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# Existence Solution for Coupled System of Langevin Fractional Differential Equations of Caputo Type with Riemann–Stieltjes Integral Boundary Conditions

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**Abstract:** By employing Shauder fixed-point theorem, this work tries to obtain the existence results for the solution of a nonlinear Langevin coupled system of fractional order whose nonlinear terms depend on Caputo fractional derivatives. We study this system subject to Stieltjes integral boundary conditions. A numerical example explaining our result is attached.

**Keywords:** coupled system; fractional Langevin equations; Stieltjes integral boundary conditions; Shauder fixed-point theorem; existence

**MSC:** 26A33; 34A08; 34A12; 34B15

**Citation:** Salem, A.; Almaghamsi, L. Existence Solution for Coupled System of Langevin Fractional Differential Equations of Caputo Type with Riemann–Stieltjes Integral Boundary Conditions. *Symmetry* **2021**, *13*, 2123. https://doi.org/10.3390/sym13112123

Academic Editor: Jan Awrejcewicz

Received: 18 October 2021
Accepted: 5 November 2021
Published: 8 November 2021



## 1. Introduction

The property of a mathematical object to remain unchanged after an operation or a transformation is called invariance. Symmetry is a type of invariance. Invariants usually reflect intrinsic properties of the object of study [1]. The reflection principle is invariably presented as a consequence of the strong Markov property. The reflection principle is one of the most important properties of Brownian motion. Brownian motion sample paths satisfy the Markov property, symmetry, reflection principle, invariance scaling, time inversion [2,3], and new symmetry nominated as the quasi-time-reversal invariance [4]. It turned out to be of the Brownian motion is the clef to confirm the period symmetry for diffusion operations on the circle.

It is noteworthy that the Brownian motion is closely depicted through the Langevin equation when the force of random fluctuation is proposed to be white noise. If the force of random fluctuation is not white noise, the particle motion is portrayed by the generalized Langevin equation [5]. In a fractal medium, varied popularizations of the Langevin equation have been suggested to depict dynamical operations. In general, the differential equations of integer order can not accurately characterize the experiential and area measurement data, as a different approach, differential equation models of fractional order are now being used [6]. Based on the fractional Langevin equation, Mainradi and Pironi [7] had reintroduced Brownian motion. Analytical expressions of the correlation functions were obtained using the two fluctuation-dissipation theorems and fractional calculus approaches. The fractional Langevin equation has been received the attention of many scientists due to its extremely useful applications in different fields of science and has been dealt with under different conditions (see [8–14]).

Undoubtedly, the close connection between Langevin equations, Brownian motion, and the symmetry principles, observed through the previous discussion, encourages any author to study these equations, their solutions, and the properties of their solutions in various fields. Therefore, in this investigation, we address the fraction Langevin coupled system of fractional Caputo type. We aim to establish sufficient conditions the existence results to the next system

$$^{c}D^{\beta_i}(^{c}D^{\alpha_i}+\lambda_i)v_i(t)=g_i(t,v_j(t),^{c}D^{\gamma_i}v_j(t)), \qquad t\in[0,1], i\neq j \tag{1}$$

subjected to the specific boundary conditions

$$\begin{aligned} &v_i(0)=0, \qquad ^{c}D^{\alpha_i}v_i(0)+^{c}D^{\alpha_i}v_i(1)=0,\\ &v_i(1)=\int_0^1 v_1(s)dx_i(s)+\int_0^1 v_2(s)dy_i(s). \end{aligned} \tag{2}$$

where for $i=1,2$, $v_i(t)$ are the displacements of two particles in the unit interval $[0,1]$, $^{c}D^{\beta_i}$ and $^{c}D^{\alpha_i}$ are the fractional derivatives in the sense of Caputo, $0<\alpha_i<1$, $1<\beta_i\leq 2$, $0<\gamma_i<\alpha_j$, $\lambda_i\in\mathbb{R}$ and the integrals implied by the conditions are the Stieltjes integral with respect to the functions $x_i(s),y_i(s):[0,1]\to\mathbb{R}, i=1,2$ and $g_i:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which are continuous and might include an extrinsic force field, a position-dependent phenomenological fluid friction coefficient, the strength of the stochastic force, or random fluctuation force (may be white noise term or not).

The Riemann–Stieltjes integrals are beneficial and valuable mathematical tools when acting with random and discrete variables synchronously. They have several applications in statistics and physics but are restricted in their applications with Stochastic processes.

The existence theorem is a method that allows us to determine whether a solution to a differential equation exists that fulfills specified initial or boundary conditions. Indeed, it is too hard to give an exact solution to the differential equation in its general form. Thus, contributors consider the existence, uniqueness, numerical algorithms, etc., to discuss the solution and its properties. In the past few decades, interest has greatly increased in the study of the existence theory of solution to differential equations, especially to fractional ones, see [15–18] and the references cited therein.

Fractional differential equations have recently received a lot of attention because of their wide range of applications in engineering, physics, chemistry, biology, and other domains. In the last few decades, researchers have been interested in differential equations of fractional order. It arises from fractional-order derivatives offering powerful tools for the description of memory and inherited properties of different materials and processes in different fields of science and engineering [19–22].

The coupled system of differential equations with fractional order is considered an important and valuable point to study because of its many applications [23,24]. It is notable that the nonlinear term in (1) is dependent on the fractional derivative of the unknown function. As far as we know, this paper seems to be the first work to deal with this case on a fractional Langevin coupled system of Caputo type under the conditions described in (2).

Our action plan is in the following manner: In the second section, we introduce some lemmas needed in our main results. In the third section, the existence results by using Schauder fixed point theorem are proven. An example showing our results is attached in the final section.

## 2. Preliminaries and Relevant Lemmas

Here, we begin by introducing various notations, fundamental facts, and definitions that will be used in the next sections. For more details see [25,26].

**Definition 1.** *Suppose that $u:[0,\infty)\to\mathbb{R}$ is a continuous function, the R-L integral with order $\sigma>0$ is presented as*

$$I^{\sigma}u(t)=\int_0^t\frac{(t-\tau)^{\sigma-1}}{\Gamma(\sigma)}u(\tau)d\tau$$

*assuming the integral's right-hand side exists, and* $\Gamma(\sigma)$ *is the Euler gamma function known as*

$$\Gamma(\sigma) = \int_0^\infty \tau^{\sigma-1} e^{-\tau} d\tau, \qquad \alpha > 0.$$

**Definition 2.** *The Caputo derivative of order* $\sigma > 0$ *to the function* $u : [0, \infty) \to \mathbb{R}$ *is introduced as*

$$^{c}D^{\sigma} u(t) = \frac{1}{\Gamma(m-\sigma)} \int_0^t (t-\tau)^{m-\sigma-1} u^{(m)}(\tau) d\tau$$

*where* $m - 1 < \sigma \leq m$ *and* $m \in \mathbb{N}$*, on the assumption that the R-H-S exists and is finite. Remember that the value of Caputo derivative is zero for any constant.*

**Lemma 1.** *Let* $\alpha$ *and* $\beta$ *be positive reals. Then,*

$$I^{\alpha} I^{\beta} u(t) = I^{\alpha+\beta} u(t)$$

*where u is a continuous function.*

**Lemma 2.** *Assume that* $m \in \mathbb{N}$ *and* $m - 1 < \alpha \leq m$*. Then,*

$$I^{\alpha}\, {}^{c}D^{\alpha} v(t) = v(t) + a_0 + a_1 t + \cdots + a_{m-1} t^{m-1}$$

*where v is a continuous function.*

**Definition 3.** *Let* $p[c,d] = \{c = t_0, t_1, \cdots, t_n = d\}$ *be a partition of the interval* $[c,d]$*. Define* $V_p(h)$ *as*

$$V_p(h) = \sum_{r=1}^{n} \Big| h(t_r) - h(t_{r-1}) \Big|.$$

*Then, we state that the function* $h : [c,d] \Rightarrow \mathbb{R}$ *is of bounded variation (*$h \in BV[c,d]$*) if the set* $\{V_p(h) : p \in P[c,d]\}$ *is bounded above, and* $\sup V_p(h)$ *is called a total variation of h and denoted by* $V_c^d h$*.*

**Lemma 3.** *The Stieltjes–Riemann integral* $\int_c^d g(\tau) dh(\tau)$ *exists if* $g(t) \in C([c,d], \mathbb{R})$ *and* $h : [c,d] \Rightarrow \mathbb{R}$ *are functions of bounded variations.*

**Lemma 4.** *Let g and h be as in Lemma 3 and* $K = \max_{t \in [c,d]} |g(t)|$*. Then,*

$$\left| \int_c^d g(\tau) dh(\tau) \right| \leqslant K V_c^d h(t).$$

**Corollary 1.** *If the function h be monotonically on* $[c,d]$ *(decreasing or increasing ), then the Stieltjes–Riemann integral* $\int_c^d g(\tau) dh(\tau)$ *exists and*

$$\left| \int_c^d g(\tau) dh(\tau) \right| \leqslant K \Big| h(d) - h(c) \Big|$$

*such that each of K and the function g is as in Lemma 4.*

For more information on the Riemann–Stieltjes integral and functions of bounded variation, we recommend reading [27–29].

**Lemma 5.** *For $i = 1, 2$, let*

$$\Lambda_i(t) = \frac{(2t - 1 - \alpha_i)t^{\alpha_i}}{1 - \alpha_i},$$
$$\Delta_i(t) = \frac{(1 - t)t^{\alpha_i}}{(1 - \alpha_i)\Gamma(\alpha_i + 1)}.$$

*Then the problem*

$$^cD^{\beta_i}(^cD^{\alpha_i} + \lambda_i)v_i(t) = g_i(t), \qquad t \in [0, 1] \tag{3}$$
$$v_i(0) = 0, \qquad ^cD^{\alpha_i}v_i(0) + ^c D^{\alpha_i}v_i(1) = 0,$$
$$v_i(1) = \int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s). \tag{4}$$

*has a unique solution illustrated by*

$$v_i(t) = I^{\alpha_i+\beta_i}g_i(t) - \lambda_i I^{\alpha_i}v_i(t) + \Lambda_i(t)(\lambda_i I^{\alpha_i}v_i(1) - I^{\alpha_i+\beta_i}g_i(1)) + \Delta_i(t)I^{\beta_i}g_i(1)$$
$$+ (\Lambda_i(t) + \lambda_i\Delta_i(t))\left(\int_0^1 v_1(s))dx_i(s) + \int_0^1 v_2(s)dy_i(s)\right), \tag{5}$$

*where $0 < \alpha_i < 1$, $1 < \beta_i \leq 2$, $\lambda_i \in \mathbb{R}$ and $g_i \in C[0, 1]$.*

**Proof.** By operating $I^{\beta_i}$ followed by $I^{\alpha_i}$ on both sides of (3) and using Lemma 2, we get

$$^cD^{\alpha_i}v_i(t) = A_i + B_i t + I^{\beta_i}g_i(t) - \lambda_i v_i(t), \tag{6}$$
$$v_i(t) = A_i\frac{t^{\alpha_i}}{\Gamma(\alpha_i + 1)} + B_i\frac{t^{\alpha_i+1}}{\Gamma(\alpha_i + 2)} + I^{\alpha_i+\beta_i}g_i(t) - \lambda_i I^{\alpha_i}v_i(t) + C_i,$$

The first condition (4) gives $C_i = 0$ and the second and third conditions give

$$2A_i + B_i = -I^{\beta_i}g_i(1) + \lambda_i\left(\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)\right)$$
$$\frac{A_i}{\Gamma(\alpha + 1)} + \frac{B_i}{\Gamma(\alpha + 2)} = -I^{\alpha_i+\beta_i}g_i(1) + \lambda_i I^{\alpha_i}v_i(1) + \int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)$$

By solving the previous equations, we get

$$A_i = \frac{1}{1 - \alpha_i}\{\Gamma(\alpha_i + 2)(I^{\alpha_i+\beta_i}g_i(1) - \lambda_i I^{\alpha_i}v_i(1)) - I^{\beta_i}g_i(1)$$
$$+ [\lambda_i - \Gamma(\alpha_i + 2)][\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)]\},$$
$$B_i = \frac{1}{1 - \alpha_i}\{2\Gamma(\alpha_i + 2)(\lambda_i I^{\alpha_i}v_i(1) - I^{\alpha_i+\beta_i}g_i(1)) + (1 + \alpha_i)I^{\beta_i}g_i(1)$$
$$+ [2\Gamma(\alpha_i + 1) - \lambda_i][\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)]\}.$$

Substituting the above values of $A_i, B_i$ in (3) to obtain the desired results. □

Postulate the space of all continuous functions on the unit interval $[0, 1]$ is denoted by $C(I)$. We define the space

$$V_i = \{v_i(t) \in C(I) \text{ and } ^cD^{\gamma_j}v_i(t) \in C(I)\}, \quad i = 1, 2$$

with the norm

$$\| v_i \|_{V_i} = \max_{t\in I} | v_i(t) | + \max_{t\in I} |^c D^{\gamma_j}v_i(t) |, \quad i \neq j = 1, 2.$$

Consider the class $\Psi$ of continuous functions which defined as

$$\Psi = \{\hat{g}_i : \hat{g}_i = g_i(t, v_j(t), cD^{\gamma_i}v_j(t)),\ (t, v_j(t), {}^cD^{\gamma_i}v_j(t)) \in I \times \mathbb{R} \times \mathbb{R}, i, j = 1, 2, i \neq j\}.$$

**Lemma 6.** *Suppose that $\hat{g}_i \in \Psi; i = 1, 2$. Then, $(v_1, v_2) \in V_1 \times V_2$ is a solution of the problem (1) if and only if $(v_1, v_2) \in V_1 \times V_2$ is a solution of the next coupled system*

$$\begin{aligned} v_i(t) &= I^{\alpha_i+\beta_i}\hat{g}_i(t) - \lambda_i I^{\alpha_i}v_i(t) + \Lambda_i(t)(\lambda_i I^{\alpha_i}v_i(1) - I^{\alpha_i+\beta_i}\hat{g}_i(1)) + \Delta_i(t)I^{\beta_i}\hat{g}_i(1) \\ &\quad + (\Lambda_i(t) + \lambda_i\Delta_i(t))(\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)). \end{aligned} \tag{7}$$

**Proof.** Let $(v_1, v_2) \in V_1 \times V_2$ be a solution of the problem (1). By the proof of Lemma 3, we can get that $(v_1, v_2)$ is a solution of (7).

Conversely, let $(v_1, v_2) \in V_1 \times V_2$ be a solution of the system (7). Then, from (5), we have

$$\begin{aligned} {}^cD^{\beta_i}({}^cD^{\alpha_i}v_i(t) + \lambda_i v_i(t)) &= {}^cD^{\beta_i}\Bigg(I^{\beta_i}\hat{g}_i(t) + \frac{\Gamma(\alpha_i+2)}{(1-\alpha_i)}(2t-1)\big[\lambda_i I^{\alpha_i}v_1(1) - I^{\alpha_i+\beta_i}\hat{g}_i(1)\big] \\ &\quad + \frac{1}{(1-\alpha_i)}((1+\alpha_i)t - 1)I^{\beta_i}\hat{g}_i(1) + \frac{1}{(1-\alpha_i)}\big[(1+\alpha_i)(2\Gamma(\alpha_i+2) + \lambda_i)t \\ &\quad - (\lambda_i + \Gamma(\alpha_i+2)\big]\big[\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)\big]\Bigg) = \hat{g}_i(t), \qquad i = 1, 2. \end{aligned}$$

The Lemma's proof is completed. □

Let the operator $T : V_1 \times V_2 \to V_1 \times V_2$ be defined as $T(v_1, v_2) = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ where, for $i = 1, 2$, we have

$$\begin{aligned} (T_i v_i)(t) &= I^{\alpha_i+\beta_i}\hat{g}_i(t) - \lambda_i I^{\alpha_i}v_i(t) + \Lambda_i(t)(\lambda_i I^{\alpha_i}v_i(1) - I^{\alpha_i+\beta_i}\hat{g}_i(1)) + \Delta_i(t)I^{\beta_i}\hat{g}_i(1) \\ &\quad + (\Lambda_i(t) + \lambda_i\Delta_i(t))(\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)) \end{aligned}$$

and

$$\begin{aligned} {}^cD^{\gamma_j}(T_i v_i)(t) &= I^{\alpha_i+\beta_i-\gamma_j}\hat{g}_i(t) - \lambda_i I^{\alpha_i-\gamma_j}v_i(t) + {}^cD^{\gamma_j}\Lambda_i(t)(\lambda_i I^{\alpha_i}v_i(1) - I^{\alpha_i+\beta_i}\hat{g}_i(1)) \\ &\quad + {}^cD^{\gamma_j}\Delta_i(t)I^{\beta_i}\hat{g}_i(1) + ({}^cD^{\gamma_j}\Lambda_i(t) + \lambda_i^c D^{\gamma_j}\Delta_i(t))\left(\int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s)\right). \end{aligned}$$

Hence, in view of the Lemma 7, the operator $T$ has a fixed point which is consistent with a solution of the system (1).

Now, before beginning to prove the basic part in this research, for $i, j = 1, 2, i \neq j$, we define a set of values that we need later.

$$\eta_i = V_0^1 x_i + V_0^1 y_i, \tag{8}$$
$$\varphi_i = \varphi_i(0) + \varphi_i(\gamma_j), \tag{9}$$
$$\mathcal{M}_i = \mathcal{M}_i(0) + \mathcal{M}_i(\gamma_j), \tag{10}$$
$$\mathcal{C}_i = \mathcal{C}_i(0) + \mathcal{C}_i(\gamma_j) \tag{11}$$

where

$$\varphi_i(\gamma_j) = \frac{1}{\Gamma(\alpha_i+\beta_i-\gamma_j+1)} + \frac{A_i(\gamma_j)}{\Gamma(\alpha_i+\beta_i+1)} + \frac{B_i(\gamma_j)}{\Gamma(\beta_i+1)},$$

$$\mathcal{M}_i(\gamma_j) = \int_0^1 \left[ \left( \frac{1}{\Gamma(\alpha_i+\beta_i-\gamma_j)} + \frac{A_i(\gamma_j)}{\Gamma(\alpha_i+\beta_i)} \right)(1-s)^{\alpha_i+\beta_i-\gamma_j-1} + \frac{B_i(\gamma_j)}{\Gamma(\beta_i)}(1-s)^{\beta_i-1} \right] \phi_i(s)ds$$

$$\mathcal{C}_i(\gamma_j) = \frac{\lambda_i}{\Gamma(\alpha_i-\gamma_j+1)} + \frac{\lambda_i A_i(\gamma_j)}{\Gamma(\alpha_i+1)} + \eta_i(A_i(\gamma_j)+\lambda_i B_i(\gamma_j)).$$

Such that, from [30], we have

1. $A_i(\gamma_j) = \max_{t\in[0,1]} |D^{\gamma_j}\Lambda_i| = \frac{\Gamma(\alpha_i+2)}{(1-\alpha_i)\Gamma(\alpha_i-\gamma_j+2)} \begin{cases} 1-\alpha_i+\gamma_j & \text{if} \quad \alpha_i \le \dfrac{1}{2}+\gamma_j \\ \left(\dfrac{\alpha_i-\gamma_j}{2}\right)^{\alpha_i-\gamma_j} & \text{if} \quad \alpha_i > \dfrac{1}{2}+\gamma_j \end{cases}$
2. $B_i(\gamma_j) = \max_{t\in[0,1]} |D^{\gamma_j}\Delta_i| = \frac{1}{(1-\alpha_i)\Gamma(\alpha_i-\gamma_j+2)} \left[ \frac{(1+\alpha_i)(\alpha_i-\gamma_j)^{\alpha_i-\gamma_j}}{(1+\alpha_i-\gamma_j)^{1+\alpha_i-\gamma_j}} + \gamma_j \right].$

## 3. Existence Results

Let us suppose the assumptions below

**($\mathcal{H}_1$)** For $i = 1,2$ and $g_i \in \Psi$. Then,

$$|g_i(t,\zeta,\xi)| \le \phi_i(t) + a_i|\zeta|^{\varrho_i} + b_1|\xi|^{\sigma_i}$$

for all $(t,\zeta,\xi) \in [0,1]\times\mathbb{R}\times\mathbb{R}$ where $\phi_i(t) \in L[0,1]$, $a_i, b_i \geqslant 0$ and $0 < \varrho_i, \sigma_i < 1$.

**($\mathcal{H}_2$)** For $i = 1,2$ and $g_i \in \Psi$. Then,

$$|g_i(t,\zeta,\xi)| \le \phi_i(t) + a_i|\zeta|^{\varrho_i} + b_1|\xi|^{\sigma_i}$$

for all $(t,\zeta,\xi) \in [0,1]\times\mathbb{R}\times\mathbb{R}$ where $\phi_i(t) \in L[0,1]$, $a_i, b_i > 0$ and $\varrho_i, \sigma_i > 1$.

**Theorem 1.** *Suppose that $x_i(s), y_i(s) : [0,1] \to \mathbb{R}$, $i = 1,2$ are functions of bounded variation and one of the previous assumptions holds. Then, the problem (1) has a solution.*

**Proof.** The linear operator $T : V_1 \times V_2 \to V_1 \times V_2$ is well-defined if it has a unique expression. Since $v_i \in V_i$ has a unique expression of continuous function $g_i$ for $i = 1,2$ in the form of Lemma 6, it is clear that the operator $T$ is well-defined.

Let us consider that ($\mathcal{H}_1$) is satisfied. Define the ball $\Omega$ as

$$\Omega = \{(v_1(t),v_2(t)) | (v_1(t),v_2(t)) \in V_1\times V_2, \| (v_1(t),v_2(t)) \| \leqslant L, t \in I\}$$

such that

$$L \ge \max_{i=1,2} \left\{ (3\varphi_i a_i)^{\frac{1}{1-\varrho_i}}, (3\varphi_i b_i)^{\frac{1}{1-\sigma_i}}, \frac{3\mathcal{M}_i}{1-3\mathcal{C}_i} \right\}$$

From Lemma 2 and the properties of fractional calculus, we get

$$\begin{aligned} |(T_i v_i)(t)| \le{}& \int_0^1 \left[ \frac{1+A_i(0)}{\Gamma(\alpha_i+\beta_i)}(1-s)^{\alpha_i+\beta_i-1} + \frac{B_i(0)}{\Gamma(\beta_i)}(1-s)^{\beta_i-1} \right] \phi_i(s)ds \\ &+ (a_i L^{\varrho_i} + a_j L^{\sigma_i}) \left( \frac{1+A_i(0)}{\Gamma(\alpha_i+\beta_i+1)} + \frac{B_i(0)}{\Gamma(\beta_i+1)} \right) \\ &+ L\left( \frac{\lambda_i(1+A_i(0))}{\Gamma(\alpha_i+1)} + \eta_i(A_i(0)+\lambda_i B_i(0)) \right) \\ ={}& \mathcal{M}_i(0) + (a_i L^{\varrho_i} + b_i L^{\sigma_i})\varphi_i(0) + L\mathcal{C}_i(0). \end{aligned}$$

Similarly,

$$|({}^cD^{\gamma_j}T_iv_i)(t)| = \mathcal{M}_i(\gamma_j) + (a_iL^{\varrho_i} + b_iL^{\sigma_i})\varphi_i(\gamma_j) + L\mathcal{C}_i(\gamma_j).$$

So,

$$\| T_iu \|_{V_i} \leq \mathcal{M}_i + (a_iL^{\varrho_i} + b_iL^{\sigma_i})\varphi_i + L\mathcal{C}_i \leq L$$

where $\varphi_i$, $\mathcal{M}_i$ and $\mathcal{C}_i$ are defined as in (9)–(11), respectively. Then, we get that $T(\Omega) \subseteq \Omega$ provided that $\mathcal{C}_i < 1/3$.

Now with regard to the second condition $(\mathcal{H}_2)$, we lay down that

$$L \leqslant \min_{i=1,2}\left\{ \left(\frac{1}{3\varphi_i a_i}\right)^{\frac{1}{\varrho_i - 1}}, \left(\frac{1}{3\varphi_i b_i}\right)^{\frac{1}{\sigma_i - 1}}, \frac{3\mathcal{M}_i}{3\mathcal{C}_i - 1} \right\}$$

By following the same steps as above we obtain

$$\| T_iu \|_{V_i} \leq \mathcal{M}_i + (a_iL^{\varrho_i} + b_iL^{\sigma_i})\varphi_i + L\mathcal{C}_i \leq L$$

which implies that $T(\Omega) \subseteq \Omega$ provided that $\mathcal{C}_i > 1/3$.

For showing that $T$ is a completely continuous, we take $N_i = \max_{t\in I} \hat{g}_i(t)$ for any $v_i, v_j \in \Omega, i, j = 0, 1, i \neq j$ such that $0 < t_2 < t_1 < 1$, then we have

$$\begin{aligned}
&|(T_iv_i)(t_1) - (T_iv_i)(t_2)| \\
&\leq \int_0^{t_2} \frac{(t_1 - s)^{\alpha_i+\beta-1} - (t_2 - s)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i + \beta_i)}\hat{g}_i(s)ds + \int_{t_2}^{t_1} \frac{(t_1 - s)^{\alpha_i+\beta-1}}{\Gamma(\alpha_i + \beta_i)}\hat{g}_i(s)ds \\
&+ |\lambda_i| \int_0^{t_2} \frac{(t_2 - s)^{\alpha_i-1} - (t_1 - s)^{\alpha_i-1}}{\Gamma(\alpha_i)}v_i(s)ds + |\lambda_i| \int_{t_2}^{t_1} \frac{(t_1 - s)^{\alpha_i-1}}{\Gamma(\alpha_i)}v_i(s)ds \\
&+ |\Lambda_i(t_1) - \Lambda_i(t_2)| \left( \frac{\lambda_i}{\Gamma(\alpha_i)} \int_0^1 (1 - s)^{\alpha_i-1}v_i(s)ds + \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_0^1 (1 - s)^{\alpha_i+\beta_i-1}\hat{g}_i(s)ds \right) \\
&+ |\Delta_i(t_1) - \Delta_i(t_2)| \int_0^1 \frac{(1 - s)^{\beta_i-1}}{\Gamma(\beta_i)}\hat{g}_i(s)ds \\
&+ (|\Lambda_i(t_1) - \Lambda_i(t_2)| + |\lambda_i||\Delta_i(t_1) - \Delta_i(t_2)|) \left( \int_0^1 v_1(s)dx_i(s) + \int_0^1 v_2(s)dy_i(s) \right) \\
&\leq \frac{N_i(t_1^{\alpha_i+\beta_i} - t_2^{\alpha_i+\beta_i})}{\Gamma(\alpha_i + \beta_i + 1)} + \frac{2|\lambda_i|L(t_1 - t_2)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\
&+ \left[ \frac{2|\lambda_i|L}{\Gamma(\alpha_i + 1)} + \frac{2N_i}{\Gamma(\alpha_i + \beta_i + 1} + \frac{N_i}{\Gamma(\alpha_i + 1)\Gamma(\beta_i + 1)} + L\eta_i\left(2 + \frac{|\lambda_i|}{\Gamma(\alpha_i + 1)}\right) \right] \frac{t_1^{\alpha_i+1} - t_2^{\alpha_i+1}}{1 - \alpha_i} \\
&+ \left[ \frac{(1 + \alpha_i)|\lambda_i|L}{\Gamma(\alpha_i + 1)} + \frac{(1 + \alpha_i)N_i}{\Gamma(\alpha_i + \beta_i + 1)} + \frac{N_i}{\Gamma(\alpha_i + 1)\Gamma(\beta_i + 1)} + \frac{L\eta_i(\Gamma(\alpha_i + 2) + |\lambda_i|)}{\Gamma(\alpha_i + 1)} \right] \frac{t_1^{\alpha_i} - t_2^{\alpha_i}}{1 - \alpha_i}.
\end{aligned}$$

By using the relation ${}^cD^{\gamma}t^{\alpha} = \frac{\Gamma(\alpha+1)t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}$, we find that

$$
\begin{aligned}
&|^c D^{\gamma_j} T_i v_i(t_1) -^c D^{\gamma_j} T_i v_i(t_2)| \\
&\leq \int_0^{t_2} \frac{(t_1-s)^{\alpha_i+\beta_i-\gamma_j-1} - (t_2-s)^{\alpha_i+\beta_i-\gamma_j-1}}{\Gamma(\alpha_i+\beta_i-\gamma_j)} |\hat{g}_i(s)| ds + \int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha_i+\beta_i-\gamma_j-1}}{\Gamma(\alpha_i+\beta_i-\gamma_j)} |\hat{g}_i(s)| ds \\
&+ |\lambda_i| \int_0^{t_2} \frac{(t_2-s)^{\alpha_i-\gamma_j-1} - (t_1-s)^{\alpha_i-\gamma_j-1}}{\Gamma(\alpha_i-\gamma_j)} |v_i(s)| ds + |\lambda_i| \int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha_i-\gamma_j-1}}{\Gamma(\alpha_i-\gamma_j)} |v_i(s)| ds \\
&+ |^c D^{\gamma_j} \Lambda_i(t_1) -^c D^{\gamma_j} \Lambda_i(t_2)| \left( |\lambda_i| \int_0^1 \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} |v_i(s)| ds + \int_0^1 \frac{(1-s)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)} |\hat{g}_i(s)| ds \right) \\
&+ |^c D^{\gamma_j} \Delta_i(t_1) -^c D^{\gamma_j} \Delta_i(t_2)| \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} |\hat{g}_i(s)| ds \\
&+ (|^c D^{\gamma_j} \Lambda_i(t_1) -^c D^{\gamma_j} \Lambda_i(t_2)| + |\lambda_i| |^c D^{\gamma_j} \Delta_i(t_1) -^c D^{\gamma_j} \Delta_i(t_2)|) \\
&\times \left( \int_0^1 v_1(s) dx_i(s) + \int_0^1 v_2(s) dy_i(s) \right) \\
&\leqslant \frac{N_i (t_1^{\alpha_i+\beta_i-\gamma_j} - t_2^{\alpha_i+\beta_i-\gamma_j})}{\Gamma(\alpha_i+\beta_i-\gamma_j+1)} + \frac{2|\lambda_i| L (t_1-t_2)^{\alpha_i-\gamma_j}}{\Gamma(\alpha_i-\gamma_j+1)} \\
&+ \left[ 2|\lambda_i| L + \frac{2\Gamma(\alpha_i+1) N_i}{\Gamma(\alpha_i+\beta_i+1} + \frac{N_i}{\Gamma(\alpha_i+1)\Gamma(\beta_i+1)} + L\eta_i (2\Gamma(\alpha_i+1) + |\lambda_i|) \right] \\
&\times \frac{(1+\alpha_i)(t_1^{\alpha_i-\gamma_j+1} - t_2^{\alpha_i-\gamma_j+1})}{(1-\alpha_i)\Gamma(1-\alpha_i-\gamma_j+2)} \\
&+ \left[ (1+\alpha_i)|\lambda_i| L + \frac{\Gamma(\alpha_i+2) N_i}{\Gamma(\alpha_i+\beta_i+1} + \frac{N_i}{\Gamma(\beta_i+1)} + L\eta_i (\Gamma(\alpha_i+2) + |\lambda_i|) \right] \frac{t_1^{\alpha_i-\gamma_j} - t_2^{\alpha_i-\gamma_j}}{(1-\alpha_i)\Gamma(\alpha_i-\gamma_j+1)}.
\end{aligned}
$$

Accordingly, we conclude that $T\Omega$ is an equicontinuous set. It is also clearly bounded since $T\Omega \subseteq \Omega$. The Shauder fixed point theorem shows that the solution exists in $\Omega$. □

**Remark 1.** *The radius of the ball $\Omega$ depends on the values of $\varrho_i$ and $\sigma_i$ which have several bounds with respect to one. Here, we mentioned two cases when $0 < \varrho_i, \sigma_i < 1$ and $\varrho_i, \sigma_i > 1$. One might take another values to them. For instance, $\varrho_i = \sigma_i = 1$. In this case, we have to provide that $(a_i + b_i)\varphi_i + \mathcal{C}_i < 1$ and the radius $L \geq \mathcal{M}_i/(1 - (a_i + b_i)\varphi_i - \mathcal{C}_i)$.*

## 4. An Illustrative Example

A Hamiltonian model for a simple dynamical system connected to the environment is used to create a theoretical Langevin equation [31]. The Langevin equation uses Newton's law to address the dynamics of a Brownian particle by combining the influence of Stokes fluid friction and thermal fluctuations in the particle's proximity into a random force with appropriately assigned attributes. The simplest fractional Langevin equation to a dynamical system unlock to the environment takes the style (1) where $v_i(t)$ are the displacements of the particle, $\lambda_i$ are dissipation parameters, and $g_i$, in the general style, the particle at time $t$ modeled by the position $v_i(t)$ and velocity ${}^c D^r v_i(t)$ . The functions $g_i$ might include an extrinsic force field, a position-dependent phenomenological fluid friction coefficient, the strength of the stochastic force, or random fluctuation force (may be white noise term or not). In the following example, we assume that there is no Gaussian white noise term that agrees with its mean equal zero and the functions $g_i$ contain external force proportional to the power of the displacement and velocity as in the following example.

**Example 1.** *Postulate the next problem*

$$
\begin{aligned}
{}^cD^{\frac{3}{2}}({}^cD^{\frac{1}{3}} + \frac{1}{30})u(t) &= (t-\frac{1}{4})^2((v(t))^{\frac{3}{2}} + ({}^cD^{\frac{1}{10}}v(t))^2), \qquad t \in [0,1],\\
{}^cD^{\frac{7}{4}}({}^cD^{\frac{1}{4}} + \frac{1}{40})v(t) &= (t-\frac{1}{4})^2((u(t))^{\frac{9}{5}} + ({}^cD^{\frac{1}{20}}u(t))^{\frac{7}{4}}), \qquad t \in [0,1].
\end{aligned}
$$

$$
\begin{aligned}
u(0) = 0, \quad u(1) &= \int_0^1 u(s)dx_1(s) + \int_0^1 v(s)dy_1(s), \quad {}^cD^{\frac{1}{3}}u(0) + {}^cD^{\frac{1}{3}}u(1) = 0,\\
v(0) = 0, \quad v(1) &= \int_0^1 u(s)dx_2(s) + \int_0^1 v(s)dy_2(s), \quad {}^cD^{\frac{1}{4}}u(0) + {}^cD^{\frac{1}{4}}u(1) = 0.
\end{aligned}
$$

*We can observe that* $\phi_1(t) = \phi_1(t) = 0$, $a_i = b_i = \frac{9}{16}$, $\mathcal{C}_1 = 0.2995493010 < \frac{1}{3}$ *and* $\mathcal{C}_2 = 0.2497957216 < \frac{1}{3}$. *Then, the problem has a solution if* $L < 0.2861152180$ *subject to that we take* $x_1(t) = y_1(t) = \frac{1}{32}(x^2-1)^2$, *which is monotonically decreasing function,* $x_2(t) = y_2(t) = \frac{1}{32}x^2$ *which monotonically increasing function. Accordingly* $\eta_1 = \eta_2 = \frac{1}{16}$.

## 5. Conclusions

In our contribution, we presented a discussion of the existence results for a fractional Langevin coupled system of Caputo type under specific boundary conditions involving Stieltjes integral boundary. We applied Shauder fixed-point theorem to get our result, and we gave an example that illustrates this result.

**Author Contributions:** All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (D-045-130-1442). The authors, therefore, acknowledge with thanks DSR technical and financial support.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We are very grateful to the anonymous referees for their valuable and useful suggestions.

**Conflicts of Interest:** The authors declare that they have no competing of interests.

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