Article

Global Bounds for the Generalized Jensen Functional with Applications

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Abstract: In this article we give sharp global bounds for the generalized Jensen functional $J_n(g, h; \mathbf{p}, x)$. In particular, exact bounds are determined for the generalized power mean in terms from the class of Stolarsky means. As a consequence, we obtain the best possible global converses of quotients and differences of the generalized arithmetic, geometric and harmonic means.

Keywords: Jensen functional; A-G-H inequalities; global bounds; power means; convex functions

MSC: 26D07(26D15)

1. Introduction

Recall that the Jensen functional $J_n(h; \mathbf{p}, x)$ is defined on an interval $I \subseteq \mathbb{R}$ by

$$J_n(h; \mathbf{p}, x) := \sum_{i=1}^n p_i h(x_i) - h\left(\sum_{i=1}^n p_i x_i\right),$$

where $h : I \rightarrow \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\mathbf{p} = \{p_i\}_1^n$ is a positive weight sequence.

If h is a convex function on I then the inequality

$$0 \leq J_n(h; \mathbf{p}, x)$$

holds for each $\mathbf{x} \in I^n$ and any positive weight sequence $\mathbf{p}$.

If h is a concave function on I then the above inequality is reversed. Those inequalities play a fundamental role in many parts of mathematical analysis and applications. For example, the well-known $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequality, Holder's inequality, Ky Fan inequality, etc., are proven by the help of Jensen's inequality (cf. [1–6]).

Our aim in this paper is to find the simplest constant C such that

$$0 \leq J_n(h; \mathbf{p}, x) \leq C,$$

for any choice of $\mathbf{p}, x$ and thus make this inequality symmetrical.

This will be done by assuming that $\mathbf{x} \in [a, b]^n \subset I^n$, and we shall find some *global bounds* for the generalized Jensen functional

$$J_n(g, h; \mathbf{p}, x) := g\left(\sum_{i=1}^n p_i h(x_i)\right) - g\left(h\left(\sum_{i=1}^n p_i x_i\right)\right),$$

that is, the bounds not depending on $\mathbf{p}$ or $\mathbf{x}$ but only on a, b and functions g and h .

In this sense, a typical result is given by the part of Theorem 1 (below).

Citation: Simić, S.; Bin-Mohsin, B. Global Bounds for the Generalized Jensen Functional with Applications. *Symmetry* **2021**, *13*, 2105. <https://doi.org/10.3390/sym13112105>

Academic Editor: Nicusor Minculete

Received: 17 October 2021

Accepted: 30 October 2021

Published: 6 November 2021

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For $\mathbf{x} \in [a, b]^n \subset I^n$, let $h : I \rightarrow J$ be convex and $g : J \rightarrow \mathbb{R}$ be an increasing function. Then

$$0 \leq J_n(g, h; \mathbf{p}, x) \leq \max_p [g(ph(a) + (1-p)h(b)) - g(h(pa + (1-p)b))].$$

Our global bounds will be entirely presented in terms of elementary means. Recall that the *mean* is a map $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with a property

$$\min(x, y) \leq M(x, y) \leq \max(x, y),$$

for each $x, y \in \mathbb{R}_+$.

In the sequel we shall use the class of so-called Stolarsky (or extended) two-parametric mean values, defined for positive values of $x, y, x \neq y$ by the following

$$E_{r,s}(x, y) = \begin{cases} \left(\frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)}, & rs(r-s) \neq 0 \\ \exp\left(\frac{-1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0 \\ \left(\frac{x^s - y^s}{s(\log x - \log y)} \right)^{1/s}, & s \neq 0, r = 0 \\ \sqrt{xy}, & r = s = 0, \\ x, & y = x > 0. \end{cases}$$

In this form it was introduced by Keneth Stolarsky in [7].

Most of the classical two variable means are special cases of the class E . For example,

$$A(x, y) = E_{1,2}(x, y) = \frac{x + y}{2}$$

is the arithmetic mean;

$$G(x, y) = E_{0,0}(x, y) = E_{-r,r}(x, y) = \sqrt{xy}$$

is the geometric mean;

$$L(x, y) = E_{0,1}(x, y) = \frac{x - y}{\log x - \log y}$$

is the logarithmic mean;

$$I(x, y) = E_{1,1}(x, y) = (x^x / y^y)^{\frac{1}{x-y}} / e$$

is the identric mean, etc.

More generally, the r -th power mean

$$A_r(x, y) = \left(\frac{x^r + y^r}{2} \right)^{1/r}$$

is equal to $E_{r,2r}(x, y)$.

Using the class of Stolarsky means enables our results to be presented in a condensed and applicable way. For example, we give some results regarding $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequalities, where

$$\mathcal{A}(\mathbf{p}, x) := \sum_1^n p_i x_i;$$

$$\mathcal{G}(\mathbf{p}, x) := \prod_1^n x_i^{p_i};$$

$$\mathcal{H}(\mathbf{p}, x) := \left(\sum_1^n p_i / x_i \right)^{-1},$$

are the generalized arithmetic, geometric and harmonic means, respectively.

Let $\mathbf{x} \in [a, b]^n, 0 < a < b$. Then

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{H}(\mathbf{p}, x) \leq 2(A(a, b) - G(a, b));$$

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{G}(\mathbf{p}, x) \leq A(a, b) - L(a, b) + L(a, b) \log \frac{L(a, b)}{G(a, b)};$$

$$1 \leq \frac{\mathcal{A}(\mathbf{p}, x)}{\mathcal{H}(\mathbf{p}, x)} \leq \left(\frac{A(a, b)}{G(a, b)} \right)^2;$$

$$1 \leq \frac{\mathcal{G}(\mathbf{p}, x)}{\mathcal{H}(\mathbf{p}, x)} \leq \frac{I(a, b)L(a, b)}{G^2(a, b)};$$

$$1 \leq \frac{\mathcal{A}(\mathbf{p}, x)}{\mathcal{G}(\mathbf{p}, x)} \leq \frac{I(a, b)L(a, b)}{G^2(a, b)},$$

where A, G, H, L, I stands for the arithmetic, geometric, harmonic, logarithmic and identric means of positive numbers a and b , respectively.

All bounds above are the best possible.

2. Results and Proofs

Our results concerning global bounds for the generalized Jensen functional are given in the following two assertions.

Theorem 1. 1. For continuous functions g, h let $h : I \rightarrow J$ be convex and $g : J \rightarrow \mathbb{R}$ be an increasing function or $h : I \rightarrow J$ be concave and $g : J \rightarrow \mathbb{R}$ be a decreasing function. Then

$$0 \leq J_n(g, h; \mathbf{p}, x) \leq \max_p [g(ph(a) + (1-p)h(b)) - g(h(pa + (1-p)b))].$$

2. If $h : I \rightarrow J$ is convex and $g : J \rightarrow \mathbb{R}$ is a decreasing function or $h : I \rightarrow J$ is concave and $g : J \rightarrow \mathbb{R}$ is an increasing function. Then

$$0 \leq -J_n(g, h; \mathbf{p}, x) \leq \max_p [g(h(pa + (1-p)b)) - g(ph(a) + (1-p)h(b))].$$

Proof. We shall prove only the part 1. The proof of part 2 of this theorem is analogous.

Therefore, if h is a convex function on J we have $\sum_1^n p_i h(x_i) \geq h(\sum_1^n p_i x_i)$. Since g is an increasing function, it follows that

$$J_n(g, h; \mathbf{p}, x) = g\left(\sum_1^n p_i h(x_i)\right) - g\left(h\left(\sum_1^n p_i x_i\right)\right) \geq 0.$$

Similarly, if h is a concave function on J we have $\sum_1^n p_i h(x_i) \leq h(\sum_1^n p_i x_i)$. Since g is a decreasing function, it follows again that

$$J_n(g, h; \mathbf{p}, x) \geq 0.$$

On the other hand, since $a \leq x_i \leq b$, there exist non-negative numbers $\lambda_i, \mu_i; \lambda_i + \mu_i = 1$, such that $x_i = \lambda_i a + \mu_i b, i = 1, 2, \dots, n$.

Hence,

$$\begin{aligned} J_n(g, h; \mathbf{p}, x) &= g\left(\sum_1^n p_i h(x_i)\right) - g\left(h\left(\sum_1^n p_i x_i\right)\right) = g\left(\sum_1^n p_i h(\lambda_i a + \mu_i b)\right) - g\left(h\left(\sum_1^n p_i (\lambda_i a + \mu_i b)\right)\right) \\ &\leq g\left(\sum_1^n p_i (\lambda_i h(a) + \mu_i h(b))\right) - g\left(h\left(a \sum_1^n p_i \lambda_i + b \sum_1^n p_i \mu_i\right)\right) \end{aligned}$$

$$= g(ph(a) + (1-p)h(b)) - g(h(pa + (1-p)b)) := F(p; a, b) \leq \max_p F(p; a, b),$$

where we denoted $\sum_1^n p_i \lambda_i := p \in [0, 1]$.

The second case with concave h and decreasing g leads to the same result. $\square$

Note that the function $F(p; a, b)$ is continuous in p and non-negative with $F(0; a, b) = F(1; a, b) = 0$. Therefore, $\max_p F(p; a, b)$ exists. Another and sometimes difficult problem is to evaluate its exact value (see Open Problem below).

For this cause, we give an estimation of $J_n(g, h; \mathbf{p}, x)$ with a unique maximum, which could be easily calculated. This method can be applied to the second part of Theorem 1, as well.

Theorem 2. 1. Under the conditions of the first part of Theorem 1, assume firstly that g is a convex function on J . Then

$$0 \leq J_n(g, h; \mathbf{p}, x) \leq \max_p [pf(a) + (1-p)f(b) - f(pa + (1-p)b)],$$

where $f := g \circ h$.

2. Assuming that $f = g \circ h$ is a concave function, we have

$$0 \leq J_n(g, h; \mathbf{p}, x) \leq \max_p [g(ph(a) + (1-p)h(b)) - (pf(a) + (1-p)f(b))].$$

Now, both maximums can be easily determined by the standard technique.

Proof. By the first part of Theorem 1, we found that there exists $p \in [0, 1]$ such that

$$J_n(g, h; \mathbf{p}, x) \leq g(ph(a) + (1-p)h(b)) - g(h(pa + (1-p)b)).$$

If additionally g is convex on J , then

$$g(ph(a) + (1-p)h(b)) \leq p(g \circ h)(a) + (1-p)(g \circ h)(b).$$

Hence,

$$\begin{aligned} J_n(g, h; \mathbf{p}, x) &\leq p(g \circ h)(a) + (1-p)(g \circ h)(b) - (g \circ h)(pa + (1-p)b) \\ &= pf(a) + (1-p)f(b) - f(pa + (1-p)b) \leq \max_p [pf(a) + (1-p)f(b) - f(pa + (1-p)b)]. \end{aligned}$$

Consequently, if $g \circ h$ is a concave function on J , we have

$$g(h(pa + (1-p)b)) = (g \circ h)(pa + (1-p)b) \geq p(g \circ h)(a) + (1-p)(g \circ h)(b),$$

and

$$J_n(g, h; \mathbf{p}, x) \leq \max_p [g(ph(a) + (1-p)h(b)) - (pf(a) + (1-p)f(b))].$$

$\square$

3. Applications

The results above are the source of a number of interesting inequalities. For instance, taking $g(x) = \log x$ in Theorem 1, we are enabled to determine converses of the quotient

$$\frac{\sum p_i h(x_i)}{h(\sum p_i x_i)}.$$

Or, taking $g(x) = h^{-1}(x)$, we can estimate the difference

$$\mathcal{A}_h(\mathbf{p}, x) - \mathcal{A}(\mathbf{p}, x),$$

where

$$\mathcal{A}_h(\mathbf{p}, x) := h^{-1}(\sum p_i h(x_i)),$$

is the quasi-arithmetic mean and

$$\mathcal{A}_x(\mathbf{p}, x) = \mathcal{A}(\mathbf{p}, x) = \sum p_i x_i,$$

is the generalized arithmetic mean.

We shall specialize this argument for the class of generalized power means $\mathcal{B}_s(\mathbf{p}, x)$ of order $s \in \mathbb{R}$, where

$$\mathcal{B}_s(\mathbf{p}, x) := (\sum p_i x_i^s)^{1/s}.$$

Some important particular cases are

$$\mathcal{B}_{-1}(\mathbf{p}, x) = (\sum_1^n p_i / x_i)^{-1} := \mathcal{H}(\mathbf{p}, x);$$

$$\mathcal{B}_0(\mathbf{p}, x) = \lim_{s \rightarrow 0} \mathcal{B}_s(\mathbf{p}, x) = \prod_1^n x_i^{p_i} := \mathcal{G}(\mathbf{p}, x);$$

$$\mathcal{B}_1(\mathbf{p}, x) = \sum_1^n p_i x_i := \mathcal{A}(\mathbf{p}, x),$$

that is, the generalized harmonic, geometric and arithmetic means, respectively.

It is well-known that $\mathcal{B}_s(\mathbf{p}, x)$ is monotone increasing in $s \in \mathbb{R}$ (cf. [4]).

Therefore,

$$\mathcal{H}(\mathbf{p}, x) \leq \mathcal{G}(\mathbf{p}, x) \leq \mathcal{A}(\mathbf{p}, x),$$

represents the famous $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequality.

As an application of Theorem 1, we shall estimate the difference $\mathcal{B}_s(\mathbf{p}, x) - \mathcal{A}(\mathbf{p}, x)$.

Theorem 3. Let $\mathbf{x} \in [a, b]^n \subset I^n$, $0 < a < b$.

Then

$$0 \leq \mathcal{B}_s(\mathbf{p}, x) - \mathcal{A}(\mathbf{p}, x) \leq \frac{s-1}{s} (E_{s,1}(a, b) - E_{s,s-1}^{-1}(1/a, 1/b)), \quad s > 1;$$

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{B}_s(\mathbf{p}, x) \leq \frac{1-s}{s} (E_{1,s}(a, b) - E_{1-s,-s}(a, b)), \quad 0 < s < 1;$$

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{B}_s(\mathbf{p}, x) \leq \frac{s-1}{s} (E_{1-s,-s}(a, b) - E_{1,s}(a, b)), \quad s < 0.$$

Proof. Let $h(x) = x^s$, $g(x) = x^{1/s}$, $s \in \mathbb{R} \setminus \{0\}$.

If $s > 1$, then h is a convex function and g is monotone increasing on $(0, \infty)$. Hence, by the first part of Theorem 1, we obtain

$$0 \leq \mathcal{B}_s(\mathbf{p}, x) - \mathcal{A}(\mathbf{p}, x) \leq \max_p ((pa^s + (1-p)b^s)^{1/s} - (pa + (1-p)b) := M_s(p_0; a, b).$$

This maximum is easy to calculate and we obtain

$$p_0 a^s + (1-p_0) b^s = \left(\frac{b^s - a^s}{s(b-a)} \right)^{s/(s-1)} = E_{s,1}^s(a, b).$$

Therefore,

$$p_0 = \frac{b^s - E_{s,1}^s(a, b)}{b^s - a^s}; \quad 1 - p_0 = \frac{E_{s,1}^s(a, b) - a^s}{b^s - a^s},$$

and

$$p_0 a + (1-p_0) b = \frac{ab^s - ba^s}{b^s - a^s} + \frac{b-a}{b^s - a^s} E_{s,1}^s(a, b).$$

Since,

$$E_{s,1}^s(a, b) = \frac{b^s - a^s}{s(b - a)} E_{s,1}(a, b),$$

we obtain

$$\begin{aligned} M_s(p_0; a, b) &= (p_0 a^s + (1 - p_0) b^s)^{1/s} - (p_0 a + (1 - p_0) b) \\ &= E_{s,1}(a, b) - \left(\frac{(1/a)^{s-1} - (1/b)^{s-1}}{(1/a)^s - (1/b)^s} + \frac{1}{s} E_{s,1}(a, b) \right) \\ &= \frac{s-1}{s} (E_{s,1}(a, b) - E_{s,s-1}^{-1}(1/a, 1/b)). \end{aligned}$$

In cases $0 < s < 1$ and $s < 0$ one should apply the second part of Theorem 1, since then h is concave and g is increasing in the first case and h is convex and g is decreasing in the second case. Proceeding as above, the result follows. $\square$

As a consequence, we obtain some converses of the $\mathcal{A}(\mathbf{p}, x) - \mathcal{G}(\mathbf{p}, x) - \mathcal{H}(\mathbf{p}, x)$ inequality.

Corollary 1. Let $\mathbf{x} \in [a, b]^n \subset I^n$, $b > a > 0$.

Then

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{H}(\mathbf{p}, x) \leq 2(A(a, b) - G(a, b)).$$

Proof. Putting $s = -1$, we obtain

$$\begin{aligned} 0 &\leq \mathcal{A}(\mathbf{p}, x) - \mathcal{B}_{-1}(\mathbf{p}, x) = \mathcal{A}(\mathbf{p}, x) - \mathcal{H}(\mathbf{p}, x) \\ &\leq 2(E_{2,1}(a, b) - E_{1,-1}(a, b)) = 2(A(a, b) - G(a, b)). \end{aligned}$$

$\square$

Corollary 2. Let $\mathbf{x} \in [a, b]^n \subset I^n$, $b > a > 0$.

Then

$$0 \leq \mathcal{A}(\mathbf{p}, x) - \mathcal{G}(\mathbf{p}, x) \leq L(a, b) \log \frac{L(a, b)I(a, b)}{G^2(a, b)}.$$

Proof. Letting $s \rightarrow 0$, we have

$$\begin{aligned} \mathcal{A}(\mathbf{p}, x) - \mathcal{G}(\mathbf{p}, x) &= \lim_{s \rightarrow 0} (\mathcal{A}(\mathbf{p}, x) - \mathcal{B}_s(\mathbf{p}, x)) \\ &\leq \lim_{s \rightarrow 0} \left(\frac{1-s}{s} (E_{1,s}(a, b) - E_{1-s,-s}(a, b)) \right). \end{aligned}$$

After somewhat laborous calculation using Taylor series, the result follows. $\square$

Remark 1. Estimating the Jensen functional

$$J_n(e^x; \mathbf{p}, \mathbf{x}) = \sum_1^n p_i e^{x_i} - e^{\sum_1^n p_i x_i}$$

for $\mathbf{x} \in [a, b]^n \subset \mathbb{R}^n$, and then changing variables $x_i \rightarrow \log x_i$; $a \rightarrow \log a$, $b \rightarrow \log b$, we obtain the same result.

Open problem Find the exact upper global bound for

$$\mathcal{G}(\mathbf{p}, x) - \mathcal{H}(\mathbf{p}, x).$$

The next proposition gives global bounds for the quotient of two power means.

Theorem 4. For $s > t$ and $\mathbf{x} \in [a, b]^n \subset \mathbb{R}_+^n$, we have

$$1 \leq \frac{\mathcal{B}_s(\mathbf{p}, \mathbf{x})}{\mathcal{B}_t(\mathbf{p}, \mathbf{x})} \leq \frac{E_{s,s-t}(a, b)}{E_{t,t-s}(a, b)}.$$

Both bounds are the best possible.

Proof. Applying the method from the proof of Theorem 1, we obtain

$$x_i^t = \lambda_i a^t + \mu_i b^t, \quad \lambda_i + \mu_i = 1, \quad i = 1, 2, \dots, n.$$

In the cases $s > t > 0$ or $s > 0, t < 0$, we have that the function $x^{s/t}$ is convex. Hence,

$$x_i^s = (\lambda_i a^t + \mu_i b^t)^{s/t} \leq \lambda_i (a^t)^{s/t} + \mu_i (b^t)^{s/t} = \lambda_i a^s + \mu_i b^t,$$

and

$$\frac{\mathcal{B}_s(\mathbf{p}, \mathbf{x})}{\mathcal{B}_t(\mathbf{p}, \mathbf{x})} = \frac{(\sum_1^n p_i x_i^s)^{1/s}}{(\sum_1^n p_i x_i^t)^{1/t}} \leq \frac{(a^s \sum_1^n p_i \lambda_i + b^s \sum_1^n p_i \mu_i)^{1/s}}{(a^t \sum_1^n p_i \lambda_i + b^t \sum_1^n p_i \mu_i)^{1/t}} = \frac{(pa^s + qb^s)^{1/s}}{(pa^t + qb^t)^{1/t}},$$

where we put

$$\sum_1^n p_i \lambda_i := p, \quad \sum_1^n p_i \mu_i := q; \quad p + q = 1.$$

Therefore, it follows that

$$\frac{\mathcal{B}_s(\mathbf{p}, \mathbf{x})}{\mathcal{B}_t(\mathbf{p}, \mathbf{x})} \leq \max_p \frac{(pa^s + qb^s)^{1/s}}{(pa^t + qb^t)^{1/t}} = \frac{(p_0 a^s + q_0 b^s)^{1/s}}{(p_0 a^t + q_0 b^t)^{1/t}}.$$

By standard means we obtain that this maximum satisfies the equation

$$\frac{s(p_0 a^s + q_0 b^s)}{a^s - b^s} = \frac{t(p_0 a^t + q_0 b^t)}{a^t - b^t},$$

that is,

$$p_0 = \frac{1}{s-t} \left(\frac{sb^s}{b^s - a^s} - \frac{tb^t}{b^t - a^t} \right); \quad q_0 = \frac{1}{s-t} \left(\frac{ta^t}{b^t - a^t} - \frac{sa^s}{b^s - a^s} \right).$$

Consequently,

$$p_0 a^t + q_0 b^t = \frac{s}{s-t} \frac{a^t b^s - a^s b^t}{b^s - a^s} = \frac{s}{s-t} \frac{(ab)^t (b^{s-t} - a^{s-t})}{b^s - a^s},$$

and

$$p_0 a^s + q_0 b^s = \frac{t}{s-t} \frac{a^t b^s - a^s b^t}{b^t - a^t} = \frac{t}{t-s} \frac{(ab)^s (b^{t-s} - a^{t-s})}{b^t - a^t}.$$

Hence,

$$(p_0 a^t + q_0 b^t)^{1/t} = G^2(a, b) / E_{s,s-t}(a, b);$$

$$(p_0 a^s + q_0 b^s)^{1/s} = G^2(a, b) / E_{t,t-s}(a, b),$$

and we finally obtain

$$\max_p \frac{(pa^s + qb^s)^{1/s}}{(pa^t + qb^t)^{1/t}} = \frac{(p_0 a^s + q_0 b^s)^{1/s}}{(p_0 a^t + q_0 b^t)^{1/t}} = \frac{E_{s,s-t}(a, b)}{E_{t,t-s}(a, b)}.$$

In the third case, for $s > t > 0$, we have

$$1 \leq \frac{\mathcal{B}_{-t}(\mathbf{p}, x)}{\mathcal{B}_{-s}(\mathbf{p}, x)} = \frac{\mathcal{B}_s(\mathbf{p}, 1/x)}{\mathcal{B}_t(\mathbf{p}, 1/x)} \leq \frac{E_{s,s-t}(1/a, 1/b)}{E_{t,t-s}(1/a, 1/b)} \\ = \frac{E_{s,s-t}(a, b)}{E_{t,t-s}(a, b)},$$

since

$$E_{u,v}(1/a, 1/b) = E_{u,v}(a, b)/G^2(a, b).$$

It is obvious that 1 is the best possible lower global bound. To prove that $M_{s,t}(a, b) := E_{s,s-t}(a, b)/E_{t,t-s}(a, b)$ is also the best possible global bound, denote by $N_{s,t}(a, b)$ an arbitrary upper bound. Then the relation

$$\frac{\mathcal{B}_s(\mathbf{p}, x)}{\mathcal{B}_t(\mathbf{p}, x)} \leq N_{s,t}(a, b),$$

holds for any $\mathbf{p}$ and $\mathbf{x}$.

Putting $x_1 = x_2 = \dots = x_{n-1} = a, x_n = b, p_n = q_0$, we obtain

$$M_{s,t}(a, b) = \frac{(p_0 a^s + q_0 b^s)^{1/s}}{(p_0 a^t + q_0 b^t)^{1/t}} = \frac{\mathcal{B}_s(\mathbf{p}, x)}{\mathcal{B}_t(\mathbf{p}, x)} \leq N_{s,t}(a, b),$$

and the proof is complete. $\square$

Some important consequences of this theorem are given in the following

Corollary 3. For $s > 1$, we have

$$\mathcal{A}(\mathbf{p}, x) \leq \mathcal{B}_s(\mathbf{p}, x) \leq \frac{E_{s,s-1}(a, b)}{E_{1,1-s}(a, b)} \mathcal{A}(\mathbf{p}, x).$$

Corollary 4. For $s > 0$, we have

$$\mathcal{G}(\mathbf{p}, x) \leq \mathcal{B}_s(\mathbf{p}, x) \leq \frac{E_{s,s}(a, b)E_{s,0}(a, b)}{G^2(a, b)} \mathcal{G}(\mathbf{p}, x).$$

Corollary 5. For $s > -1$, we have

$$\mathcal{H}(\mathbf{p}, x) \leq \mathcal{B}_s(\mathbf{p}, x) \leq \frac{E_{s+1,s}(a, b)E_{s+1,1}(a, b)}{G^2(a, b)} \mathcal{H}(\mathbf{p}, x).$$

In the last two corollaries we used the identity

$$E_{-u,-v}(a, b)E_{u,v}(a, b) = G^2(a, b).$$

Finally, putting $s = 1$ in Corollary 4 and $s = 0, s = 1$ in Corollary 5, since $E_{2,1}(a, b) = A(a, b)$, $E_{1,0}(a, b) = L(a, b)$, $E_{1,1}(a, b) = I(a, b)$, we obtain global converses of the $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequality.

Corollary 6.

$$\mathcal{G}(\mathbf{p}, x) \leq \mathcal{A}(\mathbf{p}, x) \leq \frac{L(a, b)I(a, b)}{G^2(a, b)} \mathcal{G}(\mathbf{p}, x);$$

$$\mathcal{H}(\mathbf{p}, x) \leq \mathcal{A}(\mathbf{p}, x) \leq \left(\frac{A(a, b)}{G(a, b)} \right)^2 \mathcal{H}(\mathbf{p}, x);$$

$$\mathcal{H}(\mathbf{p}, x) \leq \mathcal{G}(\mathbf{p}, x) \leq \frac{L(a, b)I(a, b)}{G^2(a, b)} \mathcal{H}(\mathbf{p}, x).$$

Therefore, a sort of tight symmetry is established for these inequalities.

4. Conclusions

We give a method for two-sided estimations of the generalized Jensen functional $J_n(g, h; \mathbf{p}, \mathbf{x})$, with applications to the general means. In particular, sharp converses of the famous

$\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequality are obtained. Further investigations can be undertaken on more general settings, i.e., $J_n(f, g, h; \mathbf{p}, \mathbf{x}) := f(\sum_1^n p_i h(x_i)) - g(h(\sum_1^n p_i x_i))$ or even $F(\sum_1^n p_i h(x_i), h(\sum_1^n p_i x_i))$, with properly chosen functions f, g, h and $F(x, y)$.

Author Contributions: Theoretical part, S.S.; numerical part with examples, B.B.-M. All authors have read and agreed to the published version of the manuscript.

Funding: Bandar Bin-Mohsin is supported by Researchers Supporting Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the referees for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interests.

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