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On a Generalized Convolution Operator

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Abstract: Recently in the paper [*Mediterr. J. Math.* **2016**, *13*, 1535–1553], the authors introduced and studied a new operator which was defined as a convolution of the three popular linear operators, namely the Sălăgean operator, the Ruscheweyh operator and a fractional derivative operator. In the present paper, we consider an operator which is a convolution operator of only two linear operators (with lesser restricted parameters) that yield various well-known operators, defined by a symmetric way, including the one studied in the above-mentioned paper. Several results on the subordination of analytic functions to this operator (defined below) are investigated. Some of the results presented are shown to involve the familiar Appell function and Hurwitz–Lerch Zeta function. Special cases and interesting consequences being in symmetry of our main results are also mentioned.

Keywords: analytic functions; convolution; subordination; convex functions

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1. Introduction, Motivation and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote a class of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, $k \in \mathbb{N} = \{1, 2, \ldots\}$, let

$$\mathcal{H}[a,k] = \left\{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots \right\}.$$

We denote a subclass of $\mathcal{H}[0,1]$ by $\mathcal{A}$ whose members are of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{U}. \tag{1}$$

Additionally, let $\mathcal{K}$ denote a subclass of $\mathcal{A}$ whose members are convex (univalent) in $\mathbb{U}$ which is equivalent to

$$f \in \mathcal{A},\quad \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0,\ z \in \mathbb{U}. \tag{2}$$

Further, let $\mathcal{S}^*$ denote a subclass of $\mathcal{A}$ of starlike functions, which is symmetric to $\mathcal{K}$ by relation $f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*$.

For two functions $p, q \in \mathcal{H}(\mathbb{U})$, we say p is *subordinate* to q, or q is *superordinate* to p in $\mathbb{U}$ and write $p(z) \prec q(z), z \in \mathbb{U}$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$, and $|\omega(z)| < 1, z \in \mathbb{U}$ such that $p(z) = q(\omega(z)), z \in \mathbb{U}$. Furthermore, if the function q is univalent in $\mathbb{U}$, then we have following symmetry:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}). \tag{3}$$

A convolution (or Hadamard product) $*$ of the functions $g_1(z)$ and $g_2(z)$ of the form:

$$g_1(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g_2(z) = \sum_{k=0}^{\infty} b_k z^k \tag{4}$$

is defined by

$$g_1(z) * g_2(z) = (g_1 * g_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (g_2 * g_1)(z). \tag{5}$$

For $m \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and for $\mu > -1$, $\lambda > 0$, a linear operator $\mathcal{J}_{\lambda,\mu}^m : \mathcal{A} \to \mathcal{A}$ is defined in [1] (see also [2–4]) by

$$\begin{cases} \mathcal{J}_{\lambda,\mu}^m f(z) & = f(z), & m = 0, \\ \mathcal{J}_{\lambda,\mu}^m f(z) & = \frac{\mu+1}{\lambda} z^{1-\frac{\mu+1}{\lambda}} \int\limits_0^z t^{\frac{\mu+1}{\lambda}-2} \mathcal{J}_{\lambda,\mu}^{m+1} f(t) \mathrm{d}t, & m = -1, -2, \dots, \\ \mathcal{J}_{\lambda,\mu}^m f(z) & = \frac{\lambda}{\mu+1} z^{2-\frac{\mu+1}{\lambda}} \frac{\mathrm{d}}{\mathrm{d}t} \left(z^{\frac{\mu+1}{\lambda}-1} \mathcal{J}_{\lambda,\mu}^{m-1} f(z) \right), & m = 1, 2, \dots . \end{cases} \tag{6}$$

We note that the operator $\mathcal{J}_{\lambda,\mu}^m$ is a multiplier operator and the series expansion of $\mathcal{J}_{\lambda,\mu}^m f(z)$ for f of the form (1) is given symmetrical by

$$\mathcal{J}_{\lambda,\mu}^m f(z) = z + \sum_{k=2}^{\infty} \left(1 + \frac{\lambda(k-1)}{\mu+1} \right)^m a_k z^k. \tag{7}$$

Without any loss of generality, we may replace $\mu > -1$ and $\lambda > 0$ by one parameter $\delta = \lambda/(\mu+1) \in \mathbb{C}$ in (7), we then obtain (with complex m)

$$\mathcal{J}_{\lambda,\mu}^m f(z) =: \mathcal{J}_{\delta}^m f(z) = z + \sum_{k=2}^{\infty} [1 + \delta(k-1)]^m a_k z^k, \tag{8}$$

where only principal branch of powers is considered.

In [5], the authors defined an operator, in a symmetry to $\mathcal{J}_{\lambda,\mu}^m$, called the Srivastava–Attiya operator $G_{\mu,b}$ which for f of the form (1) is given as

$$G_{\mu,b} f(z) = z + \sum_{k=2}^{\infty} \left\{ \frac{b+1}{b+k} \right\}^{\mu} a_k z^k, \tag{9}$$

where $b, \mu \in \mathbb{C}$ $(b \neq -2, -3, \dots)$.

It is easy to see that for $\delta = 1/1+b$ and $m = -\mu$, the operator given by (8) becomes the Srivastava–Attiya operator given by (9), and we have

$$G_{\mu,b} = \mathcal{J}_{1/1+b}^{-\mu}.$$

In [6], Carlson and Schaffer defined a linear operator $L(a,c) : \mathcal{A} \to \mathcal{A}$ by

$$L(a,c) f(z) = \phi(a,c;z) * f(z), \tag{10}$$

where

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \; (z \in \mathbb{U}; a, c \in \mathbb{C}(c \neq 0, -1, -2, \dots)), \tag{11}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{for} \quad n = 0, a \neq 0, \\ a(a+1)\dots(a+n-1) & \text{for} \quad n \in \mathbb{N} = \{1, 2, 3, \dots\}, \end{cases}$$

where Γ is the Gamma function. The Carlson-Schaffer operator $L(a,c)$ contains the Ruscheweyh operator [7] given by

$$D^{\lambda}f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \ \ (\lambda > -1; z \in \mathbb{U}),$$

because $L(\lambda+1;1)f(z) = D^{\lambda}f(z)$. If $\lambda = n \in 0,1,2,\ldots$, then

$$D^{n}f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!},$$

and D^n is then the Ruscheweyh differential operator.

For the purpose of this paper, we consider a generalized form of operator $\mathcal{F}$ which is defined by

$$\mathcal{F}f(z) := \mathcal{F}^{m}_{\delta}(a,c)f(z) = \phi(a,c;z) * \mathcal{J}^{m}_{\delta}f(z),$$

where $\mathcal{J}^{m}_{\delta}$ and $\phi(a,c;z)$ are, respectively, given by (8) and (11). For the function f of the form (1), we have

$$\mathcal{F}f(z) := \mathcal{F}^{m}_{\delta}(a,c)f(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}[1+\delta(k-1)]^{m} a_k z^k, \tag{12}$$

$$(a,c \in \mathbb{C}(c \neq 0,-1,-2,\ldots), m,\delta \in \mathbb{C}),$$

which is in symmetry to the Carlson-Schaffer operator $L(a,c)$ given in (10), (11).

By putting $a = 2, c = 2-\lambda$ and $\delta = 0$ in (12), we have the series representation

$$\Omega^{\lambda}_{z}f(z) := \mathcal{F}^{m}_{0}(2,2-\lambda)f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(1+k)}{\Gamma(1+k-\lambda)} a_k z^k$$

$$(\lambda \neq 2,3,\ldots; z \in \mathbb{U}),$$

which also arises by the application of the differintegral operator Ω^{λ}_{z} defined in [8], (see also [3]) to the function $f \in \mathcal{A}$. The operator in [3] does not yield the Srivastava–Attiya operator (9), and therefore the operator defined above by (12) is not symmetric to the operator defined in [3].

Observe that for $\Re(a) > 0$, $m \in \mathbb{C}$ and $f \in \mathcal{A}$, the operator defined by (1.12) satisfies the relation

$$\mathcal{F}^{m}_{1/a}(a+1,c)f(z) = \mathcal{F}^{m+1}_{1/a}(a,c)f(z). \tag{13}$$

For $m \in \mathbb{Z}$ and $\delta > 0$, the operator $\mathcal{F}^{m}_{\delta}(a,c)$ in (12) is the one that was used in [1,9] in slightly varied forms. Furthermore, the operator $\mathcal{F}^{m}_{\delta}(a,c)$ generalizes various known operators used in Geometric Function Theory and we exhibit such relationships here :

$$\mathcal{F}^{m}_{\lambda}(c,c) = \mathcal{D}^{m}_{\lambda} \ \ (m \in \mathbb{N}_0),$$

Al-Oboudi [10],

$$\mathcal{F}^{m}_{1}(c,c) = \mathcal{D}^{m} \ \ (m \in \mathbb{N}_0),$$

Sălăgean [11],

$$\mathcal{F}^{m}_{\lambda/(\mu+1)}(c,c) = \mathcal{I}^{m}(\lambda,\mu) \ \ (\mu > -1, \lambda > 0; m \in \mathbb{N}_0),$$

Cătaş [12],

$$\mathcal{F}^{m}_{\lambda/(\mu+1)}(c,c)f(z) = \mathcal{J}^{m}_{\lambda,\mu}f(z) \ \ (\mu > -1, \lambda > 0; m \in \mathbb{Z}),$$

Sharma et al. [4],

$$\mathcal{F}^{m}_{1/(1+b)}(c,c)f(z) = G_{-m,b}f(z) \ \ (b > -1),$$

Srivastava and Attiya [5],

$$\mathcal{F}_{\delta}^{0}(a,c) = L(a,c),$$

Carlson and Schaffer [6],

$$\mathcal{F}_{1/2}^{-\alpha}(c,c) = \mathcal{P}^{\alpha} \quad (\alpha \in \mathbb{Z}^{+},$$

Jung et al. [13].

For appropriate values of the parameters in (12) when $a = 1+\nu\ (\nu > -1), b = 2-\lambda\ (\lambda < 2), \delta = 1$ and $m = n+1\ (n \in \mathbb{N}_0)$, we can obtain the operator defined in [14,15]. It is interesting to note that one requires the convolution of only two known linear operators as defined in (12) to define the various operators discussed above including a symmetric operator introduced recently in [14] which is a convolution of three well-known operators. Our aim in this paper is to study some subordination properties of the generalized operator $\mathcal{F} = \mathcal{F}_{\delta}^{m}(a,c)$. Several results are established using well-known lemmas and some of the results also involve the Appell function and the Hurwitz–Lerch Zeta function. Special cases and interesting consequences of our main results are also mentioned.

Let $\mathcal{P}[A,B]$, $-1 \le B \le 1, -1 \le A \le 1$, denote a class of functions $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}. \tag{14}$$

The class $\mathcal{P}[A,B]$ was introduced and studied by Janowski [16] and, in particular, we denote

$$\mathcal{P}[1-2\alpha,-1] = \mathcal{P}(\alpha), \quad (\alpha \le 1).$$

2. Key Lemmas

To obtain our results, we need the following lemmas.

Lemma 1 (Hallenbeck and Ruscheweyh [17] ([18], Thm. 3.1b, p.71)**.** *Let h be convex univalent in $\mathbb{U}$ with $h(0) = a$, $\gamma \neq 0$ and $\Re(\gamma) \ge 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t)\, t^{(\gamma/n)-1} \mathrm{d}t \quad (z \in \mathbb{U}).$$

The function q is convex univalent and is the best (a,n)-dominant in the sense that if $p(z) \prec q_1(z)$, then $q(z) \prec q_1(z)$.

The Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is an analytic function in $\mathbb{U}$ and is defined for $a,b,c \in \mathbb{C}$ $(c \neq 0,-1,-2,\ldots)$ by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in \mathbb{U}).$$

Following results for the function ${}_2F_1(a,b;c;z)$ are well-known.

Lemma 2 ([19,20])**.** *Let $a,b,c \in \mathbb{C}$ $(c \neq 0,-1,-2,\ldots)$, then the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ satisfies the following identities:*

(i) ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t \ \ (\Re(c) > \Re(b) > 0),$

(ii) ${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$

Lemma 3 ([21]). *Let $F, G \in \mathcal{H}(\mathbb{U})$ be any convex univalent functions in $\mathbb{U}$. If $f \prec F$ and $g \prec G$, then*

$$f * g \prec F * G \quad \text{in } \mathbb{U}.$$

Lemma 4 ([22]). *If*

$$p_i \in \mathcal{P}(\alpha_i)\ (i = 1, 2; \alpha_i \leq 1),$$

then

$$p_1 * p_2 \in \mathcal{P}(\alpha_3),$$

where $\alpha_3 = 1 - 2(1 - \alpha_1)(1 - \alpha_2)$.

Lemma 5 ([23]). *Let $p \in \mathcal{P}(\alpha)$. Then for given α, $0 \leq \alpha < 1$, we have*

$$\Re(p(z)) > 2\alpha - 1 + \frac{2(1-\alpha)}{1+|z|}\ (z \in \mathbb{U}).$$

Lemma 6. *Let $0 < \alpha \leq \beta$. If $\beta \geq 2$ or $\alpha + \beta \geq 3$, then the function in (11)*

$$\phi(\alpha, \beta; z) = \sum_{n=1}^{\infty} \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} z^n\ (z \in \mathbb{U}) \tag{15}$$

is convex univalent.

Lemma 6 is a special case of Theorem 2.12 or Theorem 2.13 contained in [7].
Making use of Lemma 4, we proved the following result.

Lemma 7 ([4]). *Let $-1 \leq B_i < A_i \leq 1$ and $p_i \in \mathcal{P}[A_i, B_i]$ $(i = 1, 2, \ldots, n)$, then (for $n \geq 2$)*

$$p_1 * p_2 * \ldots * p_n \in \mathcal{P}[A, B], \tag{16}$$

where

$$B = (-1)^{n+1} \prod_{i=1}^{n} B_i,\ A - B = \prod_{i=1}^{n} (A_i - B_i). \tag{17}$$

Furthermore,

$$p_1 * p_2 * \ldots * p_n \in \mathcal{P}(\alpha), \tag{18}$$

where

$$\alpha = 1 - 2^{n-1} \prod_{i=1}^{n} \left(1 - \frac{1 - A_i}{1 - B_i}\right) < 1.$$

3. Main Results

We begin by finding a subordination property of the operator $\mathcal{F}$ which is contained in the following theorem.

Theorem 1. *Assume that a function $f \in \mathcal{A}$, $l \in (0, 1]$ and $-1 \leq D < C \leq 1$. If the operator $\mathcal{F}_{\delta}^{m+1}(a, c)$ satisfies*

$$\left(\mathcal{F}_{\delta}^{m+1}(a, c) f(z)\right)' \prec \left(\frac{1 + Cz}{1 + Dz}\right)^l\ (z \in \mathbb{U}),$$

then for the operator $\mathcal{F}$ defined in (12),

$$(\mathcal{F}f(z))' \prec q(z), \tag{19}$$

where

$$q(z) = F_1[1/\delta; -l, l; 1/\delta + 1; -Cz, -Dz]\ (\Re(\delta) > 0)$$

is the best dominant in (19) and F_1 is the Appell function (([24] pp. 22–23); see also ([25], Section 1.6) defined by

$$\begin{aligned} F_1[a;b,c;d;x,y] &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \\ &= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(d)_k}{}_2F_1[a+k,c;d+k;y]\frac{x^k}{k!} \quad (\max\{|x|,|y|\}<1). \end{aligned} \tag{20}$$

Proof. Let $p(z) = (\mathcal{F}f(z))'$, then $p \in \mathcal{H}[1,1]$ and with the use of the identity:

$$\mathcal{F}_\delta^{m+1}(a,c)f(z) = (1-\delta)\mathcal{F}f(z) + \delta z(\mathcal{F}f(z))',$$

we have

$$\left(\mathcal{F}_\delta^{m+1}f(z)\right)' = p(z) + \delta z p'(z) \prec \left(\frac{1+Cz}{1+Dz}\right)^l, \tag{21}$$

which is a subordination by a convex univalent function. Applying Lemma 1, changing suitably the variables and making use of the identities (i) and (ii) of Lemma 2, we obtain that

$$\begin{aligned} p(z) &\prec q(z) \\ &= \frac{1}{\delta}z^{-1/\delta}\int_0^z t^{1/\delta-1}\left(\frac{1+Ct}{1+Dt}\right)^l \mathrm{d}t \\ &= \frac{1}{\delta}\int_0^1 s^{1/\delta-1}\left(\frac{1+Csz}{1+Dsz}\right)^l \mathrm{d}s \end{aligned} \tag{22}$$

and the function $q(z)$ is the best dominant.

Using now the series expansion:

$$(1-x)^{-\alpha} = \sum_{k=1}^{\infty}(\alpha)_k\frac{x^k}{k!} \quad (|x|<1)$$

for each of the binomial factors occurring in the last integrand of (22) and changing the order of summation and integration (permissible under the assumed conditions mentioned above) and interpreting the resulting series in terms of the Appell function defined above by (20), we get

$$p(z) \prec q(z) = F_1[1/\delta; -l, l; 1/\delta+1; -Cz, -Dz].$$

This proves the result that

$$(\mathcal{F}f(z))' \prec q(z),$$

where $q(z)$ is a convex univalent function. □

On using the relation (13), we obtain the following result from Theorem 1 provided that $\Re(a) > 0$.

Corollary 1. *Let a function $f \in \mathcal{A}$ and $l \in (0,1]$, $-1 \le D < C \le 1$. If $\Re(a) > 0$ and the operator $\mathcal{F}_{1/a}^m(a+1,c)$ satisfies*

$$(\mathcal{F}_{1/a}^m(a+1,c)f(z))' \prec \left(\frac{1+Cz}{1+Dz}\right)^l \ (z \in \mathbb{U}),$$

then

$$(\mathcal{F}_{1/a}^m(a,c)f(z))' \prec r(z), \tag{23}$$

where

$$r(z) = F_1[a; -l, l; a+1; 1+Cz, 1+Dz],$$

is the best dominant in (23) and F_1 is the Appell function defined by (20).

Theorem 2. *Let a function $f \in \mathcal{A}$ and $-1 \leq F < E \leq 1$. If for $\Re(a) > 0$, the operator $\mathcal{F}_\delta^m(a+1,c)$ satisfies*

$$(\mathcal{F}_\delta^m(a+1,c)f(z))' \prec \frac{1+Ez}{1+Fz} \; (z \in \mathbb{U}), \tag{24}$$

then for the operator $\mathcal{F}$ defined in (12), we have

$$(\mathcal{F}f(z))' = (\mathcal{F}_\delta^m(a,c)f(z))' \prec Q(z) \; (z \in \mathbb{U}), \tag{25}$$

where

$$Q(z) = \frac{{}_2F_1\left(1,1;a+1;\frac{Fz}{1+Fz}\right) + \frac{a}{a+1}Ez\,{}_2F_1\left(1,1;a+2;\frac{Fz}{1+Fz}\right)}{1+Fz}$$

is the best dominant in (25).

Proof. Let $P(z) = (\mathcal{F}f(z))'$, then $P \in \mathcal{H}[1,1]$ and with the use of the identity:

$$\mathcal{F}_\delta^m(a+1,c)f(z) = \left(1 - \frac{1}{a}\right)\mathcal{F}f(z) + \frac{1}{a}z(\mathcal{F}f(z))', \tag{26}$$

we have

$$(\mathcal{F}_\delta^m(a+1,c)f(z))' = P(z) + \frac{1}{a}zP'(z) \prec \frac{1+Ez}{1+Fz},$$

which by Lemma 1, and by the change of variables followed by the use of the identities (i) and (ii) of Lemma 2 gives

$$\begin{aligned} P(z) &\prec Q(z) \\ &= az^{-a}\int_0^z t^{a-1}\frac{1+Et}{1+Ft}\mathrm{d}t \\ &= a\int_0^1 s^{a-1}\frac{1+Esz}{1+Fsz}\mathrm{d}s \\ &= a\left[\int_0^1 \frac{s^{a-1}}{1+Fsz}\mathrm{d}s + Ez\int_0^1 \frac{s^a}{1+Fsz}\mathrm{d}s\right] \\ &= \frac{{}_2F_1\left(1,1;a+1;\frac{Fz}{1+Fz}\right) + \frac{a}{a+1}Ez\,{}_2F_1\left(1,1;a+2;\frac{Fz}{1+Fz}\right)}{1+Fz}; \end{aligned} \tag{27}$$

and the function Q is the best dominant. This proves the result that

$$(\mathcal{F}f(z))' \prec Q(z),$$

where Q is a convex univalent function. □

Applying the cases when $F \neq 0$ and $F = 0$ to the expression (27), we obtain the following result (28) with the use of (i) and (ii) of Lemma 2. Furthermore, if we write the expression (27) as

$$Q(z) = \int_0^1 \mathcal{G}(s,z)\mathrm{d}\mu(s),$$

where $\mathcal{G}(s,z) = \frac{1+Esz}{1+Fsz}$ $(0 \leq s \leq 1)$ and $\mathrm{d}\mu(s) = as^{a-1}\mathrm{d}s$, so that $\int_0^1 \mathrm{d}\mu(s) = 1$, we obtain for $|z| \leq r < 1$,

$$\Re(Q(z)) \geq \int_0^1 \frac{1-Esr}{1-Fsr}\mathrm{d}\mu(s) = Q(-r).$$

Since, for $|z| \leq r < 1$, we have $\Re\left(\frac{1+Ez}{1+Fz}\right) \geq \frac{1-Er}{1-Fr}$, therefore letting $r \to 1^-$, we obtain the following result from Theorem 2.

Corollary 2. *Let a function $f \in \mathcal{A}$ and $-1 \leq F < E \leq 1$. If $\Re(a) > 0$, the operator $\mathcal{F}_\delta^m(a+1,c)$ satisfies the condition (24), then*

$$(\mathcal{F}f(z))' \prec Q(z) = \begin{cases} \frac{1+Ez}{1+Fz} - \frac{\left(\frac{E}{F}-1\right)\left[{}_2F_1\left(1,1;a+1;\frac{Fz}{1+Fz}\right)-1\right]}{1+Fz}, & F \neq 0, \\ 1 + \frac{a}{a+1}Ez, & F = 0, \end{cases} \quad z \in \mathbb{U}. \tag{28}$$

Furthermore,

$$\Re(Q(z)) > \rho, \tag{29}$$

where

$$\rho = \begin{cases} \frac{E}{F} + \frac{\left(1-\frac{E}{F}\right)\,{}_2F_1\left(1,1;a+1;\frac{F}{F-1}\right)}{1-F}, & F \neq 0, \\ 1 - \frac{a}{a+1}E, & F = 0. \end{cases}$$

The result is best possible.

Additionally, on applying a special case when $a = 1$, we obtain from the expression (27), the following result involving the Sălăgean operator $\mathcal{D}^m$ for $m \in \mathbb{N}_0$ with the use of the identity:

$${}_2F_1\left(1,1;2;\frac{Fz}{1+Fz}\right) = \frac{(1+Fz)\log(1+Fz)}{Fz} \quad (F \neq 0).$$

Corollary 3 ([4], Corollary 2.2, p. 54)**.** *If for $-1 \leq F < E \leq 1$,*

$$\left(\mathcal{D}^{m+1}f(z)\right)' \prec \frac{1+Ez}{1+Fz}, z \in \mathbb{U},$$

then

$$(\mathcal{D}^m f(z))' \prec \begin{cases} \frac{1+Ez}{1+Fz} - (F-1)\left\{\frac{\log(1+Fz)}{Fz} - \frac{1}{1+Fz}\right\}, & F \neq 0, \\ 1 + \frac{1}{2}Ez, & F = 0, \end{cases} \quad z \in \mathbb{U}.$$

The result is best possible.

We next prove the following theorem using Lemma 5.

Theorem 3. *Let a function $f \in \mathcal{A}$ and $-1 \leq F < E \leq 1$. If $\Re(a) > 0$ and the operator $\mathcal{F}_\delta^m(a+1,c)$ satisfies the condition (24), then*

$$(\mathcal{F}f(z))' \in \mathcal{P}(\gamma),$$

where

$$\gamma = 1 + \frac{E-F}{1-F}\left[{}_2F_1\left(1,1;a+1;\frac{1}{2}\right) - 2\right]. \tag{30}$$

Proof. Let $\left(\mathcal{F}_\delta^m(a+1,c)f(z)\right)' \prec \frac{1+Ez}{1+Fz}$ $(z \in \mathbb{U})$, then we have

$$\Re\left\{(\mathcal{F}_\delta^m(a+1,c\}f(z))'\right) > \frac{1-E}{1-F} := \alpha \;\; (z \in \mathbb{U}), \tag{31}$$

where $0 \leq \alpha < 1$. Using the identity (26), we write

$$\begin{aligned} z^{a-1}(\mathcal{F}_\delta^m(a+1,c)f(z))' &= z^{a-1}(\mathcal{F}f(z))' + \frac{z^a}{a}(\mathcal{F}f(z))'' \\ &= \left(\frac{z^a}{a}(\mathcal{F}f(z))'\right)', \end{aligned}$$

which yields the result

$$(\mathcal{F}f(z))' = \frac{a}{z^a}\int_0^z t^{a-1}(\mathcal{F}_\delta^m(a+1,c)f(t))'\mathrm{d}t \ (z \in \mathbb{U}).$$

Hence, on putting $t = uz$ $(z \in \mathbb{U})$ and on using Lemma 5 with the condition (31), we obtain that

$$\begin{aligned}
\Re\Big((\mathcal{F}f(z))'\Big) &= a\int_0^1 u^{a-1}\Re\Big((\mathcal{F}_\delta^m(a+1,c)f(uz))'\Big)\mathrm{d}u \\
&\geq a\int_0^1 u^{a-1}\left(2\alpha - 1 + \frac{2(1-\alpha)}{1+u|z|}\right)\mathrm{d}u \\
&> a\int_0^1 u^{a-1}\left(2\alpha - 1 + \frac{2(1-\alpha)}{1+u}\right)\mathrm{d}u \\
&= 1 - 2(1-\alpha)\left[1 - a\int_0^1 \frac{u^{a-1}}{1+u}\mathrm{d}u\right] \\
&= 1 + (1-\alpha)\left[{}_2F_1\left(1,1;a+1;\frac{1}{2}\right) - 2\right] = \gamma,
\end{aligned}$$

where $0 \leq \gamma < 1$. This proves Theorem 3 on putting the value of α from (31). □

Before stating and proving our next result, we recall the Hurwitz–Lerch Zeta function $\Phi(z;s;a)$ defined by

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} z^n/(n+a)^s, \quad z \in \mathbb{U}, \tag{32}$$

for some $s \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $a > 0$, [26]. We consider $\Phi(z,s,a)$ also for $|z| = 1$ when $\Re s > 1$.

Theorem 4. *Let $f_i \in \mathcal{A}$ $(i = 1,2,\ldots,n)$ and if*

$$(\mathcal{F}f_i(z))' \prec \frac{1+A_i z}{1+B_i z} \ (-1 \leq B_i < A_i \leq 1;\ i = 1,2,\ldots,n; z \in \mathbb{U}), \tag{33}$$

then

$$(\mathcal{F}f_1 * \mathcal{F}f_2 * \ldots * \mathcal{F}f_n)'(z) \prec h(z) \ (z \in \mathbb{U}), \tag{34}$$

where

$$h(z) = 1 + (A-B)\sum_{k=1}^{\infty}\frac{(-B)^{k-1}z^k}{(k+1)^{n-1}} = 1 - \frac{A-B}{B}(\Phi(-Bz, n-1, 1) - 1) \tag{35}$$

is convex in $\mathbb{U}$, and A, B are given by (17) in Lemma 7. The function $\Phi(-Bz,n-1,1)$ is the Hurwitz−Lerch Zeta function defined above by (32).

Proof. Observe that if $n = 1$, the result is trivial. For $n \geq 2$ and for $i = 1,2,\ldots,n$, let

$$\omega_i(z) = \frac{\mathcal{F}f_i(z)}{z}.$$

Then ω_i are analytic in $\mathbb{U}$ with $\omega_i(0) = 1$. Using (33), we obtain

$$\omega_i(z) + z\omega_i'(z) = (\mathcal{F}f_i(z))' \prec \frac{1+A_i z}{1+B_i z} \ (-1 \leq B_i < A_i \leq 1;\ i = 1,2,\ldots,n;\ z \in \mathbb{U}).$$

Hence, by Lemma 1 (for the case when $n = \gamma = 1$), we have

$$\frac{\mathcal{F}f_i(z)}{z} \prec q_i(z) = \frac{1}{z}\int_0^z \frac{1+A_i t}{1+B_i t}\mathrm{d}t \ (z \in \mathbb{U}), \tag{36}$$

where q_i is convex in $\mathbb{U}$ and is the best dominant. Applying now Lemma 3 to the subordination (36) for $i = 1, 2, \ldots, n-1$, and to the subordination (33) for $i = n$, we obtain

$$\begin{aligned}
& \frac{\mathcal{F}f_1(z)}{z} * \ldots * \frac{\mathcal{F}f_{n-1}(z)}{z} * (\mathcal{F}f_n(z))' \qquad (37)\\
\prec\ & \frac{1}{z}\int_0^z \frac{1+A_1 t}{1+B_1 t}\mathrm{d}t * \ldots * \frac{1}{z}\int_0^z \frac{1+A_{n-1}t}{1+B_{n-1}t}\mathrm{d}t * \frac{1+A_n z}{1+B_n z} \qquad (38)\\
=\ & \left(1 + (A_1 - B_1)\sum_{n=1}^{\infty}\frac{(-B_1)^{n-1}z^n}{n+1}\right) * \cdots * \left(1 + (A_{n-1} - B_{n-1})\sum_{n=1}^{\infty}\frac{(-B_{n-1})^{n-1}z^n}{n+1}\right)\\
& * \left(1 + (A_n - B_n)\sum_{n=1}^{\infty}(-B_n)^{n-1}z^n\right)\\
=\ & h(z),
\end{aligned}$$

where h is convex in $\mathbb{U}$ being the convolution of functions which are convex in $\mathbb{U}$ and is given by (35) in terms of the Hurwitz–Lerch Zeta function (32). The left-hand side of the above subordination in (37) is evidentially

$$(\mathcal{F}f_1(z) * \mathcal{F}f_2(z) * \ldots * \mathcal{F}f_n(z))'.$$

This proves Theorem 4. □

Results on the convolution of finite number of analytic functions have also been investigated earlier by considering a different operator in [27] (see also [4,28,29]).

Theorem 5. *Let a function $f \in \mathcal{A}$ and assume that $|A|, |B|, |C|, |D| \leq 1$. If*

$$\{L(a,c)f(z)\}' \prec \frac{1+Az}{1+Bz} \qquad (39)$$

and

$$\frac{1}{z}\mathcal{J}_\delta^m f(z) \prec \frac{1+Cz}{1+Dz}, \qquad (40)$$

then

$$\{\mathcal{F}f(z)\}' \prec \frac{1-(AD+BC-AC)z}{1-BDz}. \qquad (41)$$

Proof. We first observe that we have subordination under a convex univalent function in both (39) and in (40). Therefore applying Lemma 3, we obtain

$$\{L(a,c)f(z)\}' * \left\{\frac{1}{z}\mathcal{J}_\delta^m f(z)\right\} \prec \frac{1+Az}{1+Bz} * \frac{1+Cz}{1+Dz}.$$

In view of (12), the left-hand side turns out to be the operator $\{\mathcal{F}f(z)\}'$, while the right-hand side is

$$\begin{aligned}
& \frac{1+Az}{1+Bz} * \frac{1+Cz}{1+Dz}\\
=\ & \left\{1 - \frac{(B-A)z}{1+Bz}\right\} * \left\{1 - \frac{(D-C)z}{1+Dz}\right\}\\
=\ & \{1 - (B-A)(z - Bz^2 + B^2z^3 - B^3z^4 + \cdots)\} * \{1 - (D-C)(z - Dz^2 + D^2z^3 - D^3z^4 + \cdots)\}\\
=\ & 1 + (B-A)(D-C)\frac{z}{1-BDz}\\
=\ & \frac{1-(AD+BC-AC)z}{1-BDz}.
\end{aligned}$$

This gives (41). □

From Theorem 5 and from the assertion that $f' * (g/z) = (f/z) * g' = (f * g)'$, we directly obtain the following corollary.

Corollary 4. *Assume that* $|A|, |B|, |C|, |D| \leq 1$, *and let*

$$\{\mathcal{J}_\delta^m f(z)\}' \prec \frac{1+Az}{1+Bz}$$

and

$$\frac{1}{z}L(a,c)f(z) \prec \frac{1+Cz}{1+Dz},$$

then

$$\{\mathcal{F}f(z)\}' \prec \frac{1-(AD+BC-AC)z}{1-BDz}.$$

Again on using the relation (13), we obtain the following result from Theorem 5 provided that $\Re(a) > 0$.

Corollary 5. *Let a function* $f \in \mathcal{A}$ *and assume that* $|A|, |B|, |C|, |D| \leq 1$. *If for* $\Re(a) > 0$,

$$\{L(a+1,c)f(z)\}' \prec \frac{1+Az}{1+Bz} \tag{42}$$

and

$$\frac{1}{z}\mathcal{J}_{1/a}^{m-1} f(z) \prec \frac{1+Cz}{1+Dz}, \tag{43}$$

then

$$\{\mathcal{F}_{1/a}^m(a,c)f(z)\}' \prec \frac{1-(AD+BC-AC)z}{1-BDz}. \tag{44}$$

Theorem 6. *Let a function* $f \in \mathcal{A}$ *and assume that* $|A|, |B|, |E|, |F| \leq 1$. *If the function* f *satisfies the condition (39) and*

$$(\mathcal{J}_\delta^m f(z))' \prec \frac{1+Ez}{1+Fz}, \tag{45}$$

then

$$\{\mathcal{F}f(z)\}' \prec \theta(z), \tag{46}$$

where

$$\theta(z) = 1 + \frac{(A-B)(E-F)}{BF}\sum_{k=1}^{\infty}\frac{(BFz)^k}{k+1} = 1 + \frac{(A-B)(E-F)}{BF}(\Phi(BFz,1,1)-1) \quad (z \in \mathbb{U}) \tag{47}$$

is convex in $\mathbb{U}$ *and the function* $\Phi(BFz,1,1)$ *is the Hurwitz−Lerch Zeta function defined above by (32).*

Proof. Let

$$\omega(z) = \frac{\mathcal{J}_\delta^m f(z)}{z},$$

which is analytic in $\mathbb{U}$ with $\omega(0) = 1$, and on using now (45), we obtain

$$\omega(z) + z\omega'(z) = (\mathcal{J}_\delta^m f(z))' \prec \frac{1+Ez}{1+Fz}\ (-1 \leq F < E \leq 1; z \in \mathbb{U}).$$

Hence, by Lemma 1 (for the case when $n = \gamma = 1$), we obtain

$$\frac{\mathcal{J}_\delta^m f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z \frac{1+Et}{1+Ft}\mathrm{d}t\ (z \in \mathbb{U}), \tag{48}$$

where q is convex in $\mathbb{U}$ and is the best dominant. Now on applying Lemma 3 to the subordination conditions (45) and (39), we obtain

$$\{L(a,c)f(z)\}' * \frac{\mathcal{J}_\delta^m f(z)}{z} \prec \frac{1+Az}{1+Bz} * \frac{1}{z}\int_0^z \frac{1+Et}{1+Ft}\mathrm{d}t.$$

The left-hand side simplifies to $\{\mathcal{F}f(z)\}'$, while the right-hand side is

$$\begin{aligned} &\frac{1+Az}{1+Bz} * \frac{1}{z}\int_0^z \frac{1+Et}{1+Ft}\mathrm{d}t \\ = \;& \left\{\frac{A}{B} + \frac{(B-A)/B}{1+Bz}\right\} * \left\{\frac{E}{F} + \frac{F-E}{F}\frac{\log(1+Fz)}{Fz}\right\} \\ = \;& \left(1 + (A-B)\sum_{n=1}^{\infty}(-B)^{n-1}z^n\right) * \left(1 + (E-F)\sum_{n=1}^{\infty}\frac{(-F)^{n-1}z^n}{n+1}\right) \\ = \;& \theta(z), \end{aligned}$$

where θ is given by (47). This proves the result (46). □

Lastly, we prove the following result.

Theorem 7. *Let a function $f \in \mathcal{A}$ and assume that $\nu(z)$ is convex univalent and $0 < c \leq a$. If $a \geq 2$ or $a + c \geq 3$ and operator $\mathcal{F}$ defined in (12) satisfies*

$$\mathcal{F}f(z) \prec \nu(z), \tag{49}$$

then operator $\mathcal{J}_\delta^m$ defined in (8) satisfies

$$\mathcal{J}_\delta^m f(z) \prec \nu(z) * \sum_{n=0}^{\infty}\frac{(c)_n}{(a)_n}z^{n+1}. \tag{50}$$

Proof. Because (49) is a subordination under a convex univalent function and by Lemma 6

$$\sum_{n=0}^{\infty}\frac{(c)_n}{(a)_n}z^{n+1}$$

is a convex univalent too. Therefore applying Lemma 3. we obtain

$$\mathcal{F}f(z) * \sum_{n=0}^{\infty}\frac{(c)_n}{(a)_n}z^{n+1} \prec \nu(z) * \sum_{n=0}^{\infty}\frac{(c)_n}{(a)_n}z^{n+1},$$

where on the left-hand side we have $\mathcal{J}_\delta^m f(z)$. This gives (50). □

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