

Review

# Interplay between Spacetime Curvature, Speed of Light and Quantum Deformations of Relativistic Symmetries

Angel Ballesteros <sup>1,\*</sup> , Giulia Gubitosi <sup>2,3</sup> and Flavio Mercati <sup>1</sup><sup>1</sup> Departamento de Física, Universidad de Burgos, 09001 Burgos, Spain; fmercati@ubu.es<sup>2</sup> Dipartimento di Fisica “Ettore Pancini”, Università di Napoli Federico II, 80126 Napoli, Italy; giulia.gubitosi@unina.it<sup>3</sup> National Institute of Nuclear Physics, Sezione di Napoli, 80126 Napoli, Italy

\* Correspondence: angelb@ubu.es

**Abstract:** Recent work showed that  $\kappa$ -deformations can describe the quantum deformation of several relativistic models that have been proposed in the context of quantum gravity phenomenology. Starting from the Poincaré algebra of special-relativistic symmetries, one can toggle the curvature parameter  $\Lambda$ , the Planck scale quantum deformation parameter  $\kappa$  and the speed of light parameter  $c$  to move to the well-studied  $\kappa$ -Poincaré algebra, the (quantum) (A)dS algebra, the (quantum) Galilei and Carroll algebras and their curved versions. In this review, we survey the properties and relations of these algebras of relativistic symmetries and their associated noncommutative spacetimes, emphasizing the nontrivial effects of interplay between curvature, quantum deformation and speed of light parameters.

**Keywords:** quantum groups; Poincaré group; (Anti)-de Sitter; Galilei group; carroll symmetries; curvature; deformation; Planck scale; noncommutative spacetimes; quantum gravity; phenomenology



**Citation:** Ballesteros, A.; Gubitosi, G.; Mercati, F. Interplay between Spacetime Curvature, Speed of Light and Quantum Deformations of Relativistic Symmetries. *Symmetry* **2021**, *13*, 2099. <https://doi.org/10.3390/sym13112099>

Academic Editor: Joseph L. Buchbinder

Received: 9 October 2021

Accepted: 30 October 2021

Published: 5 November 2021

**Publisher’s Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



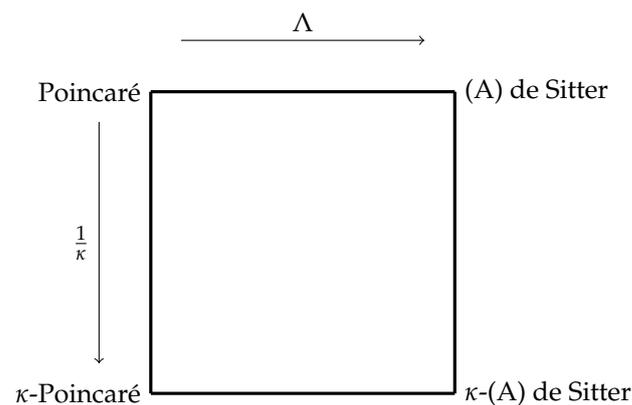
**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Deformations of relativistic symmetries have been playing a prominent role in the study of phenomenologically relevant effects of quantum gravity in a “non-quantum” and “non-gravitational” regime, such that both the Planck constant  $\hbar$  and the Newton constant  $G$  are negligible, but their ratio is not, thus leaving the Planck energy  $E_P = \sqrt{\frac{c^5 \hbar}{G}}$  finite [1,2].

In this context, a much studied formalism that provides a rigorous mathematical framework for the deformed symmetry models is that of  $\kappa$ -deformations [3–10], which turn the Lie algebra describing the Poincaré symmetries of special relativity into a Hopf algebra and where the quantum deformation parameter  $\kappa$  is assumed to be of the order of the Planck energy [11]. Despite these models being originally derived as a contraction of the quantum (Anti)-de Sitter algebra in the limit of vanishing cosmological constant  $\Lambda$ , the great majority of the subsequent work focussed exclusively on the  $\Lambda = 0$  case.

Nevertheless, some preliminary analyses [12–17] pointed out that nontrivial effects are to be expected due to the interplay between the cosmological constant  $\Lambda$  and the quantum deformation parameter  $\kappa$ , and these effects might have significant implications for phenomenological analyses that focus on an astrophysical setup where the cosmological expansion is non-negligible [18]. This interplay emerges because the two parameters govern two kinds of deformation of the Poincaré algebra, respectively, a classical deformation, turning the Poincaré algebra into a new Lie algebra describing (Anti)-de Sitter symmetries [19], and a quantum deformation, turning the Poincaré algebra into a deformed Hopf algebra (see Figure 1). When both deformations are present, the Poincaré algebra turns into a  $\kappa$ -deformed (Anti)-de Sitter Hopf algebra, and novel features emerge that are governed by products of the two deformation parameters, so that they disappear in both the flat  $\Lambda \rightarrow 0$  and the classical  $\kappa^{-1} \rightarrow 0$  limits [20–25].

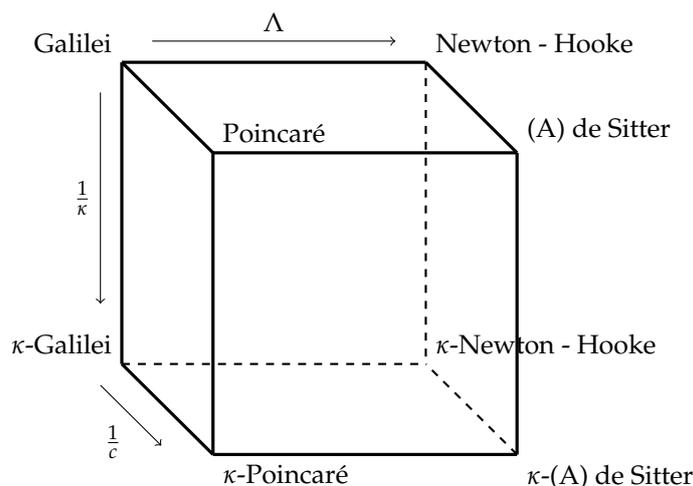


**Figure 1.** The various algebras of relativistic symmetries emerging in the regimes set by different combinations of the cosmological constant  $\Lambda$  and the quantum deformation parameter  $\kappa$ . The arrows point in the direction where the indicated parameter becomes nonzero. We see that the (Anti-)de Sitter algebra and the  $\kappa$ -Poincaré algebra are both deformations of the Poincaré algebra, one being a classical deformation and the other a quantum deformation, respectively.

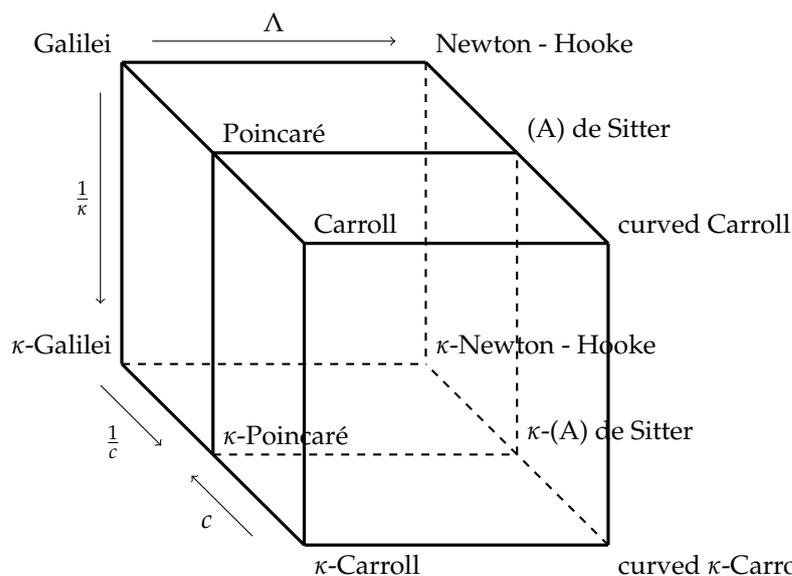
Very recent work analyzed yet another direction of classical deformation, this time governed by the speed of light  $c$  (see Figures 2 and 3). The novel feature of this deformation, with respect to the classical deformation governed by  $\Lambda$ , is that it can work in two different directions: starting from the Poincaré Lie algebra, one can perform two kinds of contractions, one where  $c^{-1} \rightarrow 0$  and one where  $c \rightarrow 0$ , which lead to the Galilei and Carroll Lie algebras and groups, respectively [26–29]. These two contractions can also be performed in the presence of the cosmological constant  $\Lambda$  and of the quantum deformation parameter  $\kappa$ , as was shown very recently [30], thus providing us with a quite rich structure of possible algebras of relativistic symmetries, shown in Figure 3. We recall that Galilean symmetries with  $\Lambda \neq 0$  are known in the literature as Newton–Hooke algebras [27].

In this review, we survey the properties and relations of all of these algebras, emphasizing the different effects the three deformation parameters have and how they interact with one another. While the technical results on which we base our discussion have appeared in previous works, which are referenced to in the appropriate sections; this is the first time that a systematic picture of the properties and relations of these algebras is provided.

The plan of this review is the following. In Section 2, we revisit the quantum deformation procedure, turning the Poincaré Lie algebra into the  $\kappa$ -Poincaré Hopf algebra. In Section 3, we revisit the classical deformation procedure that turns the Poincaré algebra into the (Anti-)de Sitter algebra with non-vanishing cosmological constant and show how the quantum deformation procedure applies to the latter. The interplay between the effects of curvature and of quantum deformation are discussed. In Section 4, we perform the two classical contraction procedures governed by the speed of light, leading to the Galilean and Carrollian limits of the classical (Anti-)de Sitter algebra. Here, we discuss how the two classical deformations, governed by the speed of light and curvature, interact. Section 5 looks at the full picture, where all of the three parameters are in play. The different features of the various algebras are revisited from the noncommutative spacetime point of view in Section 6. Final remarks are provided in Section 7.



**Figure 2.** The various algebras of relativistic symmetries emerging in the regimes set by different combinations of the cosmological constant  $\Lambda$ , the speed of light  $c$  and the quantum deformation parameter  $\kappa$ . The arrows point in the direction where the indicated parameter becomes nonzero. In addition to those shown in the previous picture, here we also see the classical deformation direction governed by the speed of light  $c$ , linking special-relativistic-like symmetries and Galilean-like symmetries.



**Figure 3.** The various algebras of relativistic symmetries emerging in the regimes set by different combinations of the cosmological constant  $\Lambda$ , the speed of light  $c$  and the quantum deformation parameter  $\kappa$ . The arrows point in the direction where the indicated parameter becomes nonzero. In addition to those shown in the previous pictures, here we also see a new direction in which the classical deformation governed by the speed of light  $c$  can work, linking special-relativistic-like symmetries and Carrollian-like symmetries.

**2. The  $\kappa$ -Poincaré Model**

We start by briefly reviewing the classical (3 + 1)-dimensional Poincaré Lie algebra  $\mathfrak{p}(3 + 1)$ , using a language that will prove useful for discussing its quantum deformation. This algebra is defined by the commutation relations:

$$\begin{aligned}
 [J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, \\
 [K_a, P_0] &= P_a, & [K_a, P_b] &= \delta_{ab} P_0, & [K_a, K_b] &= -\epsilon_{abc} J_c, \\
 [P_0, P_a] &= 0, & [P_a, P_b] &= 0, & [P_0, J_a] &= 0,
 \end{aligned}
 \tag{1}$$

where in the so-called kinematical basis  $\{P_0, P_a, K_a, J_a\}$  ( $a = 1, 2, 3$ ) are the generators of time translations, space translations, boosts and rotations, respectively. The sum over repeated indices is assumed and, for the moment, the speed of light  $c$  is set to 1. As for any Lie algebra, the universal enveloping algebra  $U(\mathfrak{p}(3+1))$  of the Poincaré algebra is a Hopf algebra endowed with a primitive (non-deformed) coproduct:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \forall X \in \mathfrak{p}(3+1). \quad (2)$$

For the generators of spacetime translations, this coproduct algebraically encodes the linear addition law for momenta that characterize the usual special relativistic kinematics.

In this group-theoretical setting, Minkowski spacetime  $\mathbf{M}^{3+1}$  can be constructed from the Poincaré Lie group as the homogeneous space:

$$\mathbf{M}^{3+1} \equiv \text{ISO}(3,1)/\text{SO}(3,1), \quad (3)$$

where the isotropy subgroup is the Lorentz group  $\text{SO}(3,1)$ . Explicitly, a 5-dimensional faithful representation  $\rho$  for a generic element  $X$  of the Poincaré Lie algebra is given by:

$$\rho(X) = x^\alpha \rho(P_\alpha) + \zeta^a \rho(K_a) + \theta^a \rho(J_a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x^0 & 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ x^1 & \zeta^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \zeta^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \zeta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \quad (4)$$

If we parametrize an element  $G$  of the Poincaré group  $\text{ISO}(3,1)$  in the form:

$$G = \exp(x^0 \rho(P_0)) \exp(x^1 \rho(P_1)) \exp(x^2 \rho(P_2)) \exp(x^3 \rho(P_3)) \\ \times \exp(\zeta^1 \rho(K_1)) \exp(\zeta^2 \rho(K_2)) \exp(\zeta^3 \rho(K_3)) \exp(\theta^1 \rho(J_1)) \exp(\theta^2 \rho(J_2)) \exp(\theta^3 \rho(J_3)), \quad (5)$$

the  $(3+1)$ -dimensional Minkowski spacetime  $\mathbf{M}^{3+1}$  can be constructed as a coset space (note that the Lorentz subgroup is located at the rightmost side in the exponentials above), whose points are labeled by the usual Minkowski coordinates  $x^\alpha$  associated to translations. From a Hopf-algebraic point of view, this means that there is a pairing:

$$\langle x^\alpha, P_\beta \rangle = \delta_\beta^\alpha, \quad (6)$$

between Poincaré translation generators and the Minkowski coordinates  $x^\alpha$ .

The representation theory of the Poincaré Lie algebra is characterized by its Casimir operators (see, for instance [31]): the quadratic one:

$$\mathcal{C} = P_0^2 - \mathbf{P}^2, \quad (7)$$

whose realization on momentum space gives rise to the energy-momentum dispersion relation, and the quartic one  $\mathcal{W}$ , constructed in terms of the components of the Pauli-Lubanski four vector in the form:

$$\mathcal{W} = W_0^2 - \mathbf{W}^2, \quad \text{where} \quad (8) \\ W_0 = \mathbf{J} \cdot \mathbf{P} \quad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c.$$

### 2.1. The $\kappa$ -Poincaré Quantum Algebra

The  $\kappa$ -Poincaré algebra [3] (see also [4,5]) is a quantum Poincaré algebra, that is, a Hopf algebra deformation (see [32,33]) of the Poincaré algebra in terms of a quantum deformation parameter  $\kappa^{-1}$ . The essential feature of quantum deformations is that, in general, the deformation affects both the defining commutation rules of the algebra (which turn out to be nonlinear) and the coproduct map (for which the linear rule of superposition of generators is broken).

The deformed commutation rules and the deformed coproducts have to be compatible in the sense that the latter have to be a homomorphism map for the former. Moreover, quantum deformations are smooth in the sense that in the vanishing deformation parameter limit, the quantum algebra reduces to the initial Lie algebra. All these conditions restrict the number of possible inequivalent quantum deformations of a Lie algebra. For the Poincaré case, the classification of all its possible quantum deformations was presented in [34], and the analogue classification in the quantum group setting was given in [35].

The  $\kappa$ -Poincaré algebra is a very specific Hopf algebra deformation of the Poincaré algebra, which was obtained through quantum group contraction techniques [36–38] from the so-called Drinfel'd–Jimbo quantum deformation of the (Anti)-de Sitter Lie algebra [39,40]. Explicitly, its commutation rules are given by a non-deformed sector:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, \\ [K_a, P_0] &= P_a, & [K_a, K_b] &= -\epsilon_{abc} J_c, & & \\ [P_0, P_a] &= 0, & [P_a, P_b] &= 0, & [P_0, J_a] &= 0, \end{aligned} \quad (9)$$

together with the following deformed commutators:

$$[K_a, P_b] = \delta_{ab} \left( \frac{\kappa}{2} \left( 1 - e^{-2P_0/\kappa} \right) + \frac{1}{2\kappa} \mathbf{P}^2 \right) - \frac{1}{\kappa} P_a P_b. \quad (10)$$

The deformed coproduct map for the  $\kappa$ -Poincaré algebra reads:

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(J_a) &= J_a \otimes 1 + 1 \otimes J_a, \\ \Delta(P_a) &= P_a \otimes 1 + e^{-P_0/\kappa} \otimes P_a, \\ \Delta(K_a) &= K_a \otimes 1 + e^{-P_0/\kappa} \otimes K_a + \frac{1}{\kappa} \epsilon_{abc} P_b \otimes J_c. \end{aligned} \quad (11)$$

We stress that the  $\kappa^{-1} \rightarrow 0$  limit of all these expressions leads to the non-deformed Hopf algebra structure of the Poincaré algebra.

It is also worth emphasizing that this is an “essential” deformation in the sense that the theory of quantum universal enveloping algebras ensures that there does not exist any change of basis that transforms the deformed coproduct (11) into the non-deformed one (2). On the other hand, it is possible to find a (nonlinear) change of basis transforming the deformed commutation rules (9) and (10) into the non-deformed ones (1). As expected, this transformation to the so-called “classical basis” [41] for  $\kappa$ -Poincaré provides a (quite cumbersome) deformed coproduct and shows that, in order to prevent inconsistencies, all models defined through quantum deformations have to accommodate the full Hopf algebra structure (commutation rules + coproduct) as their underlying symmetry.

Some features of this quantum deformation of the Poincaré algebra deserve some attention. Firstly, the existence of deformed commutation rules (10) implies that Casimir operators have to also be  $\kappa$ -deformed. In particular, the deformed quadratic Casimir is found to be:

$$C_\kappa = 4\kappa^2 \sinh^2(P_0/2\kappa) - e^{P_0/\kappa} \mathbf{P}^2, \quad (12)$$

and obviously its  $\kappa \rightarrow \infty$  limit leads to (7). When the corresponding momentum space representation of the  $\kappa$ -Poincaré algebra is considered [41–43], this Casimir gives rise to a deformed dispersion relation, which is the cornerstone of the quantum gravity phenomenology of the  $\kappa$ -Poincaré model (see [44] for a review on the role of  $\kappa$ -Poincaré in Doubly Special Relativity models).

The deformation of the Pauli-Lubanski Casimir (8) reads (see [23] and references therein):

$$\mathcal{W}_\kappa = \left( \cosh(P_0/\kappa) - \frac{1}{4\kappa^2} e^{P_0/\kappa} \mathbf{P}^2 \right) W_{\kappa,0}^2 - \mathbf{W}_\kappa^2, \quad \text{where}$$

$$W_{\kappa,0} = e^{P_0/(2\kappa)} \mathbf{J} \cdot \mathbf{P}, \quad W_{\kappa,a} = -J_a \kappa \sinh(P_0/\kappa) + e^{P_0/\kappa} \epsilon_{abc} \left( K_b + \frac{1}{2\kappa} \epsilon_{bkl} J_k P_l \right) P_c. \quad (13)$$

Similar to what happens in the non-deformed Poincaré case, Casimir operators label the irreducible representations of the  $\kappa$ -Poincaré Hopf algebra. The spin zero representation was already given in [3], where the corresponding  $\kappa$ -Klein–Gordon equation was proposed. Irreducible representations for arbitrary spin were constructed in [4] for both the massive and the massless cases, and the  $\kappa$ -Dirac equation was introduced in [5,45,46].

Secondly, the deformed coproduct of the  $\kappa$ -Poincaré algebra provides a non-primitive addition law for momenta:

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_a) &= P_a \otimes 1 + e^{-P_0/\kappa} \otimes P_a, \end{aligned} \quad (14)$$

which encodes in algebraic terms the nontrivial properties of the geometry of the associated momentum space. These expressions imply that the momentum sector of the  $\kappa$ -Poincaré algebra is a Hopf subalgebra, since the coproducts of momenta generators depend only on themselves. As we will see in the following Section, this is no longer the case when the spacetime curvature  $\Lambda$  is considered. Finally, it is worth to mention that the Lorentz generators do not close a Hopf subalgebra, since the coproducts (11) for the boost generators include translations. Quantum Poincaré and (A)dS algebras endowed with quantum Lorentz subgroup have recently been characterized in [47].

## 2.2. The $\kappa$ -Poincaré Lie Bialgebra and $\kappa$ -Minkowski Spacetime

The ambiguity in the selection of the basis of quantum algebra does not affect the Lie bialgebra structure  $\delta$  associated to the  $\kappa$ -Poincaré algebra. In fact, this is an object that characterizes any quantum deformation in a unique way since it does not depend on changes of basis of the type

$$X' = X'(P_0, P_a, J_a, K_a, \kappa) \quad \text{with} \quad \lim_{\kappa \rightarrow \infty} X' = X, \quad \text{for} \quad X \equiv \{P_0, P_a, J_a, K_a\}. \quad (15)$$

This Lie bialgebra structure is obtained by taking the skew-symmetric part of the first order in  $1/\kappa$  of the deformed coproduct (11), and reads:

$$\begin{aligned} \delta(P_0) &= \delta(J_a) = 0, \\ \delta(P_a) &= \frac{1}{\kappa} P_a \wedge P_0, \\ \delta(K_1) &= \frac{1}{\kappa} (K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2), \\ \delta(K_2) &= \frac{1}{\kappa} (K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3), \\ \delta(K_3) &= \frac{1}{\kappa} (K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1). \end{aligned} \quad (16)$$

This cocommutator map  $\delta: \mathfrak{p}(3+1) \rightarrow \mathfrak{p}(3+1) \otimes \mathfrak{p}(3+1)$  is defined on the undeformed Poincaré algebra, and can be obtained from the classical  $r$ -matrix that characterizes the  $\kappa$ -deformation:

$$r = \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3), \quad (17)$$

through  $\delta(X) = [1 \otimes X + X \otimes 1, r]$ , where  $r$  is a solution of the modified classical Yang–Baxter equation. From this perspective, the  $r$ -matrix is the “minimal” object that defines a given quantum deformation: from it, the first order deformation of the coproduct can be

obtained, and the semiclassical counterpart of the associated quantum group (a Poisson–Lie group) is uniquely defined. In the approach presented here, Lie bialgebra structures are used as the defining objects for quantum deformations, and the type of interplay among all the parameters arising in them can be already studied at the Lie bialgebra level (in particular, the theory of quantum group contractions is based on the contraction theory for Lie bialgebras [38]). A detailed description of Lie bialgebras and their role in quantum group theory can be found in [32], and a complete presentation of kinematical Lie bialgebras has been given in [48].

We also stress that the Hopf subalgebra structure of the momentum sector (14) is reflected at the Lie bialgebra level in the form

$$\delta(P_0) = 0, \quad \delta(P_1) = \frac{1}{\kappa} P_1 \wedge P_0, \quad \delta(P_2) = \frac{1}{\kappa} P_2 \wedge P_0. \quad (18)$$

This sub-Lie bialgebra structure for the momentum sector can be dualized to give rise to the so-called  $\kappa$ -Minkowski Lie algebra

$$[X^0, X^a] = -\frac{1}{\kappa} X^a, \quad [X^a, X^b] = 0. \quad (19)$$

This algebra can be identified with the one defining the  $\kappa$ -Minkowski non-commutative spacetime [3,6–8]. Moreover, the  $\kappa$ -Poincaré momentum space can be constructed as an orbit of a certain linear action of the  $\kappa$ -Minkowski Lie group [49–51]. Such an orbit turns out to be (a half of) the (3 + 1) de Sitter space with curvature  $1/\kappa^2$ , and the deformed dispersion relation of the model can be thought of as the distance to the origin in such curved momentum space [42].

### 2.3. Applications

The  $\kappa$ -Poincaré model and its associated quantum geometry has been extensively used in the literature in order to study different explicit models dealing with both mathematical and physical features of quantum geometry which are expected to arise at the Planck scale. Without aiming to be exhaustive, some of the facets of  $\kappa$ -Poincaré algebra and  $\kappa$ -Minkowski spacetime that have been analyzed in the literature are the following ones (see also references therein):

- Deformed dispersion relations and Doubly Special Relativity [52–58], in particular the first paper associating deformed dispersion relations to  $\kappa$ -Poincaré/ $\kappa$ -Minkowski [52] and the review [54].
- $\kappa$ -deformed models of Relative Locality [42,43,59–66], see also the first papers defining the theory of Relative Locality [1,2,67].
- There is an interesting string of works on the representation theory of  $\kappa$ -Minkowski commutation relations [68–72].
- Another aspect of interest is the differential geometry of  $\kappa$ -Minkowski spacetime (and generalizations), and its relationship with the  $\kappa$ -Poincaré group and with star products [73–78].
- There is a vast literature on how to construct classical (in the sense of  $\hbar = 0$ ) and quantum noncommutative field theories that are symmetric under the  $\kappa$ -Poincaré group and are based on different versions of  $\kappa$ -Minkowski spacetime. A non-exhaustive list is [79–104], and references therein.
- A crucial issue is what limits to the spacetime localizability of observables does a  $\kappa$ -deformed theory imply [71,80,105–107]. Related to this is the possibility of deformations or the fuzziness of light cones [108,109].
- An important consequence of  $\kappa$ -deformed spacetime symmetries and noncommutative spacetimes is the emergence of a curvature of momentum space and related deformations of phase space [42,50,110–113].
- Finally, a recent line of research led to the development of a  $\kappa$ -deformed noncommutative version of the spaces of worldlines [105,114].

It is worth emphasizing that most of the above-mentioned techniques and models have been exclusively developed for the  $\kappa$ -Poincaré case. Therefore, the approach that we summarize in the following sections provides the basis for the generalization of all these results and models when the cosmological constant parameter  $\Lambda$  is non-vanishing and/or for the Galilean and Carrollian limits when  $c \rightarrow \infty$  and  $c \rightarrow 0$ , respectively.

### 3. Interplay between Curvature and Quantum Effects

If one aims to study the effects of quantum-deformed relativistic symmetries in a cosmological context (as is, e.g., the case in studies of the propagation of signals from astrophysical sources [54]), the most natural option consists of the generalization of the  $\kappa$ -Poincaré model to allow for a nonvanishing cosmological constant  $\Lambda$ . This leads to a quantum-deformed (Anti)-de Sitter (hereafter (A)dS) model.

Works in (1 + 1) and (2 + 1) dimensions already suggested that there is a nontrivial interplay between the quantum deformation and curvature. In particular, once the quantum deformation is taken into account, the effects that are classically associated with spacetime curvature acquire a new energy-dependence. For example, the travel time of massless particles acquire an energy dependence that depends on the curvature and the quantum deformation parameter in a nontrivial way [13,15–17]. While the phenomenology of the  $\kappa$ -(A)dS model in (3 + 1) dimensions still has to be worked out, preliminary studies show that, in this case, the interplay between quantum deformation and curvature can be even more virulent, as we will discuss in this section.

Despite the fact that the  $\kappa$ -Poincaré algebra was initially obtained as the quantum group contraction associated to the flat  $\Lambda \rightarrow 0$  limit of the quantum  $so(3,2)$  algebra [3,115], neither the relation among the generators of such  $so(3,2)$  quantum deformation and the kinematical generators  $\{P_0, P_a, K_a, J_a\}$  nor the explicit role played by the cosmological constant  $\Lambda$  in the quantum case were explored. This lack of information prevented any physical interpretation, as well as the construction of the corresponding quantum (A)dS spacetimes in terms of local coordinates. This started to change recently, since a series of papers have filled this gap by constructing the fully explicit  $\kappa$ -(A)dS model [23] and its associated noncommutative spacetime [25]. The main features of the former will be summarized in this section following the presentation of the  $\kappa$ -Poincaré model given in the previous section, while the latter will be presented in Section 6. We stress that, throughout this construction, the curvature  $\Lambda$  will always be made explicit as a “classical” curvature parameter whose  $\Lambda \rightarrow 0$  limit leads exactly to the  $\kappa$ -Poincaré model.

#### 3.1. (Anti-)de Sitter Symmetries as a Classical Deformation of Poincaré Symmetries

Before going to the quantum-deformed (A)dS model, we briefly show that the standard (A)dS algebra can be seen as a classical deformation of the Poincaré algebra. This is based on writing the (A)dS Lie algebra in (3 + 1)D in the following manner:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J_c, & [J_a, P_b] &= \epsilon_{abc} P_c, & [J_a, K_b] &= \epsilon_{abc} K_c, \\ [K_a, P_0] &= P_a, & [K_a, P_b] &= \delta_{ab} P_0, & [K_a, K_b] &= -\epsilon_{abc} J_c, \\ [P_0, P_a] &= -\Lambda K_a, & [P_a, P_b] &= \Lambda \epsilon_{abc} J_c, & [P_0, J_a] &= 0, \end{aligned} \quad (20)$$

where  $\Lambda$  is the cosmological constant parameter. This Lie algebra is just a  $\Lambda$  deformation of (1), and the latter is obtained in the smooth  $\Lambda \rightarrow 0$  limit of (20). In this way, the three relativistic spacetimes with constant curvature are obtained as the following maximally symmetric homogeneous spaces:

- For  $\Lambda < 0$  we have the  $SO(3,2)$  symmetry algebra and the AdS spacetime  $\mathbf{AdS}^{3+1}$  is obtained as the coset space  $SO(3,2)/SO(3,1)$ .
- For  $\Lambda > 0$  we have the  $SO(4,1)$  symmetry algebra that gives rise to the de Sitter spacetime  $\mathbf{dS}^{3+1} \equiv SO(4,1)/SO(3,1)$ .
- Finally, for  $\Lambda = 0$  we recover the Poincaré algebra, and Minkowski spacetime is  $\mathbf{M}^{3+1} \equiv ISO(3,1)/SO(3,1)$ .

This approach provides (A)dS Casimir operators as a  $\Lambda$ -deformation of Poincaré invariants. The quadratic one being:

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda(\mathbf{J}^2 - \mathbf{K}^2), \quad (21)$$

and the quartic one (of Pauli-Lubanski type) reads

$$\mathcal{W} = W_0^2 - \mathbf{W}^2 - \Lambda(\mathbf{J} \cdot \mathbf{K})^2, \quad (22)$$

$$\text{where } W_0 = \mathbf{J} \cdot \mathbf{P} \quad \text{and} \quad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c.$$

Two main features of the (A)dS Lie algebra (20) are worth to be emphasized. Firstly, that space–time translation generators do not commute when  $\Lambda \neq 0$ :

$$[P_0, P_a] = -\Lambda K_a, \quad [P_a, P_b] = \Lambda \epsilon_{abc} J_c, \quad (23)$$

and therefore the translation sector does not define a Lie subalgebra. This reflects the fact that the (A)dS spacetimes are curved spaces, since spacetime translations are generators of geodesic motions on them.

Secondly, when  $\Lambda \neq 0$  the following automorphism interchanges the role of  $P_a$  and  $K_a$  (see [116]):

$$\tilde{P}_0 = P_0, \quad \tilde{P}_a = \sqrt{-\Lambda} K_a, \quad \tilde{K}_a = -\frac{1}{\sqrt{-\Lambda}} P_a, \quad \tilde{J}_a = J_a. \quad (24)$$

In this sense, translations and boosts play an algebraically equivalent role, although their physical meaning is indeed different. As we will see, this property will be essential in order to understand some of the features of the  $\kappa$ -(A)dS model.

### 3.2. The $\kappa$ -(A)dS Model in (3 + 1) Dimensions

We recall that the (2 + 1) dimensional  $\kappa$ -(A)dS algebra and deformed Casimir operators was already presented in [20]. The very same quantum algebra was later rediscovered in [12] as the algebra containing the cosmological constant that was proposed as a symmetry for the low energy limit of a quantum theory of gravity (see also [24] for a more recent approach). The classical  $r$ -matrix generating such a (2 + 1) quantum (A)dS deformation is:

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2). \quad (25)$$

Surprisingly enough, the cosmological constant parameter  $\Lambda$  is absent in this  $r$ -matrix, which therefore coincides with its Poincaré limit. Nevertheless, the full quantum algebra does contain  $\Lambda$  explicitly.

The quest for the generalization of (25) to the (3 + 1)-dimensional case was recently solved in [23], and the unique (modulo automorphisms) solution is:

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2), \quad (26)$$

where from now on we will use the parameter  $\eta^2 := -\Lambda$ . This is the unique skewsymmetric  $r$ -matrix for the (A)dS algebra fulfilling two conditions: the  $\Lambda \rightarrow 0$  limit of (26) is the  $\kappa$ -Poincaré  $r$ -matrix (this guarantees the appropriate flat limit of the model), and the cocommutator of the  $P_0$  generator is primitive  $\delta(P_0) = 0$  (this enables in the curved case the interpretation of  $\kappa$  as a mass).

From the  $r$ -matrix (26), the following  $\kappa$ -(A)dS cocommutator map is obtained:

$$\begin{aligned}
 \delta(P_0) &= \delta(J_3) = 0, & \delta(J_1) &= \frac{\eta}{\kappa} J_1 \wedge J_3, & \delta(J_2) &= \frac{\eta}{\kappa} J_2 \wedge J_3, \\
 \delta(P_1) &= \frac{1}{\kappa} (P_1 \wedge P_0 - \eta P_3 \wedge J_1 - \eta^2 K_2 \wedge J_3 + \eta^2 K_3 \wedge J_2), \\
 \delta(P_2) &= \frac{1}{\kappa} (P_2 \wedge P_0 - \eta P_3 \wedge J_2 + \eta^2 K_1 \wedge J_3 - \eta^2 K_3 \wedge J_1), \\
 \delta(P_3) &= \frac{1}{\kappa} (P_3 \wedge P_0 + \eta P_1 \wedge J_1 + \eta P_2 \wedge J_2 - \eta^2 K_1 \wedge J_2 + \eta^2 K_2 \wedge J_1), \\
 \delta(K_1) &= \frac{1}{\kappa} (K_1 \wedge P_0 + P_2 \wedge J_3 - P_3 \wedge J_2 - \eta K_3 \wedge J_1), \\
 \delta(K_2) &= \frac{1}{\kappa} (K_2 \wedge P_0 - P_1 \wedge J_3 + P_3 \wedge J_1 - \eta K_3 \wedge J_2), \\
 \delta(K_3) &= \frac{1}{\kappa} (K_3 \wedge P_0 + P_1 \wedge J_2 - P_2 \wedge J_1 + \eta K_1 \wedge J_1 + \eta K_2 \wedge J_2).
 \end{aligned} \tag{27}$$

When comparing these expressions with those that hold for  $\kappa$ -Poincaré (which are recovered in the  $\eta \rightarrow 0$  limit), several features of the new model arise, which are not present in the  $\kappa$ -Poincaré nor in the classical (A)dS limit, thus being due genuinely to the interplay between the two deformations. The most striking feature is that the  $so(3)$  subalgebra generated by rotations  $J_a$  is deformed, with a deformation governed by the ratio  $\eta/\kappa$ . Therefore, the deformation of space isotropy has to be expected as a direct consequence of the interplay between curvature and quantum effects. Moreover, the cocommutator for the translations sector does no longer define a sub-Lie bialgebra structure, and involves the Lorentz sector. Related to this, the expressions for  $\delta(P_i)$  and  $\delta(K_i)$  can be interchanged under the automorphism (24). Therefore, deformed space translations and boosts are expected to play similar algebraic roles within the  $\kappa$ -(A)dS model.

We recall that the cocommutator (27) provides the first order in the quantum deformation. In [23], by making use of a Poisson version of the so-called “quantum duality principle” presented in [117], full expressions for the (Poisson) analogue of the full  $\kappa$ -(A)dS model were explicitly obtained. Here, we recall only some of them in order to illustrate the previous remarks. In particular, the rotations sector is deformed into a quantum  $so(3)$  algebra with a deformation parameter given by  $\eta/\kappa = \sqrt{-\Lambda}/\kappa$ :

$$\begin{aligned}
 \Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\
 \Delta(J_1) &= J_1 \otimes e^{\frac{\eta}{\kappa} J_3} + 1 \otimes J_1, & \Delta(J_2) &= J_2 \otimes e^{\frac{\eta}{\kappa} J_3} + 1 \otimes J_2,
 \end{aligned} \tag{28}$$

and whose deformed brackets read:

$$\{J_1, J_2\} = \frac{e^{2\frac{\eta}{\kappa} J_3} - 1}{2\eta/\kappa} - \frac{\eta}{2\kappa} (J_1^2 + J_2^2), \quad \{J_1, J_3\} = -J_2, \quad \{J_2, J_3\} = J_1. \tag{29}$$

The coproduct for the translations sector, that in principle would provide the deformed composition law for momenta in the corresponding DSR model, as seen for the  $\kappa$ -Poincaré case in the previous section, reads:

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta(P_1) &= P_1 \otimes \cosh(\eta J_3 / \kappa) + e^{-P_0 / \kappa} \otimes P_1 - \eta K_2 \otimes \sinh(\eta J_3 / \kappa) \\
&\quad - \frac{\eta}{\kappa} P_3 \otimes J_1 + \frac{\eta^2}{\kappa} K_3 \otimes J_2 + \frac{\eta^2}{\kappa^2} (\eta K_1 - P_2) \otimes J_1 J_2 e^{-\frac{\eta}{\kappa} J_3} \\
&\quad - \frac{\eta^2}{\kappa^2} (\eta K_2 + P_1) \otimes (J_1^2 - J_2^2) e^{-\frac{\eta}{\kappa} J_3}, \\
\Delta(P_2) &= P_2 \otimes \cosh(\eta J_3 / \kappa) + e^{-P_0 / \kappa} \otimes P_2 + \eta K_1 \otimes \sinh(\eta J_3 / \kappa) \\
&\quad - \frac{\eta}{\kappa} P_3 \otimes J_2 - \frac{\eta^2}{\kappa} K_3 \otimes J_1 - \frac{\eta^2}{\kappa^2} (\eta K_2 + P_1) \otimes J_1 J_2 e^{-\frac{\eta}{\kappa} J_3} \\
&\quad - \frac{1}{2} \frac{\eta^2}{\kappa^2} (\eta K_1 - P_2) \otimes (J_1^2 - J_2^2) e^{-\frac{\eta}{\kappa} J_3}, \\
\Delta(P_3) &= P_3 \otimes 1 + e^{-P_0 / \kappa} \otimes P_3 + \frac{1}{\kappa} (\eta^2 K_2 + \eta P_1) \otimes J_1 e^{-\frac{\eta}{\kappa} J_3} \\
&\quad - \frac{1}{\kappa} (\eta^2 K_1 - \eta P_2) \otimes J_2 e^{-\frac{\eta}{\kappa} J_3}.
\end{aligned} \tag{30}$$

As we anticipated from the Lie bialgebra structure, the deformed composition law for momenta involves the full Lorentz sector, which indicates that the construction of the associated momentum needs to include the Lorentz sector as well [116,118]. Moreover, the corresponding deformed brackets show that momenta are both non-commuting (due to  $\eta \neq 0$ ) and quantum deformed:

$$\begin{aligned}
\{P_1, P_2\} &= -\eta^2 \frac{\sinh(2\frac{\eta}{\kappa} J_3)}{2\eta / \kappa} - \frac{\eta}{2\kappa} (2P_3^2 + \eta^2 (J_1^2 + J_2^2)) - \frac{\eta^5}{4\kappa^3} e^{-2\frac{\eta}{\kappa} J_3} (J_1^2 + J_2^2)^2, \\
\{P_1, P_3\} &= \frac{1}{2} \eta^2 J_2 \left( 1 + e^{-2\frac{\eta}{\kappa} J_3} \left[ 1 + \frac{\eta^2}{\kappa^2} (J_1^2 + J_2^2) \right] \right) + \frac{\eta}{\kappa} P_2 P_3, \\
\{P_2, P_3\} &= -\frac{1}{2} \eta^2 J_1 \left( 1 + e^{-2\frac{\eta}{\kappa} J_3} \left[ 1 + \frac{\eta^2}{\kappa^2} (J_1^2 + J_2^2) \right] \right) - \frac{\eta}{\kappa} P_1 P_3.
\end{aligned} \tag{31}$$

Note also here the complicated interplay between curvature and quantum effects arising in the quantum deformation, which is expressed through products of different powers of  $1/\kappa$  and of the cosmological constant parameter  $\eta$ . Nevertheless, we stress that we have an all-order model at hand, with which all types of DSR predictions could be, in principle, computed.

Finally, we recall the (Poisson) counterpart of the second-order Casimir:

$$\begin{aligned}
C &= 2\kappa^2 \left[ \cosh(P_0 / \kappa) \cosh\left(\frac{\eta}{\kappa} J_3\right) - 1 \right] + \eta^2 \cosh(P_0 / \kappa) (J_1^2 + J_2^2) e^{-\frac{\eta}{\kappa} J_3} \\
&\quad - e^{P_0 / \kappa} (\mathbf{P}^2 + \eta^2 \mathbf{K}^2) \left[ \cosh\left(\frac{\eta}{\kappa} J_3\right) + \frac{\eta^2}{2\kappa^2} (J_1^2 + J_2^2) e^{-\frac{\eta}{\kappa} J_3} \right] \\
&\quad + 2\eta^2 e^{P_0 / \kappa} \left[ \frac{\sinh\left(\frac{\eta}{\kappa} J_3\right)}{\eta} \mathcal{R}_3 + \frac{1}{\kappa} (J_1 \mathcal{R}_1 + J_2 \mathcal{R}_2 + \frac{\eta}{2\kappa} (J_1^2 + J_2^2) \mathcal{R}_3) e^{-\frac{\eta}{\kappa} J_3} \right],
\end{aligned} \tag{32}$$

where  $\mathcal{R}_a = \epsilon_{abc} K_b P_c$ . As expected, in the  $\kappa \rightarrow \infty$  limit we obtain (21) and in the  $\eta \rightarrow 0$  limit, we obtain the  $\kappa$ -Poincaré quantum Casimir in the bicrossproduct basis (12). The Poisson version of the  $\kappa$ -(A)dS analogue of the Pauli-Lubanski fourth order Casimir (22) was also presented in [23], and the study of the representation theory of the  $\kappa$ -(A)dS Hopf algebra is still an open problem.

#### 4. Interplay between Curvature and the Speed of Light

So far, the speed of light parameter has been set to  $c = 1$ . Therefore, in order to unveil the coupling between  $\Lambda$  and  $c$ , the latter parameter has to be explicitly included in the formalism. At the classical level, it is well-known [26,27,29] that this gives rise to two

possible limits: the so-called “non-relativistic” or “Galilean” limit  $c \rightarrow \infty$  and the “ultra-relativistic” or “Carrollian” limit  $c \rightarrow 0$ . A complete study of the metrics and foliations for classical Galilei and Carroll spaces (also in the curved cases with  $\Lambda \neq 0$ ) can be found in the literature (see, for instance, [119] and references therein).

#### 4.1. The Galilean Limit of (A)dS

The Galilean limit corresponds to taking small velocities compared to the speed of light. In this limit, the light-cone opens along  $t = 0$ , producing a space with absolute time.

The interplay between the contraction procedure and curvature can be studied by looking at the contraction of the (A)dS spacetime and its algebra of symmetries. This is obtained via an Inönü–Wigner contraction procedure, induced by the algebra automorphism  $\mathcal{P}(P_0, P_a, K_a, J_a) = (P_0, -P_a, -K_a, J_a)$  (speed-space contraction), see for example [119]. Upon the rescaling

$$P_a \rightarrow \frac{P_a}{c}, \quad K_a \rightarrow \frac{K_a}{c}, \quad (33)$$

one finds that, when  $c \rightarrow \infty$  the following commutators of the (A)dS algebra are modified:

$$\begin{aligned} [K_a, P_b] &= \delta_{ab} \frac{1}{c^2} P_0 & \rightarrow [K_a, P_b] &= 0 \\ [K_a, K_b] &= -\epsilon_{abc} \frac{1}{c^2} J_c & \rightarrow [K_a, K_b] &= 0 \\ [P_a, P_b] &= \Lambda \epsilon_{abc} \frac{1}{c^2} J_c & \rightarrow [P_a, P_b] &= 0, \end{aligned} \quad (34)$$

and the Casimir reduces to:

$$\mathcal{C} = \mathbf{P}^2 - \Lambda \mathbf{K}^2. \quad (35)$$

We note that the presence of curvature does not affect the appearance of an absolute space in the Galilean limit, since the commutator between boosts and time translation vanishes. However, while in the flat  $\Lambda \rightarrow 0$  limit the translation sector is unaffected by the Galilei contraction, when curvature is present one still obtains ‘flat’ spatial slices in the Galilei limit, since the commutator between spatial translations vanishes (see [119] for details).

#### 4.2. The Carroll Limit of (A)dS

The Carroll limit corresponds to taking large space intervals. It is relevant in the strong gravity regime and close to the black hole horizon (see [120] and references therein). In contrast to the Galilean limit, in this case, the light-cone closes along the  $t$  direction.

As done in the Galilean case, we look at the contraction of the (A)dS spacetime and its algebra of symmetries. This is obtained via an Inönü–Wigner contraction procedure, induced by the algebra automorphism  $\mathcal{T}(P_0, P_a, K_a, J_a) = (-P_0, P_a, -K_a, J_a)$  (speed-time contraction), see for example [119]. Upon the rescaling:

$$P_0 \rightarrow cP_0, \quad K_a \rightarrow cK_a, \quad (36)$$

one finds that when  $c \rightarrow 0$  the following commutators of the (A)dS algebra are modified:

$$\begin{aligned} [K_a, K_b] &= -\epsilon_{abc} c^2 J_c & \rightarrow [K_a, K_b] &= 0 \\ [K_a, P_0] &= c^2 P_a & \rightarrow [K_a, P_0] &= 0, \end{aligned} \quad (37)$$

and the Casimir reduces to:

$$\mathcal{C} = P_0^2 + \Lambda \mathbf{K}^2. \quad (38)$$

Similar to the Galilean case, the most relevant feature of the Carrollian relativity, namely that of having an absolute time, is preserved in the presence of curvature. Moreover, the noncommutativity of translations, caused by spacetime curvature, is not affected in the Carrollian limit, as opposed to what happens in the Galilean case. A summary of

the different effects that the non-relativistic and the ultra-relativistic limits have on the symmetries of a given spacetime, with and without curvature, is presented in Table 1.

**Table 1.** Table with the summary of the interplay between curvature and the speed of light parameter as seen in the (A)dS algebra and its Galilean and Carrollian limits. Horizontal lines indicate that the commutator is the same for the three cases.

	Galilean Limit	(A)dS	Carrollian Limit
$[J_a, J_b]$	—————	$\epsilon_{abc}J_c$	—————
$[J_a, P_b]$	—————	$\epsilon_{abc}P_c$	—————
$[J_a, K_b]$	—————	$\epsilon_{abc}K_c$	—————
$[J_a, P_0]$	—————	0	—————
$[K_a, K_b]$	0	$-\epsilon_{abc}J_c$	0
$[K_a, P_b]$	0	$\delta_{ab}P_0$	$\delta_{ab}P_0$
$[K_a, P_0]$	$P_a$	$P_a$	0
$[P_a, P_b]$	0	$\Lambda\epsilon_{abc}J_c$	$\Lambda\epsilon_{abc}J_c$
$[P_a, P_0]$	$\Lambda K_a$	$\Lambda K_a$	$\Lambda K_a$

## 5. Interplay of the Three Parameters: Curvature, Speed of Light and Quantum Deformation

### 5.1. Zero Curvature Case: Galilei and Carroll Contraction of $\kappa$ -Poincaré

In order to study the Galilei and Carroll limits of the  $\kappa$ -Poincaré algebra, we would like to perform a contraction similar to the one used in the non-quantum case of the previous section. However, as was discussed in detail in [30], the contraction procedure of a quantum algebra (Lie bialgebra contraction) might require a rescaling of the quantum deformation parameter along with the generators in order to obtain meaningful structures.

In general, one can perform two kinds of contractions, either working at the level of the r-matrix (this is a “coboundary” contraction), or working directly at the level of the co-commutators (this is the so-called “fundamental” contraction) [38,48]. As was shown in [30], this distinction is especially relevant in the case of the Galilean limit of  $\kappa$ -Poincaré, where the two procedures are nonequivalent. In fact, after the rescaling (33), the  $\kappa$ -Poincaré r-matrix (17) reads:

$$r = \frac{c^2}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3). \quad (39)$$

This is well-behaved in the  $c \rightarrow \infty$  limit if also the quantum parameter is rescaled as  $\kappa \rightarrow \kappa/c^2$ . However, the resulting r-matrix,

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3), \quad (40)$$

produces trivial cocommutators,  $\delta(X) = 0$ , for all generators  $X$  of the algebra. So the coboundary contraction of the  $\kappa$ -Poincaré algebra produces the classical Galilei algebra. On the other hand, working directly at the level of the cocommutators (16), one can easily see that they are invariant under the rescaling (33), so that the  $c \rightarrow \infty$  limit is well-defined without need to rescale the quantum deformation parameter. The resulting  $\kappa$ -Galilei algebra contains the following modified commutators with respect to the  $\kappa$ -Poincaré algebra, which corresponds to the left column, and in which the automorphism (33) has been applied:

$$\begin{aligned} [K_a, P_b] &= \frac{\delta_{ab}}{c^2} \left[ \frac{\kappa}{2} \left( 1 - e^{-2P_0/\kappa} \right) + c^2 \frac{\bar{P}^2}{2\kappa} \right] - \frac{P_a P_b}{\kappa} &\rightarrow [K_a, P_b] &= \delta_{ab} \frac{\bar{P}^2}{2\kappa} - \frac{P_a P_b}{\kappa} \\ [K_a, K_b] &= -\frac{\epsilon_{abc}}{c^2} J_c && \rightarrow [K_a, K_b] = 0, \end{aligned} \quad (41)$$

while the coproducts are unmodified with respect to the  $\kappa$ -Poincaré case.

When performing the Carrollian limit of the  $\kappa$ -Poincaré algebra, one finds that the two procedures outlined above give equivalent results. The rescaled r-matrix reads:

$$r = \frac{1}{c\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3), \tag{42}$$

which is well-behaved in the  $c \rightarrow 0$  limit if the quantum deformation parameter is rescaled as  $\kappa \rightarrow c\kappa$ . Then the r-matrix reads:

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3), \tag{43}$$

and produces non-trivial co-commutators:

$$\begin{aligned} \delta(P_0) &= \delta(J_a) = 0, \\ \delta(P_a) &= \frac{1}{\kappa}P_a \wedge P_0, \\ \delta(K_a) &= \frac{1}{\kappa}K_a \wedge P_0. \end{aligned} \tag{44}$$

The resulting  $\kappa$ -Carroll algebra contains the following modified commutators with respect to the  $\kappa$ -Poincaré algebra:

$$\begin{aligned} [K_a, P_0] &= P_a c^2 && \rightarrow [K_a, P_0] = 0 \\ [K_a, P_b] &= c\delta_{ab} \left[ \frac{\kappa}{2c} \left( 1 - e^{-2P_0/\kappa} \right) + \frac{\vec{P}^2 c}{2\kappa} \right] - c \frac{P_a P_b}{\kappa} && \rightarrow [K_a, P_b] = \delta_{ab} \frac{\kappa}{2} \left( 1 - e^{-2P_0/\kappa} \right) \\ [K_a, K_b] &= -c^2 \epsilon_{abc} J_c && \rightarrow [K_a, K_b] = 0, \end{aligned} \tag{45}$$

while, again, the coproducts are unmodified.

We see that, in both the Galilean and Carrollian limits the commutator between boosts generators vanishes, as in the classical case. A relevant difference between the two limits is that, while in the Carrollian limit the presence of the quantum deformation does not spoil the appearance of an absolute space (signaled by the vanishing of the commutator between boosts and time translations), in the Galilean limit the mixing between time and space induced by the quantum deformation prevents the emergence of an absolute time, since the commutators between boosts and spatial translations remain non-vanishing in the transition from the  $\kappa$ -Poincaré to the  $\kappa$ -Galilei symmetries. These properties are summarized in Table 2.

**Table 2.** Summary of the properties of the  $\kappa$ -Galilei,  $\kappa$ -Poincaré and  $\kappa$ -Carroll algebras.

	$\kappa$ -Galilei	$\kappa$ -Poincaré	$\kappa$ -Carroll
$[J_a, J_b]$	—————	$\epsilon_{abc} J_c$	—————
$[J_a, P_b]$	—————	$\epsilon_{abc} P_c$	—————
$[J_a, K_b]$	—————	$\epsilon_{abc} K_c$	—————
$[J_a, P_0]$	—————	0	—————
$[K_a, K_b]$	0	$-\epsilon_{abc} J_c$	0
$[K_a, P_b]$	$\frac{\delta_{ab}}{2\kappa} \vec{P}^2 - \frac{P_a P_b}{\kappa}$	$\delta_{ab} \left[ \frac{\kappa}{2} \left( 1 - e^{-\frac{2}{\kappa} P_0} \right) + \frac{\vec{P}^2}{2\kappa} \right] - \frac{P_a P_b}{\kappa}$	$\delta_{ab} \frac{\kappa}{2} \left( 1 - e^{-2P_0/\kappa} \right)$
$[K_a, P_0]$	$P_a$	$P_a$	0
$[P_a, P_b]$	—————	0	—————
$[P_a, P_0]$	—————	0	—————

### 5.2. With Curvature: Galilei and Carroll Contraction of $\kappa$ -(A)dS

Here, we study the interplay of all of the three parameters that govern different kinds of deformations of special relativity: the speed of light, the cosmological constant and the quantum deformation parameter.

In order to do so, we look at the Galilean and Carrollian contraction of the  $\kappa$ -(A)dS algebra. This is done by following the same procedure discussed in the previous subsection for the  $\Lambda = 0$  case. The detailed formulas can be found in [30] and are schematically represented in Table 3. Here, we discuss the points that are particularly relevant. We noticed in Section 3 that an important effect of the interplay between curvature and quantum deformation is that the rotation sector gets deformed. The Galilean contraction does not spoil this feature, while the Carrollian contraction restores standard isotropy. As already observed in the  $\Lambda = 0$  case, the mixing between time and space due to the quantum deformation prevents the emergence of an absolute time in the Galilean limit, and the presence of curvature does not affect this result. Finally, one can see effects that are only relevant when all of the three parameters enter in the deformation of the Poincaré algebra: in the Galilean limit, when the curvature is non-zero, the commutator between boosts does not vanish, and is proportional to  $\sqrt{\Lambda}/\kappa$ . In general, the Carrollian limit seems to be a “milder” deformation, since it is isotropic, preserve the absoluteness of space and the vanishing commutators between boosts.

**Table 3.** Summary of the properties of curved  $\kappa$ -Galilei,  $\kappa$ -(A)dS and curved  $\kappa$ -Carroll.

	(Curved) $\kappa$ -Galilei	$\kappa$ -(A)dS	(Curved) $\kappa$ -Carroll
$[J_a, J_b]$			
$[J_a, P_b]$	anisotropy $\sim \frac{\Lambda}{\kappa}$	anisotropy $\sim \frac{\Lambda}{\kappa}$	isotropy
$[J_a, K_b]$			
$[J_a, P_0]$	-----	0	-----
$[K_a, K_b]$	$O(\frac{\sqrt{\Lambda}}{\kappa})$	$-\epsilon_{abc}J_c + O(\frac{\sqrt{\Lambda}}{\kappa})$	0
$[K_a, P_b]$	$O(\frac{\sqrt{\Lambda}}{\kappa}, \frac{1}{\kappa})$	$\delta_{ab}P_0 + O(\frac{\Lambda}{\kappa}, \frac{1}{\kappa})$	$\delta_{ab}P_0 + O(\frac{\Lambda}{\kappa}, \frac{1}{\kappa})$
$[K_a, P_0]$	$P_a$	$P_a$	0
$[P_a, P_b]$	$O(\frac{\sqrt{\Lambda}}{\kappa})$	$\Lambda\epsilon_{abc}J_c + O(\frac{\sqrt{\Lambda}}{\kappa})$	$\Lambda\epsilon_{abc}J_c$
$[P_a, P_0]$	$\Lambda K_a$	$\Lambda K_a$	$\Lambda K_a$

## 6. Noncommutative Spacetimes

Besides looking at the properties of the algebra of quantum-deformed relativistic symmetries, it is also instructive to study the properties of the associated noncommutative spacetimes, in which the interplay previously analyzed can be also illustrated. Moreover, since the Poincaré, (A)dS, Galilei and Carroll classical spacetimes are homogeneous spaces of the corresponding kinematical groups, their noncommutative analogues can be constructed as quantum homogeneous spaces of the corresponding quantum groups, although their construction procedure is rather involved from the computational viewpoint (see, for instance, [121,122]). Nevertheless, the noncommutative algebra defining a given quantum homogeneous space is just the quantization of the Poisson homogeneous space that is associated to the  $r$ -matrix defining the first-order of the quantum kinematical algebra. This Poisson homogeneous space is just the classical homogeneous space endowed with a Poisson algebra structure which can be explicitly obtained as a canonical projection of the Sklyanin Poisson bracket that is derived from the  $r$ -matrix, provided that the so called coisotropy condition holds (see [123]). In the following, we will present the explicit

expressions for the Poisson-noncommutative spacetimes corresponding to the quantum deformations presented in the previous sections. All technical aspects of this construction as well as appropriate references can be found in [25,30].

We mentioned when introducing the classical homogeneous spacetimes that their definition requires us to identify the spacetime coordinates as the group parameters of the spacetime translations  $P_\alpha$ . As we have seen in the previous section (see also Table 3), the algebra of translation generators is especially sensitive to the presence of curvature (both with and without quantum deformation). For this reason, we expect that the same happens to spacetime noncommutativity, and indeed this is the case as shown below.

### 6.1. The $\kappa$ -(A)dS Spacetime

By computing the Sklyanin bracket for the  $\kappa$ -(A)dS  $r$ -matrix (26) we obtain the semi-classical version of the  $\kappa$ -(A)dS spacetime in terms of the Poisson brackets:

$$\begin{aligned} \{x^0, x^1\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^1)}{\eta \cosh^2(\eta x^2) \cosh^2(\eta x^3)}, \\ \{x^0, x^2\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^2)}{\eta \cosh^2(\eta x^3)}, \\ \{x^0, x^3\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^3)}{\eta}, \\ \{x^1, x^2\} &= -\frac{1}{\kappa} \frac{\cosh(\eta x^1) \tanh^2(\eta x^3)}{\eta}, \\ \{x^1, x^3\} &= \frac{1}{\kappa} \frac{\cosh(\eta x^1) \tanh(\eta x^2) \tanh(\eta x^3)}{\eta}, \\ \{x^2, x^3\} &= -\frac{1}{\kappa} \frac{\sinh(\eta x^1) \tanh(\eta x^3)}{\eta}, \end{aligned} \quad (46)$$

where we defined  $\eta^2 = -\Lambda$  so that the zero-curvature limit, giving the  $\kappa$ -Minkowski Poisson homogeneous space (whose quantization is the  $\kappa$ -Minkowski noncommutative spacetime (19)), is given by the  $\eta \rightarrow 0$  limit of (46), namely:

$$\{x^0, x^a\} = -\frac{1}{\kappa} x^a, \quad \{x^a, x^b\} = 0, \quad (47)$$

and in this flat case space translations do commute among themselves. Indeed, if we take the first-order expansion in terms of  $\eta$  we get:

$$\begin{aligned} \{x^0, x^1\} &= -\frac{1}{\kappa} (x^1 + o[\eta^2]), \\ \{x^0, x^2\} &= -\frac{1}{\kappa} (x^2 + o[\eta^2]), \\ \{x^0, x^3\} &= -\frac{1}{\kappa} (x^3 + o[\eta^2]), \\ \{x^1, x^2\} &= -\frac{1}{\kappa} (\eta (x^3)^2 + o[\eta^2]), \\ \{x^1, x^3\} &= \frac{1}{\kappa} (\eta x^2 x^3 + o[\eta^2]), \\ \{x^2, x^3\} &= -\frac{1}{\kappa} (\eta x^1 x^3 + o[\eta^2]). \end{aligned} \quad (48)$$

Notice that curvature has a more prominent role in the space–space brackets, where it contributes at the order  $\frac{\sqrt{\Lambda}}{\kappa}$ , while for the time–space brackets curvature only contributes starting from the  $\frac{\Lambda}{\kappa}$  order. This behavior is similar (but not completely equal) to the properties of the algebra of translation generators, schematically described in Table 3. In fact,

the quantum-curvature effects in the commutators between space–space generators are governed by  $O(\frac{\sqrt{\Lambda}}{\kappa})$  (similar to what happens to the brackets between spatial coordinates), while for time–space commutators one has no contributions at all from quantum effects (for the time–space coordinates there is a contribution, even though it is milder than in the space–space case). The quantization of the  $\kappa$ (A)dS Poisson homogeneous spacetime was fully given in [25] by choosing a precise ordering of the generators, but the interplay between  $\Lambda$  and  $\kappa$  here presented is not modified after quantization. We recall that other noncommutative (A)dS spacetimes arising from different noncommutative geometry approaches can be found in [124–127].

### 6.2. $\kappa$ -Galilean and $\kappa$ -Carrollian Spacetimes

The Galilean and Carrollian limits of the  $\kappa$ -(A)dS spacetime (46) are obtained by appropriately rescaling spacetime coordinates to so keep the products  $x^0 P_0$  and  $x^a P_a$  invariant under contraction (see [128] for the theory of contractions of Poisson-Lie groups and [30], where these two limits have been performed on the Snyder noncommutative spacetime [129]).

Specifically, the Galilean limit is obtained by rescaling:

$$x^a \rightarrow c x^a, \quad (49)$$

and then taking the  $c \rightarrow \infty$  limit of (46). This produces a spacetime algebra which has the same commutation rules as  $\kappa$ -Minkowski for the space–time sector, and shows the residual anisotropy discussed above in Section 5.2 in the space sector:

$$\{x^a, x^0\} = \frac{1}{\kappa} x^a, \quad \{x^1, x^2\} = -\frac{\eta}{\kappa} (x^3)^2, \quad \{x^1, x^3\} = \frac{\eta}{\kappa} x^2 x^3, \quad \{x^2, x^3\} = -\frac{\eta}{\kappa} x^1 x^3. \quad (50)$$

Symplectic leaves for the space sector are just 3-spheres

$$S = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad (51)$$

which reflects the role of the deformed SO(3) sector (28) in both  $\kappa$ -(A)dS and curved (Newton-Hooke)  $\kappa$ -Galilean algebras and spaces.

The Carrollian limit is obtained as the limit  $c \rightarrow 0$  of (46), after the following rescaling is performed (notice that, as done for the algebra of symmetries, the quantum deformation parameters has to be also rescaled):

$$x^0 \rightarrow x^0/c, \quad \kappa \rightarrow c\kappa. \quad (52)$$

In this case, the space–time part of the algebra is not affected by the contraction, and remains equal to the one of  $\kappa$ -(A)dS. The most important effect of the contraction is the restoration of isotropy at the level of spatial coordinates, consistently with what found in Section 5.2 at the level of the algebra of symmetries:

$$\begin{aligned} \{x^1, x^0\} &= \frac{1}{\kappa} \frac{\tanh(\eta x^1)}{\eta \cosh^2(\eta x^2) \cosh^2(\eta x^3)}, \\ \{x^2, x^0\} &= \frac{1}{\kappa} \frac{\tanh(\eta x^2)}{\eta \cosh^2(\eta x^3)}, \\ \{x^3, x^0\} &= \frac{1}{\kappa} \frac{\tanh(\eta x^3)}{\eta}, \\ \{x^a, x^b\} &= 0. \end{aligned} \quad (53)$$

When the flat  $\Lambda \rightarrow 0$  limit is taken, in both cases one recovers the same  $\kappa$ -Minkowski Poisson algebra (47). In particular, as seen for the associated algebra of symmetries in Section 5.2, isotropy is restored also in the Galilean case. As it can be described in [30],

the quantization of all these Galilean and Carrollian Poisson homogeneous spacetimes can be fully performed by mimicking the quantization procedure used in the  $\kappa$ -(A)dS case. In particular, in the curved Galilean case the “quantum spheres”:

$$\hat{S}_{\eta/\kappa} = (\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 + \frac{\eta}{\kappa} \hat{x}^1 \hat{x}^2, \quad [\hat{S}_{\eta/\kappa}, \hat{x}^a] = 0, \quad (54)$$

are obtained as the quantization of the symplectic leaves (51), where the term depending on  $\eta/\kappa$  arises from the noncommutativity of the quantum space coordinates  $\hat{x}^a$ .

## 7. Concluding Remarks

There exist two more frameworks in which the results here presented for each of the quantum kinematical algebras and their associated noncommutative spacetimes can be rephrased.

Firstly, all the models here presented could be analyzed in terms of the associated curved momentum spaces. These are pseudo-Riemannian manifolds that can be obtained as orbits of suitable actions of the dual Poisson-Lie group associated to the  $\kappa$ -deformation. In the case of  $\kappa$ -Poincaré, as was first shown in [49], the geometry one finds is that of one half of de Sitter space. This analysis can be generalized to more general  $\kappa$ -deformations of the  $ISO(p, q)$  group and its Carrollian contractions, in which the “deformed” direction is not necessarily the “time” one (the zeroth coordinate). The result is a collection of 4-dimensional momentum spaces which always have the geometry of a homogeneous space (dS, AdS or Minkowski) and, in some cases, cover only half of said geometries, in other cases cover a whole sheet (as in the Euclidean case  $ISO_{\kappa}(4)$  [112]).

In the case of  $\kappa$ -(A)dS, the Lie bialgebra (27) dualizes to a Lie algebra which admits a 7-dimensional solvable Lie subalgebra that includes the duals of the translation and boost generators. Therefore the smallest generalization of momentum space is 7-dimensional, and includes three additional coordinates associated to “hyperbolic angular momentum” [116]. The geometry of these momentum spaces is half of the  $(6 + 1)$ -dimensional de Sitter space in the case of  $\kappa$ -dS, and half of a space with  $SO(4, 4)$  invariance for  $\kappa$ -(A)dS. The Galilean and Carrollian limits of these momentum spaces have not been studied yet, and are worth further investigation.

Secondly, an alternative viewpoint is provided by the construction of the corresponding noncommutative spaces of worldlines associated to all the  $\kappa$ -deformations here presented. In particular, for the (A)dS and Poincaré cases, the spaces of time-like worldlines are obtained as homogeneous spaces of cosets of the corresponding Lie group with respect to the 4D isotropy subgroup of the worldline of a particle located at the origin of the spacetime and having zero velocity, which is generated by the subalgebra of symmetries given by  $\mathfrak{h} = \{J_1, J_2, J_3, P_0\}$  (see [114] and references therein).

In the Poincaré case, the classical 6D space of time-like worldlines  $\mathcal{W}$  has been explicitly constructed, and has been endowed with a Poisson homogeneous structure associated to the  $\kappa$ -Poincaré  $r$ -matrix (17). As was shown in [114], this structure provides a Poisson algebra on the space of worldlines coordinates, that can be quantized, giving rise to the quantum space of worldlines associated to the  $\kappa$ -Poincaré symmetry. This noncommutative space of time-like worldlines provides an alternative (and physically sound) framework for the description of the spacetime fuzziness encoded in quantum deformations [105]. The construction of the noncommutative spaces of worldlines associated to the  $\kappa$ -(A)dS,  $\kappa$ -Galilean and  $\kappa$ -Carrollian algebras can be attempted by adopting a similar approach, thus providing a complementary perspective for the analysis of the interplay between the quantum deformation parameter  $\kappa$ , the curvature parameter  $\Lambda$  and the speed of light  $c$ .

**Author Contributions:** All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work has been partially supported by Ministerio de Ciencia e Innovación (Spain) under grants MTM2016-79639-P (AEI/FEDER, UE) and PID2019-106802GB-I00/AEI/10.13039/

501100011033, by Junta de Castilla y León (Spain) under grants BU229P18 and BU091G19, as well as by the Action CA18108 QG-MM from the European Cooperation in Science and Technology (COST).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

- Amelino-Camelia, G.; Freidel, L.; Kowalski-Glikman, J.; Smolin, L. The principle of relative locality. *Phys. Rev. D* **2011**, *84*, 084010. [[CrossRef](#)]
- Amelino-Camelia, G.; Freidel, L.; Kowalski-Glikman, J.; Smolin, L. Relative locality: A deepening of the relativity principle. *Gen. Rel. Grav.* **2011**, *43*, 2547–2553. [[CrossRef](#)]
- Lukierski, J.; Ruegg, H.; Nowicki, A.; Tolstoy, V.N.  $q$ -deformation of Poincaré algebra. *Phys. Lett. B* **1991**, *264*, 331–338. [[CrossRef](#)]
- Giller, S.; Kosinski, P.; Majewski, M.; Maslanka, P.; Kunz, J. More about the  $q$ -deformed Poincaré algebra. *Phys. Lett. B* **1992**, *286*, 57–62. [[CrossRef](#)]
- Lukierski, J.; Nowicki, A.; Ruegg, H. New quantum Poincaré algebra and  $\kappa$ -deformed field theory. *Phys. Lett. B* **1992**, *293*, 344–352. [[CrossRef](#)]
- Maslanka, P. The  $n$ -dimensional  $\kappa$ -Poincaré algebra and group. *J. Phys. A Math. Gen.* **1993**, *26*, L1251–L1253. [[CrossRef](#)]
- Zakrzewski, S. Quantum Poincaré group related to the  $\kappa$ -Poincaré algebra. *J. Phys. A Math. Gen.* **1994**, *27*, 2075–2082. [[CrossRef](#)]
- Majid, S.; Ruegg, H. Bicrossproduct structure of  $\kappa$ -Poincaré group and non-commutative geometry. *Phys. Lett. B* **1994**, *334*, 348–354. [[CrossRef](#)]
- Lukierski, J.; Ruegg, H. Quantum  $kappa$ -Poincaré in any dimension. *Phys. Lett.* **1994**, *B329*, 189–194. [[CrossRef](#)]
- Lukierski, J. Kappa-Deformations: Historical Developments and Recent Results. *J. Phys. Conf. Ser.* **2017**, *804*, 012028. [[CrossRef](#)]
- Majid, S. Hopf algebras for physics at the Planck scale. *Class. Quantum Gravity* **1988**, *5*, 1587–1606. [[CrossRef](#)]
- Amelino-Camelia, G.; Smolin, L.; Starodubtsev, A. Quantum symmetry, the cosmological constant and Planck-scale phenomenology. *Class. Quantum Gravity* **2004**, *21*, 3095–3110. [[CrossRef](#)]
- Marciano, A.; Amelino-Camelia, G.; Bruno, N.R.; Gubitosi, G.; Mandanici, G.; Melchiorri, A. Interplay between curvature and Planck-scale effects in astrophysics and cosmology. *J. Cosmol. Astropart. Phys.* **2010**, *6*, 30. [[CrossRef](#)]
- Bianchi, E.; Rovelli, C. A Note on the geometrical interpretation of quantum groups and non-commutative spaces in gravity. *Phys. Rev. D* **2011**, *84*, 027502. [[CrossRef](#)]
- Amelino-Camelia, G.; Marciano, A.; Matassa, M.; Rosati, G. Deformed Lorentz symmetry and relative locality in a curved/expanding spacetime. *Phys. Rev. D* **2012**, *86*, 124035. [[CrossRef](#)]
- Rosati, G.; Amelino-Camelia, G.; Marciano, A.; Matassa, M. Planck-scale-modified dispersion relations in FRW spacetime. *Phys. Rev. D* **2015**, *92*, 124042. [[CrossRef](#)]
- Barcaroli, L.; Gubitosi, G. Kinematics of particles with quantum-de Sitter-inspired symmetries. *Phys. Rev. D* **2016**, *93*, 124063. [[CrossRef](#)]
- Amelino-Camelia, G.; Rosati, G.; Bedić, S. Phenomenology of curvature-induced quantum-gravity effects. *Phys. Lett. B* **2021**, *820*, 136595 [[CrossRef](#)]
- Herranz, F.J.; Santander, M. (Anti)de Sitter/Poincaré symmetries and representations from Poincaré/Galilei through a classical deformation approach. *J. Phys. A Math. Theor.* **2008**, *41*, 015204. [[CrossRef](#)]
- Ballesteros, A.; Herranz, F.J.; del Olmo, M.A.; Santander, M. Quantum (2 + 1) kinematical algebras: A global approach. *J. Phys. A Math. Gen.* **1994**, *27*, 1283–1297. [[CrossRef](#)]
- Ballesteros, A.; Bruno, N.R.; Herranz, F.J. A non-commutative Minkowskian spacetime from a quantum AdS algebra. *Phys. Lett. B* **2003**, *574*, 276–282. [[CrossRef](#)]
- Ballesteros, A.; Herranz, F.J.; Meusburger, C. A (2 + 1) non-commutative Drinfel'd double spacetime with cosmological constant. *Phys. Lett. B* **2014**, *732*, 201–209. [[CrossRef](#)]
- Ballesteros, A.; Herranz, F.J.; Musso, F.; Naranjo, P. The  $\kappa$ -(A)dS quantum algebra in (3 + 1) dimensions. *Phys. Lett. B* **2017**, *766*, 205–211. [[CrossRef](#)]
- Rosati, G.  $\kappa$ -de Sitter and  $\kappa$ -Poincaré symmetries emerging from Chern-Simons (2 + 1)D gravity with a cosmological constant. *Phys. Rev. D* **2017**, *96*, 066027. [[CrossRef](#)]
- Ballesteros, A.; Gutierrez-Sagredo, I.; Herranz, F.J. The  $\kappa$ -(A)dS noncommutative spacetime. *Phys. Lett. B* **2019**, *796*, 93–101. [[CrossRef](#)]
- Lévy-Leblond, J.M. Une nouvelle limite non-relativiste du group de Poincaré. *Ann. Inst. H. Poincaré* **1965**, *3*, 1–12.
- Bacry, H.; Lévy-Leblond, J.M. Possible Kinematics. *J. Math. Phys.* **1968**, *9*, 1605–1614. [[CrossRef](#)]
- Aldrovandi, R.; Barbosa, A.L.; Crispino, L.C.B.; Pereira, J.G. Non-relativistic spacetimes with cosmological constant. *Class. Quantum Gravity* **1999**, *16*, 495–506. [[CrossRef](#)]
- Duval, C.; Gibbons, G.W.; Horvathy, P.A.; Zhang, P.M. Carroll versus Newton and Galilei: Two dual non-Einsteinian concepts of time. *Class. Quantum Gravity* **2014**, *31*, 085016. [[CrossRef](#)]

30. Ballesteros, A.; Gubitosi, G.; Gutierrez-Sagredo, I.; Herranz, F.J. The  $\kappa$ -Newtonian and  $\kappa$ -Carrollian algebras and their noncommutative spacetimes. *Phys. Lett. B* **2020**, *805*, 135461. [[CrossRef](#)]
31. Tung, W.-K. *Group Theory in Physics*; World Scientific: Singapore, 1985.
32. Chari, V.; Pressley, A. *A guide to Quantum Groups*; Cambridge University Press: Cambridge, UK, 1994.
33. Majid, S. *Foundations of Quantum Group Theory*; Cambridge University Press: Cambridge, UK, 1995.
34. Zakrzewski, S. Poisson structures on the Poincaré group. *Commun. Math. Phys.* **1997**, *185*, 285–311. [[CrossRef](#)]
35. Podleś, P.; Woronowicz, S.L. On the classification of quantum Poincaré groups. *Commun. Math. Phys.* **1996**, *178*, 61–82. [[CrossRef](#)]
36. Celeghini, E.; Giachetti, R.; Sorace, E.; Tarlini, M. The quantum Heisenberg group  $H(1)_q$ . *J. Math. Phys.* **1991**, *32*, 1155–1158. [[CrossRef](#)]
37. Celeghini, E.; Giachetti, R.; Sorace, E.; Tarlini, M. Contractions of quantum groups. *Lect. Notes Math.* **1992**, *1510*, 221–244.
38. Ballesteros, A.; Gromov, N.A.; Herranz, F.J.; del Olmo, M.A.; Santander, M. Lie bialgebra contractions and quantum deformations of quasi-orthogonal algebras. *J. Math. Phys.* **1995**, *36*, 5916–5937. [[CrossRef](#)]
39. Drinfeld, V. Quantum groups. In *Proceedings of the International Congress of Mathematicians (Berkeley 1986)*; American Mathematical Society: Providence, RI, USA, 1987; pp. 798–820.
40. Jimbo, M. A q-difference analogue of  $U(\mathfrak{g})$  and the Yang-Baxter equation. *Lett. Math. Phys.* **1985**, *10*, 63–69. [[CrossRef](#)]
41. Kowalski-Glikman, J.; Nowak, S. Doubly special relativity theories as different bases of  $\kappa$ -Poincaré algebra. *Phys. Lett. B* **2002**, *539*, 126–132. [[CrossRef](#)]
42. Gubitosi, G.; Mercati, F. Relative locality in  $\kappa$ -Poincaré. *Class. Quantum Gravity* **2013**, *30*, 145002. [[CrossRef](#)]
43. Amelino-Camelia, G.; Arzano, M.; Kowalski-Glikman, J.; Rosati, G.; Trevisan, G. Relative-locality distant observers and the phenomenology of momentum-space geometry. *Class. Quantum Gravity* **2012**, *29*, 075007. [[CrossRef](#)]
44. Amelino-Camelia, G. Doubly-special relativity: Facts, myths and some key open issues. *Symmetry* **2010**, *2*, 230–271. [[CrossRef](#)]
45. Giller, S.; Gonera, C.; Kosinski, P.; Majewski, M.; Maslanka, P.; Kunz, J. On q covariant wave functions. *Mod. Phys. Lett. A* **1993**, *8*, 3785–3797. [[CrossRef](#)]
46. Nowicki, A.; Sorace, E.; Tarlini, M. The Quantum deformed Dirac equation from the kappa Poincare algebra. *Phys. Lett. B* **1993**, *302*, 419–422. [[CrossRef](#)]
47. Ballesteros, A.; Gutierrez-Sagredo, I.; Herranz, F.J. Noncommutative (A)dS and Minkowski spacetimes from quantum Lorentz subgroups. *arXiv* **2021**, arXiv:2108.02683.
48. Gutierrez-Sagredo, I.; Herranz, F.J. Cayley–Klein Lie Bialgebras: Noncommutative Spaces, Drinfel’d Doubles and Kinematical Applications. *Symmetry* **2021**, *13*, 1249. [[CrossRef](#)]
49. Kowalski-Glikman, J. de Sitter space as an arena for doubly special relativity. *Phys. Lett.* **2002**, *B547*, 291–296. [[CrossRef](#)]
50. Kowalski-Glikman, J.; Nowak, S. Doubly special relativity and de Sitter space. *Class. Quantum Gravity* **2003**, *20*, 4799–4816. [[CrossRef](#)]
51. Kowalski-Glikman, J. Living in Curved Momentum Space, *Int. J. Mod. Phys. A* **2013**, *28*, 1330014. [[CrossRef](#)]
52. Amelino-Camelia, G.; Majid, S. Waves on noncommutative space–time and gamma-ray bursts. *Int. J. Mod. Phys. A* **2000**, *15*, 4301–4323. [[CrossRef](#)]
53. Kowalski-Glikman, J.; Nowak, S. Noncommutative space–time of doubly special relativity theories. *Int. J. Mod. Phys.* **2003**, *D12*, 299–316. [[CrossRef](#)]
54. Amelino-Camelia, G. Quantum-Spacetime Phenomenology. *Living Rev. Relativ.* **2013**, *16*, 5 [[CrossRef](#)]
55. Borowiec, A.; Pachol, A.  $\kappa$ -Minkowski spacetime as the result of Jordanian twist deformation. *Phys. Rev. D* **2009**, *79*, 04501. [[CrossRef](#)]
56. Borowiec, A.; Pachol, A.  $\kappa$ -Minkowski spacetimes and DSR algebras: Fresh look and old problems. *Symmetry Integr. Geom. Methods Appl.* **2010**, *6*, 086. [[CrossRef](#)]
57. Borowiec, A.; Gupta, K.S.; Meljanac, S.; Pachol, A. Constraints on the quantum gravity scale from  $\kappa$ -Minkowski spacetime. *Europhys. Lett.* **2010**, *92*, 20006. [[CrossRef](#)]
58. Aschieri, P.; Borowiec, A.; Pachol, A. Observables and dispersion relations in  $\kappa$ -Minkowski spacetime. *J. High Energy Phys.* **2017**, *152*, 2017. [[CrossRef](#)]
59. Gubitosi, G.; Heefer, S. Relativistic compatibility of the interacting  $\kappa$ -Poincaré model and implications for the relative locality framework. *Phys. Rev. D* **2019**, *99*, 086019. [[CrossRef](#)]
60. Amelino-Camelia, G.; Loret, N.; Rosati, G. Speed of particles and a relativity of locality in  $\kappa$ -Minkowski quantum spacetime. *Phys. Lett. B* **2011**, *700*, 150–156. [[CrossRef](#)]
61. Carmona, J.M.; Cortes, J.L.; Mazon, D.; Mercati, F. Locality and the Relativity Principle Beyond Special Relativity. *Phys. Rev.* **2011**, *D84*, 085010. [[CrossRef](#)]
62. Carmona, J.M.; Cortes, J.L.; Mercati, F. Relativistic kinematics beyond Special Relativity. *Phys. Rev.* **2012**, *D86*, 084032, [[CrossRef](#)]
63. Carmona, J.M.; Cortes, J.L.; Relancio, J.J. Spacetime from locality of interactions in deformations of special relativity: The example of  $\kappa$ -Poincaré Hopf algebra. *Phys. Rev. D* **2018**, *97*, 064025. [[CrossRef](#)]
64. Amelino-Camelia, G.; Bianco, S.; Brighenti, F.; Buonocore, R.J. Causality and momentum conservation from relative locality. *Phys. Rev. D* **2015**, *91*, 084045. [[CrossRef](#)]
65. Amelino-Camelia, G.; Bianco, S.; Rosati, G. Planck-Scale-Deformed Relativistic Symmetries and Diffeomorphisms on Momentum Space, *Phys. Rev. D* **2020**, *101*, 026018. [[CrossRef](#)]

66. Amelino-Camelia, G.; Gubitosi, G.; Palmisano, G. Pathways to relativistic curved momentum spaces: De Sitter case study. *Int. J. Mod. Phys. D* **2016**, *25*, 1650027. [[CrossRef](#)]
67. Amelino-Camelia, G.; Matassa, M.; Mercati, F.; Rosati, G. Taming Nonlocality in Theories with Planck-Scale Deformed Lorentz Symmetry. *Phys. Rev. Lett.* **2011**, *106*, 071301. [[CrossRef](#)] [[PubMed](#)]
68. Dabrowski, L.; Piacitelli, G. Canonical  $\kappa$ -Minkowski Spacetime. *arXiv* **2010**, arXiv:1004.5091.
69. Agostini, A.; Lizzi, F.; Zampini, A. Generalized Weyl systems and kappa Minkowski space. *Mod. Phys. Lett.* **2002**, *A17*, 2105–2126. [[CrossRef](#)]
70. Agostini, A.  $\kappa$ -Minkowski representations on Hilbert spaces. *J. Math. Phys.* **2007**, *48*, 052305. [[CrossRef](#)]
71. Lizzi, F.; Manfredonia, M.; Mercati, F.; Poulain, T. Localization and Reference Frames in  $\kappa$ -Minkowski Spacetime. *Phys. Rev. D* **2019**, *99*, 085003. [[CrossRef](#)]
72. Carotenuto, A.; Lizzi, F.; Mercati, F.; Manfredonia, M. The Weyl-Mellin quantization map for  $\kappa$ -Minkowski Noncommutative Spacetime. *arXiv* **2020**, arXiv:2011.10628.
73. Sitarz, A. Noncommutative differential calculus on the  $\kappa$ -Minkowski space. *Phys. Lett. B* **1995**, *349*, 42–48. [[CrossRef](#)]
74. de Azcárraga, J.A.; Pérez-Bueno, J.C. Relativistic and Newtonian kappa space–times. *J. Math. Phys.* **1995**, *36*, 6879–6896. [[CrossRef](#)]
75. Mercati, F. Quantum  $\kappa$ -deformed differential geometry and field theory, *Int. J. Mod. Phys. D* **2016**, *25*, 1650053. [[CrossRef](#)]
76. Mercati, F.; Sitarz, A.  $\kappa$ -Minkowski differential calculi and star product. *arXiv* **2011**, arXiv:1105.1599.
77. Durhuus, B.; Sitarz, A. Star product realizations of  $\kappa$ -Minkowski space. *J. Noncommut. Geom.* **2013**, *7*, 605–645. [[CrossRef](#)]
78. Juric, T.; Meljanac, S.; Pikutić, D.; Štrajn, R. Toward the classification of differential calculi on  $\kappa$ -Minkowski space and related field theories. *J. High Energy Phys.* **2015**, *2015*, 55. [[CrossRef](#)]
79. Kosinski, P.; Lukierski, J.; Maslanka, P. Local  $D = 4$  field theory on kappa deformed Minkowski space. *Phys. Rev. D* **2000**, *62*, 025004. [[CrossRef](#)]
80. Kosinski, P.; Lukierski, J.; Maslanka, P.  $\kappa$ -deformed Wigner construction of relativistic wave functions and free fields on  $\kappa$ -Minkowski space. *Nucl. Phys. B Proc. Suppl.* **2001**, *102*, 161–168. [[CrossRef](#)]
81. Agostini, A.; Amelino-Camelia, G.; Arzano, M. Dirac spinors for doubly special relativity and kappa Minkowski noncommutative space–time. *Class. Quantum Gravity* **2004**, *21*, 2179–2202. [[CrossRef](#)]
82. Dimitrijević, M.; Jonke, L.; Möller, L.; Tsouchnika, E.; Wess, J.; Wohlgemant, M. Deformed field theory on  $\kappa$ -spacetime. *Eur. Phys. J. C* **2003**, *31*, 129–138. [[CrossRef](#)]
83. Kosinski, P.; Maslanka, P.; Lukierski, J.; Sitarz, A. Generalized kappa deformations and deformed relativistic scalar fields on noncommutative Minkowski space. In *Conference on Topics in Mathematical Physics, General Relativity, and Cosmology on the Occasion of the 75th Birthday of Jerzy F. Plebanski*; World Scientific: Singapore, 2006; pp. 255–277.
84. Amelino-Camelia, G.; Gubitosi, G.; Marciano, A.; Martinetti, P.; Mercati, F. A No-pure-boost uncertainty principle from spacetime noncommutativity. *Phys. Lett. B* **2009**, *671*, 298–302. [[CrossRef](#)]
85. Amelino-Camelia, G.; Gubitosi, G.; Marciano, A.; Martinetti, P.; Mercati, F.; Pranzetti, D.; Tacchi, R.A. First results of the Noether theorem for Hopf-algebra spacetime symmetries. *Prog. Theor. Phys. Suppl.* **2007**, *171*, 65–78. [[CrossRef](#)]
86. Freidel, L.; Kowalski-Glikman, J.; Nowak, S. From noncommutative  $\kappa$ -Minkowski to Minkowski space–time. *Phys. Lett. B* **2007**, *648*, 70–75. [[CrossRef](#)]
87. Arzano, M.; Marciano, A. Fock space, quantum fields and kappa-Poincare symmetries. *Phys. Rev. D* **2007**, *76*, 125005. [[CrossRef](#)]
88. Daszkiewicz, M.; Lukierski, J.; Woronowicz, M.  $\kappa$ -deformed statistics and classical fourmomentum addition law. *Mod. Phys. Lett. A* **2008**, *23*, 653–665. [[CrossRef](#)]
89. Freidel, L.; Kowalski-Glikman, J.; Nowak, S. Field theory on  $\kappa$ -Minkowski space revisited: Noether charges and breaking of Lorentz symmetry. *Int. J. Mod. Phys.* **2008**, *A23*, 2687–2718. [[CrossRef](#)]
90. Arzano, M.; Kowalski-Glikman, J.; Walkus, A. Lorentz invariant field theory on  $\kappa$ -Minkowski space. *Class. Quantum Gravity* **2010**, *27*, 025012. [[CrossRef](#)]
91. Amelino-Camelia, G.; Loret, N.; Mandanici, G.; Mercati, F. Gravity in quantum spacetime. *Int. J. Mod. Phys. D* **2010**, *19*, 2385–2392. [[CrossRef](#)]
92. Dimitrijević, M.; Jonke, L.; Pachoł, A. Gauge theory on twisted  $\kappa$ -Minkowski: Old problems and possible solutions. *Symmetry Integr. Geom. Methods Appl.* **2014**, *10*, 063. [[CrossRef](#)]
93. Jurić, T.; Meljanac, S.; Samsarov, A. Light-like  $\kappa$ -deformations and scalar field theory via Drinfeld twist. *J. Phys. Conf. Ser.* **2015**, *634*, 012005. [[CrossRef](#)]
94. Meljanac, S.; Meljanac, D.; Mercati, F.; Pikutić, D. Noncommutative spaces and Poincaré symmetry. *Phys. Lett.* **2017**, *B766*, 181–185. [[CrossRef](#)]
95. Loret, N.; Meljanac, S.; Mercati, F.; Pikutić, D. Vectorlike deformations of relativistic quantum phase-space and relativistic kinematics. *Int. J. Mod. Phys.* **2017**, *D26*, 1750123. [[CrossRef](#)]
96. Arzano, M.; Kowalski-Glikman, J. Non-commutative fields and the short-scale structure of spacetime. *Phys. Lett.* **2017**, *B771*, 222–226. [[CrossRef](#)]
97. Arzano, M. Anatomy of a deformed symmetry: Field quantization on curved momentum space, *Phys. Rev. D* **2011**, *83*, 025025. [[CrossRef](#)]
98. Poulain, T.; Wallet, J.C.  $\kappa$ -Poincaré invariant quantum field theories with KMS weight. *Phys. Rev. D* **2018**, *98*, 025002. [[CrossRef](#)]

99. Ballesteros, A.; Mercati, F. Extended noncommutative Minkowski spacetimes and hybrid gauge symmetries. *Eur. Phys. J. C* **2018**, *78*, 615. [[CrossRef](#)]
100. Poulain, T.; Wallet, J.-C.  $\kappa$ -Poincaré invariant orientable field theories at one-loop. *J. High Energy Phys.* **2019**, *1*, 64. [[CrossRef](#)]
101. Jurić, T.; Poulain, T.; Wallet, J.-C. Vacuum energy and the cosmological constant problem in  $\kappa$ -Poincaré invariant field theories. *Phys. Rev. D* **2019**, *99*, 045004. [[CrossRef](#)]
102. Mercati, F.; Sergola, M. Pauli-Jordan function and scalar field quantization in  $\kappa$ -Minkowski noncommutative spacetime. *Phys. Rev.* **2018**, *D98*, 045017. [[CrossRef](#)]
103. Mathieu, P.; Wallet, J.-C. Gauge theories on  $\kappa$ -Minkowski spaces: Twist and modular operators. *J. High Energy Phys.* **2020**, *5*, 112. [[CrossRef](#)]
104. Arzano, M.; Bevilacqua, A.; Kowalski-Glikman, J.; Rosati, G.; Unger, J.  $\kappa$ -deformed complex fields and discrete symmetries. *Phys. Rev. D* **2021**, *103*, 106015. [[CrossRef](#)]
105. Ballesteros, A.; Gubitosi, G.; Gutierrez-Sagredo, I.; Mercati, F. Fuzzy worldlines with  $\kappa$ -Poincaré symmetries. *arXiv* **2021**, arXiv:2109.09699.
106. Amelino-Camelia, G.; Astuti, V.; Rosati, G. Predictive description of Planck-scale-induced spacetime fuzziness. *Phys. Rev. D* **2013**, *87*, 084023. [[CrossRef](#)]
107. Amelino-Camelia, G.; Astuti, V.; Rosati, G. Relative locality in a quantum spacetime and the pregeometry of  $\kappa$ -Minkowski. *Eur. Phys. J. C* **2013**, *73*, 2521. [[CrossRef](#)]
108. Mercati, F.; Sergola, M. Light cone in a quantum spacetime. *Phys. Lett. B* **2018**, *787*, 105–110. [[CrossRef](#)]
109. Arzano, M.; Consoli, L.T. Signal propagation on  $\kappa$ -Minkowski spacetime and nonlocal two-point functions. *Phys. Rev. D* **2018**, *98*, 106018. [[CrossRef](#)]
110. Blaut, A.; Daszkiewicz, M.; Kowalski-Glikman, J.; Nowak, S. Phase spaces of doubly special relativity. *Phys. Lett. B* **2004**, *582*, 82–85. [[CrossRef](#)]
111. Lukierski, J.; Škoda, Z.; Woronowicz, M.  $\kappa$ -deformed covariant quantum phase spaces as Hopf algebroids. *Phys. Lett. B* **2015**, *750*, 401–406. [[CrossRef](#)]
112. Lizzi, F.; Manfredonia, M.; Mercati, F. The momentum spaces of  $\kappa$ -Minkowski noncommutative spacetime. *Nucl. Phys. B* **2020**, *958*, 115117. [[CrossRef](#)]
113. Carmona, J.M.; Cortés, J.L.; Relancio, J.J. Relativistic deformed kinematics from momentum space geometry. *Phys. Rev. D* **2019**, *100*, 104031. [[CrossRef](#)]
114. Ballesteros, A.; Gutierrez-Sagredo, I.; Herranz, F.J. Noncommutative spaces of worldlines. *Phys. Lett. B* **2019**, *792*, 175–181. [[CrossRef](#)]
115. Lukierski, J.; Nowicki, A.; Ruegg, H. Real forms of complex quantum anti-de-Sitter algebra  $U_q(Sp(4; \mathbb{C}))$  and their contraction schemes. *Phys. Lett. B* **1991**, *271*, 321–328. [[CrossRef](#)]
116. Ballesteros, A.; Gubitosi, G.; Gutiérrez-Sagredo, I.; Herranz, F.J. Curved momentum spaces from quantum (anti-)de Sitter groups in  $(3 + 1)$  dimensions. *Phys. Rev. D* **2018**, *97*, 106024. [[CrossRef](#)]
117. Ballesteros, A.; Musso, F. Quantum algebras as quantizations of dual Poisson-Lie groups. *J. Phys. A Math. Theor.* **2013**, *46*, 195203. [[CrossRef](#)]
118. Ballesteros, A.; Gubitosi, G.; Gutiérrez-Sagredo, I.; Herranz, F.J. Curved momentum spaces from quantum groups with cosmological constant. *Phys. Lett. B* **2017**, *773*, 47–53. [[CrossRef](#)]
119. Ballesteros, A.; Gubitosi, G.; Herranz, F.J. Lorentzian Snyder spacetimes and their Galilei and Carroll limits from projective geometry. *Class. Quantum Gravity* **2020**, *37*, 195021. [[CrossRef](#)]
120. Bergshoeff, E.; Gomis, J.; Rollier, B.; Rosseel, J.; Ter Veldhuis, T. Carroll versus Galilei Gravity. *J. High Energy Phys.* **2017**, *2017*, 165. [[CrossRef](#)]
121. Dijkhuizen, M.S.; Koornwinder, T.H. Quantum homogeneous spaces, duality and quantum 2-spheres. *Geom. Dedicata* **1994**, *52*, 291–315. [[CrossRef](#)]
122. Ciccoli, N. Quantum planes and quantum cylinders from Poisson homogeneous spaces. *J. Phys. A Math. Gen.* **1996**, *29*, 1487–1495. [[CrossRef](#)]
123. Ballesteros, A.; Gutierrez-Sagredo, I.; Mercati, F. Coisotropic Lie bialgebras and complementary dual Poisson homogeneous spaces. *J. Phys. A Math. Theor.* **2021**, *54*, 315203. [[CrossRef](#)]
124. Steinacker, H. Quantum anti-de Sitter space and sphere at roots of unity. *Adv. Theor. Math. Phys.* **2000**, *4*, 155–208. [[CrossRef](#)]
125. Buric, M.; Madore, J. Noncommutative de Sitter and FRW spaces. *Eur. Phys. J. C* **2015**, *75*, 502. [[CrossRef](#)]
126. Heckman, J.J.; Verlinde, H. Covariant non-commutative space-time. *Nucl. Phys. B* **2015**, *894*, 58–74. [[CrossRef](#)]
127. Manolakos, G.; Manousselis, P.; Zoupanos, G. Four-dimensional Gravity on a Covariant Noncommutative Space. *J. High Energy Phys.* **2020**, *2020*, 1. [[CrossRef](#)]
128. Ballesteros, A.; Herranz, F.J.; Del Olmo, M.A.; Santander, M. Classical deformations, Poisson-Lie contractions, and quantization of dual Lie bialgebras. *J. Math. Phys.* **1995**, *36*, 631–640. [[CrossRef](#)]
129. Snyder, H. Quantized space-time. *Phys. Rev.* **1947**, *71*, 38–41. [[CrossRef](#)]