1.

Proposition: Consider the square polynomial matrices with complex coefficients, of size n , noted $A, B, C, L, Z = O_n$ and matrix

$$M = \begin{pmatrix} A & Z & B \\ Z & A & B \\ L & L & C \end{pmatrix}$$

then $\det(M)$ is dividable by $\det(A)$.

4. Eigenvalues and Eigenmodes

In a line of Table 1 are presented the common eigenvalues, property presented before. The eigenvalues of a part of the cylinder are eigenvalues for the whole cylinder.

The eigenmodes are represented in Table 1. The properties presented before are sustained by the graphical representation on this table. We have two kinds of symmetry. If we look to the last line of the table we can see that the eigenfrequency 22,043 Hz is obtained for six different type of element. This thing is happening for all the symmetry that can be built for the cylinder.

The paper aims to show that considering the symmetries that occur within a structure can lead to the identification of properties that allow ease of calculation. For example, in the case presented in the paper, that of a cylinder, the consideration of some substructures of the cylinder allows to reduce the size of the problem of calculating the eigenvalues and eigenmodes. Thus, if we consider for example a quarter of the cylinder, all eigenvalues calculated for this substructure are found between the eigenvalues of the entire cylinder. In this, using the existing symmetries, calculations can be made on substructures reduced

in size. The calculation for these is easier and in this way a number of eigenvalues can be determined for the whole structure. In this way, knowing these eigenvalues, the size of the initial system and, consequently, the necessary computational effort can be significantly reduced. In general, the numerical solution of some problems leads to an exponential explosion of the necessary computational effort [39]. FEM is only an analysis tool, considered appropriate by the authors for analyzing the problem.

Table 1. The eigenmodes of the studied parts (Hz).

Model											
Full		Half		Quarter		Half		Quarter		An Eighth

Mode	Freq. * Hz	Mode	Freq. * Hz	Mode	Freq. * Hz	Mode	Freq. * Hz	Mode	Freq. * Hz	Mode	Freq. * Hz
7;8	7970	4	7970			4	7970	2	7970

9	7998

10;11	10,055	5	10,055	2	10,055	6;7	10,055	3	10,055	1	10,055

12;13	10,287	6	10,287	3	10,287						

Table 1. Cont.

Model																	
Full		Half		Quarter		Half		Quarter		An Eighth

14;15	12,095	7	12,095	4	12,095	8;9	12,094	4	12,094	2	12,094

16;17	12,557	8	12,557

18	12,630	9	12,630	5	12,630	10	12,630	5	12,630	3	12,630

19	15,984					11	15,984										

Table 1. Cont.

Model											
Full		Half		Quarter		Half		Quarter		An Eighth

20;21	16,297	10	16,297	6	16,297

22;23	17,204	11	17,204

24;25	17,989	12	17,989			12;13	17,989	6	17,989

26	20,815	13	20,815	7	20,815						

Table 1. Cont.

Model											
Full		Half		Quarter		Half		Quarter		An Eighth

27;28	21,977	14	21,977	8	21,977	14;15	21,977	7	21,977	4	21,977

29	22,043	15	22,043	9	22,043	16	22,043	8	22,043	5	22,043

* Freq.—represents the calculated frequency in Hz.

5. Discussion and Conclusions

The main field of applicability of these properties is in the field of civil engineering but, just as well, there are applications in all branches of engineering. Experimental measurement in order to verify such type of model are presented in [38].

The issue addressed in this paper has been relatively little studied; however, some papers in the field exist, and the importance of repetitive elements (parts) or symmetries in performing stress and strain calculations has been observed for a long time. In this paper, we have proved that for a mechanical linear elastic system consisting of two identical subsystems, the eigenfrequencies of repetitive subsystems are eigenvalues of the entire structure as well. At the same time, the eigenmodes present certain symmetries. These properties allow the simplification of the vibration calculus for this type of structure. We can first determine the eigenvalues of the symmetric structure, then these eigenvalues can be eliminated from the motion equations of motion for the whole structure. Thus, the size of the problem decreases and allows a reduced cost in computation. The symmetries presented by eigenvectors allow a quantitative understanding of vibration phenomena in the system. The modes of the whole structure can be obtained from the modes of its one substructure. The properties can be useful for an experimental procedure. It is necessary to

perform the measurement only for a half body, using the described properties is possible to know the modes for the entire structure.

If we take into account the properties demonstrated in the paper, it is enough to calculate eigenfrequencies and eigenmodes only for half of the structure, or a quarter of the structure. Appropriate boundary conditions must be imposed in the symmetry plane. The eigenpulsations of such a structure are also the eigenpulsations of the whole structure, and therefore the size of the system can be reduced using classical methods from algebra. The number of degrees of freedom decreases by almost half, and thus the calculation necessary for the determination of its eigenpulsations and its eigenmodes is significantly reduced, and as a result, the cost of the calculation and design operations of such a structure decreases.

The eigenmodes of the whole structure are obtained by the natural extension of the symmetric modes obtained for a substructure.

At the experimental level, if eigenfrequency measurements are made only for half of the structure, they are also the eigenfrequency for the whole structure.

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