Distance Fibonacci Polynomials by Graph Methods

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Abstract: In this paper we introduce and study a new generalization of Fibonacci polynomials which generalize Fibonacci, Jacobsthal and Narayana numbers, simultaneously. We give a graph interpretation of these polynomials and we obtain a binomial formula for them. Moreover by modification of Pascal's triangle, which has a symmetric structure, we obtain matrices generated by coefficients of generalized Fibonacci polynomials. As a consequence, the direct formula for generalized Fibonacci polynomials was given. In addition, we determine matrix generators for generalized Fibonacci polynomials, using the symmetric matrix of initial conditions.

Keywords: Fibonacci numbers; Fibonacci polynomials; matrix generators; Pascal's triangle

MSC: 11B39; 11B37; 11C20

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1. Introduction and Preliminary Results

Let $k \geq 1$, $n \geq 0$ be integers. Sequences defined by the $(k + 1)$ -st order linear recurrence relation

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_{k+1} a_{n-(k+1)}, \text{ for } n \geq k + 1 \quad (1)$$

where $b_i \in N \cup \{0\}$, $i \in \{1, 2, \dots, k + 1\}$, $b_{k+1} \neq 0$, with initial conditions $a_n \in N \cup \{0\}$, $n \in \{0, 1, \dots, k\}$, such that $\sum_{j=0}^k a_j^2 > 0$, are named sequences of the Fibonacci type and consequently their elements are Fibonacci type numbers.

For special cases of k , b_i , $i \in \{1, 2, \dots, k + 1\}$ and a_n , $n \in \{0, 1, \dots, k\}$ the equality (1) gives the well-known number sequences. In this paper we consider the following sequences.

1. Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ and $F_0 = 0$, $F_1 = 1$.
2. Jacobsthal sequence $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ and $J_0 = 0$, $J_1 = 1$.
3. Narayana sequence $N_n = N_{n-1} + N_{n-3}$, for $n \geq 3$ and $N_0 = 0$, $N_1 = N_2 = 1$.
4. generalized Fibonacci sequence $F(p, n) = F(p, n - 1) + F(p, n - p)$, for $p \geq 2$, $n \geq p + 1$ and $F(p, n) = n + 1$ for $n \in \{0, \dots, p - 1\}$ (see for this concept [1]).

There are a number of other well-known Fibonacci type sequences, see for example [2].

Fibonacci sequence initiated a wide theory of Fibonacci type sequences. The recurrence which defines the sequence (F_n) appears in 1202 in the book *Liber Abaci* of Italian mathematician Leonardo of Pisa, better known as Fibonacci, as a solution of the famous rabbit problem. This book was a compendium of the arithmetical and algebraic knowledge of those time and played an important role in the development of European mathematics. However, the name Fibonacci sequence for (F_n) was introduced in the 19th century by F. Lucas. It is worth mentioning that Fibonacci introduced the sequence (F_n) to European mathematics, but that sequence was described earlier by Indian mathematics in works of Pingala and later Virachanka, see for details [3].

Fibonacci sequence has many interesting applications, for example, their connections with the golden ratio. Actually, there is a huge interest of modern science in the applications of golden section, which appears nowadays in physics research of the high energy particles and theoretical physics. Fibonacci numbers are intensively studied in different areas of science by mathematics enthusiasts and many interesting generalizations of Fibonacci sequence and Fibonacci type sequences have been defined. A special direction of generalization of the Fibonacci sequence is a sequence of Fibonacci polynomials $f_n(x)$, which were introduced by Catalan in 1883 and next studied with respect to their distinct properties, see, for examples, [4].

Fibonacci polynomials $f_n(x)$ are defined by the recurrence relation of the form

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \text{ for } n \geq 2 \quad (2)$$

with initial conditions $f_0(x) = 0$, $f_1(x) = 1$. Clearly $f_n(1) = F_n$.

Generalizations of numbers or polynomials of the Fibonacci type existing in the literature are mainly concentrated on combinatorial properties using generating functions or the problem of solving the recurrence relation, called a problem of Binet's formula type, see, for example, [5–7]. Note that some generalizations we obtain by changing initial terms and preserving the recurrence equation (see [8,9]) or by modification of the recurrence, see, for instance, [10].

In the literature we can also find another definition of Fibonacci polynomials which arises from the graph theory. A question of a graph interpretation of Fibonacci type numbers and polynomials is interesting not only from the graph theoretical point of view but also from their applications and the possibility to use graphs methods as a proving tools. Some results concerning applications of graphs in studying known sequences of the Fibonacci type were given, for example, in [11–17]. Motivated by their results in this paper we use graph methods to give properties of generalized Fibonacci polynomials. Let us first recall necessary graph concepts and give some historical background.

By a graph G we mean a finite, undirected, connected, simple graph where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G , respectively. By a P_n we mean a graph with the vertex set $V(P_n) = \{t_1, \dots, t_n\}$ and the edge set $E(P_n) = \{t_i t_{i+1}; i \in \{1, \dots, n-1\}\}$, $n \geq 2$. Moreover, P_0 is the empty graph and P_1 is the graph that consists of only one vertex. Let K_x denote the complete graph on x vertices, $x \geq 1$.

Let G be a graph on $V(G) = \{v_1, \dots, v_n\}$, $n \geq 2$ and $h_n = (H_i)_{i \in \{1, \dots, n\}}$ be a sequence of vertex disjoint graphs on $V(H_i) = \{(v_i, y_1), \dots, (v_i, y_x)\}$, $x \geq 1$. By the generalized lexicographic product of the graph G and the sequence h_n we mean the graph $G[h_n]$ such that $V(G[h_n]) = \bigcup_{i=1}^n V(H_i)$ and $E(G[h_n]) = \{(v_i, y_p)(v_j, y_q); (v_i = v_j \text{ and } (v_i, y_p)(v_i, y_q) \in E(H_i)) \text{ or } v_i v_j \in E(G)\}$. If $H_i = H$ for $i \in \{1, \dots, n\}$, then $G[h_n] = G[H]$, where $G[H]$ is the well-known composition of two graphs. By H_i^c , $i \in \{1, \dots, n\}$ we will denote the copy of the graph H_i in $G[h_n]$.

Let H be an arbitrary subgraph of $G[h_n]$. By a projection of a subgraph H on the graph G we mean a subgraph $pr_G H$ induced by the set $\{v \in V(G); \text{there exists } 1 \leq i \leq n \text{ such that } (v, y_i) \in V(H)\}$.

For $x, y \in V(G)$ by $d_G(x, y)$ we mean the distance between x and y in G .

A subset of vertices is independent if no two vertices are adjacent. Moreover the empty set and a set containing exactly one vertex also are independent.

The problem of counting of independent sets in graphs is related to the Fibonacci type numbers. Fibonacci numbers appears in graphs first in the paper of H. Prodinger and R. F. Tichy [18] in the context of counting independent sets in n -vertex path P_n . They proved that the total number of independent sets in P_n is equal to F_{n+2} . This simple observation gave impetus for counting independent sets in graphs and finding graph interpretations of Fibonacci type numbers. The interest in counting independent sets in graphs was multiplied by the fact that the total number of independent sets in molecular graphs, named the Merrifield–Simmons index, has a huge application in combinational chemistry

(see, for example, the survey [19] and its references). Based on the graph interpretation of Fibonacci numbers in 1983 G. Hopkins and W. Staton defined Fibonacci polynomials of graphs being the total number of independent sets in $G[K_x]$. In consequence of their results another type of Fibonacci polynomials can be defined as follows

$$F_n(x) = F_{n-1}(x) + xF_{n-2}(x) \text{ for } n \geq 2 \quad (3)$$

with initial conditions $F_0(x) = 1$ and $F_1(x) = x + 1$.

If $x = 1$, then $F_n(1) = F_{n+2}$, see for details [20].

In spite of both $f_n(x)$ and $F_n(x)$ generalized Fibonacci numbers, $(f_n(x))$ and $(F_n(x))$ are different. For illustration let us observe the first 10 elements of these sequences.

$$(f_n(x)): (0, 1, x, x^2 + 1, x^3 + 2x, x^4 + 3x^2 + 1, x^5 + 4x^3 + 3x, x^6 + 5x^4 + 6x^2 + 1, x^7 + 6x^5 + 10x^3 + 4x, x^8 + 7x^6 + 15x^4 + 10x^2 + 1, x^9 + 8x^7 + 21x^5 + 20x^3 + 5x, \dots)$$

$$(F_n(x)): (1, x + 1, 2x + 1, x^2 + 3x + 1, 3x^2 + 4x + 1, x^3 + 6x^2 + 5x + 1, 4x^3 + 10x^2 + 6x + 1, x^4 + 10x^3 + 15x^2 + 7x + 1, 5x^4 + 20x^3 + 21x^2 + 8x + 1, x^5 + 15x^4 + 35x^3 + 28x^2 + 9x + 1, 6x^5 + 35x^4 + 56x^3 + 36x^2 + 10x + 1, \dots)$$

Analogously, as Fibonacci numbers, Fibonacci polynomials $f_n(x)$ and $F_n(x)$ also have many generalizations and interpretations, see, for example, [21]. Some of them generalize Fibonacci polynomials in the distance sense, i.e., by changing the distance between terms of the sequence. Generalization of the Fibonacci polynomials $F_n(x)$ in the distance sense was given in 2003 by I. Włoch using graph methods, see for details [22].

As a consequence of results from [22] for $n \geq 0, k \geq 1$ generalized Fibonacci polynomials $F_n(k, x)$ can be defined by a recurrence relation.

$$F_n(k, x) = F_{n-1}(k, x) + xF_{n-(k+1)}(k, x) \text{ for } n \geq k + 1 \quad (4)$$

with initial conditions $F_0(k, x) = 1$ and $F_n(k, x) = nx + 1$ for $n \in \{1, \dots, k\}$. This sequence generalizes Fibonacci polynomials $F_n(1, x) = F_n(x)$ and also Fibonacci numbers, Jacobsthal numbers and Narayana numbers, since $F_n(1, 1) = F_{n+2}$, $F_n(1, 2) = J_{n+2}$ and $F_n(2, 1) = N_{n+3}$, respectively.

In this paper using graph methods we give a new distance generalization of Fibonacci polynomials. Existing definitions of Fibonacci type polynomials are given by putting the variable x as a coefficient in the recurrence equation. In this paper we introduce another type of distance Fibonacci polynomial using the power of the variable x , which relates to the order of the recurrence equation. This implies that we can only use known methods partially.

Let $k \geq 1, n \geq 2$ be integers. Distance Fibonacci polynomials $\varphi_n(k, x)$ are defined by

$$\varphi_n(k, x) = \varphi_{n-1}(k, x) + x^k \varphi_{n-(k+1)}(k, x) \text{ for } n \geq k + 1 \quad (5)$$

with initial conditions $\varphi_n(k, x) = 1$ for $n \in \{0, 1, \dots, k - 1\}$ and $\varphi_k(k, x) = 1 + x^k$.

Note that $\varphi_n(1, x) = F_n(x)$ and consequently $\varphi_n(1, 1) = F_{n+1}$, $\varphi_n(1, 2) = J_{n+1}$, $\varphi_n(2, 1) = N_{n+2}$.

Theorem 1. Let $k \geq 1$ be an integer, t , and R be real where $|t| < R, R > 0$. The generating function for distance Fibonacci polynomials sequence $(\varphi_n(k, x))$ is

$$g(t, k) = \frac{1 + x^k t^k}{1 - t - x^k t^{k+1}}. \quad (6)$$

Proof. Assume that the generating function has the form $g(t, k) = \sum_{n=0}^{\infty} \varphi_n(k, x) t^n$. Then

$$\begin{aligned}
 g(t, k) &= \sum_{n=0}^{\infty} \varphi_n(k, x) t^n = 1 + t + \cdots + t^{k-1} + (x^k + 1)t^k + \sum_{n=k+1}^{\infty} \varphi_n(k, x) t^n \\
 &= 1 + t + \cdots + t^{k-1} + (x^k + 1)t^k + \sum_{n=k+1}^{\infty} (\varphi_{n-1}(k, x) + x^k \varphi_{n-(k+1)}(k, x)) t^n \\
 &= 1 + t + \cdots + t^{k-1} + (x^k + 1)t^k + \sum_{n=k+1}^{\infty} \varphi_{n-1}(k, x) t^n + \sum_{n=k+1}^{\infty} x^k \varphi_{n-(k+1)}(k, x) t^n \\
 &= 1 + t + \cdots + t^{k-1} + (x^k + 1)t^k + t \sum_{n=k+1}^{\infty} \varphi_{n-1}(k, x) t^{n-1} \\
 &\quad + x^k t^{k+1} \sum_{n=k+1}^{\infty} \varphi_{n-(k+1)}(k, x) t^{n-(k+1)} \\
 &= 1 + t + \cdots + t^{k-1} + t^k + t^k x^k + t \sum_{n=k}^{\infty} \varphi_n(k, x) t^n + x^k t^{k+1} \sum_{n=0}^{\infty} \varphi_n(k, x) t^n \\
 &= 1 + t^k x^k + t \sum_{n=0}^{\infty} \varphi_n(k, x) t^n + x^k t^{k+1} \sum_{n=0}^{\infty} \varphi_n(k, x) t^n \\
 &= 1 + t^k x^k + t \cdot g(t, k) + x^k t^{k+1} \cdot g(t, k).
 \end{aligned}$$

Hence we have $g(t, k) = 1 + x^k t^k + t \cdot g(t, k) + x^k t^{k+1} \cdot g(t, k)$ and $g(t, k) = \frac{1+x^k t^k}{1-t-x^k t^{k+1}}$. $\square$

For $k = 1$ we have a generating function of the sequence of Fibonacci polynomials (3)

$$g(t, x) = \frac{1 + xt}{1 - t - xt^2}. \quad (7)$$

Moreover, for special values of k and x we obtain a generating function of Fibonacci, Jacobsthal and Narayana sequences, respectively.

$$\sum_{n=0}^{\infty} F_{n+2} t^n = \frac{1+t}{1-t-t^2} \quad (8)$$

$$\sum_{n=0}^{\infty} J_{n+2} t^n = \frac{1+2t}{1-t-2t^2} \quad (9)$$

$$\sum_{n=0}^{\infty} N_{n+2} t^n = \frac{1+t^2}{1-t-t^3} \quad (10)$$

Theorem 2. Let $n \geq 0, k \geq 1$ be integers. Then

$$\sum_{i=0}^n \varphi_i(k, x) = \frac{\varphi_{n+k+1}(k, x) - 1 - x^k}{x^k}. \quad (11)$$

Proof. (by induction on n). If $n = 0$, then the equality (11) follows immediately. Assume that the formula (11) holds for n , $n \geq 1$. We shall show that it is also true for $n + 1$.

$$\begin{aligned}\sum_{i=0}^{n+1} \varphi_i(k, x) &= \sum_{i=0}^n \varphi_i(k, x) + \varphi_{n+1}(k, x) \\ &= \frac{\varphi_{n+k+1}(k, x) - 1 - x^k}{x^k} + \frac{x^k \varphi_{n+1}(k, x)}{x^k} \\ &= \frac{\varphi_{n+k+1}(k, x) + x^k \varphi_{n+1}(k, x) - 1 - x^k}{x^k} \\ &= \frac{\varphi_{n+k+2}(k, x) - 1 - x^k}{x^k},\end{aligned}$$

which ends the proof. $\square$

2. Graph Interpretation and Direct Formula of $\varphi_n(k, x)$

We give a graph interpretation of polynomials $\varphi_n(k, x)$ with respect to counting of special induced subgraphs in a lexicographic product.

Let $H \leq G$ be an induced subgraph of G such that every connected component of H is isomorphic to P_k , for a fixed $k \geq 1$. In particular H can be empty. A component of H we will name the P_k -component and the subgraph H with $p \geq 0$ components by the pP_k -subgraph.

Denote by $\mathcal{H}$ an induced subgraph of $G[h_n]$ satisfying conditions:

- (a) each connected component of $\mathcal{H}$ is isomorphic to P_k ,
- (b) for each two vertices $(v, y_i), (v, y_j) \in V(\mathcal{H})$ $i = j$,
- (c) $pr_G \mathcal{H}$ is a pP_k -subgraph of G , $p \geq 0$.

In particular $\mathcal{H}$ can be empty.

Let $\varphi_{G[h_n]}(k, x)$ denote the number of all subgraphs $\mathcal{H}$ of $G[h_n]$.

Theorem 3. Let $k \geq 1$, $n \geq 0$ be integers. Then

$$\varphi_{P_n[h_n]}(k, x) = \varphi_n(k, x). \quad (12)$$

Proof. (by induction on n) From the definition of $\mathcal{H}$ we have that at most one vertex from H_i^c , $i \in \{1, \dots, n\}$ can belong to $\mathcal{H}$.

If $n \in \{0, \dots, k-1\}$, then the empty set is the unique subgraph $\mathcal{H}$ of $P_n[h_n]$.

If $n = k$, then either $\mathcal{H}$ is empty or $\mathcal{H}$ contains exactly one P_k -component. Since the vertex from every H_i^c can be chosen in x ways, we have $x^k + 1$ subgraphs $\mathcal{H}$, so $\varphi_{P_k[h_k]}(k, x) = x^k + 1$.

Let $n \geq k + 1$ and suppose that equality holds for each $t < n$. We shall show this for n .

If $|V(\mathcal{H}) \cap V(H_n^c)| = 0$, then $\mathcal{H} \cong \mathcal{H}^*$, where $\mathcal{H}^*$ is an induced subgraph of $P_{n-1}[h_{n-1}]$ satisfying conditions (a)–(c). Using the induction's hypothesis we have $\varphi_{n-1}(k, x)$ subgraphs $\mathcal{H}$.

If $|V(\mathcal{H}) \cap V(H_n^c)| = 1$, then the P_k -component of $\mathcal{H}$ including the vertex from H_n^c contains exactly one vertex from every H_i^c , $i \in \{n - (k - 1), \dots, n\}$. Since the vertex from H_i^c , $i \in \{n - (k - 1), \dots, n\}$, can be chosen in x ways, so we have exactly x^k such P_k -components. Moreover, $\mathcal{H} = \mathcal{H}' \cup \{P_k\}$ where $\mathcal{H}'$ is a subgraph of $P_{n-(k+1)}[h_{n-(k+1)}]$ satisfying conditions (a)–(c). In particular $\mathcal{H}'$ can be empty. So by the induction's hypothesis we have $x^k \varphi_{n-(k+1)}(k, x)$ subgraphs $\mathcal{H}$.

From the above we have that $\varphi_{P_n[h_n]}(k, x) = \varphi_{n-1}(k, x) + x^k \varphi_{n-(k+1)}(k, x) = \varphi_n(k, x)$ and by the induction rule the theorem is proved. $\square$

Now we determine the direct formula for $\varphi_n(k, x)$ using the above graph interpretation.

Let $k \geq 1$, $n \geq 0$, $p \geq 0$ be integers. By $\sigma_G(n, k, p)$ we denote the total number of pP_k -subgraphs of a graph G .

Theorem 4. Let $k \geq 1, n \geq 0, x \geq 1$ be integers. Then for an arbitrary graph G on n vertices

$$\varphi_{G[h_n]}(k, x) = \sum_{p \geq 0} \sigma_G(k, n, p) x^{kp}. \quad (13)$$

Proof. Let G be a given graph on n vertices. If $n \in \{0, \dots, k-1\}$ the result is obvious. Let $n \geq k$. To determine the number $\varphi_{G[h_n]}(k, x)$ firstly we have to choose a pP_k -subgraph of the graph G . Clearly we can choose a pP_k -subgraph in $\sigma_G(k, n, p)$ ways. Let $S = \{v_i; i \in \mathcal{I} \subset \{1, \dots, pk\}\}$ be a vertex set of a pP_k -subgraph of G . Let $S' \subset V(G[h_n])$ and $S' = \{(v_i, y_t); v_i \in S \text{ and } 1 \leq t \leq x\}$. In other words, for each H_i^c such that $v_i \in S$ we choose exactly one vertex from $V(H_i^c)$ to the set S' . Clearly this vertex can be chosen in x ways. From the definition of the $G[h_n]$ we have that S' is a subgraph $\mathcal{H}$ satisfying conditions (a)–(c) with p P_k -components. Since each P_k -component from S generates x^k possible components $\mathcal{H}$, we obtain $\sigma_G(k, n, p) x^{kp}$ subsets $\mathcal{H}$ of $G[h_n]$. So we have $\sum_{p \geq 0} \sigma_G(k, n, p) x^{kp}$ subgraphs $\mathcal{H}$, which ends the proof. $\square$

Lemma 1. Let $k \geq 1, n \geq 0, p \geq 0$ be integers. Then

$$\sigma_{P_n}(k, n, p) = \binom{n - pk + 1}{p}. \quad (14)$$

Proof. Consider a path P_n with $V(P_n) = \{v_1, \dots, v_n\}$ and vertices are numbered in the natural fashion. If $n \in \{0, \dots, k-1\}$, then the result is obvious. Let $n \geq k$ and H be a pP_k -subgraph of P_n .

If $H = \emptyset$, then the result immediately follows.

Let $H \neq \emptyset$. Let $\alpha(v)$ be the function such that $\alpha(v) = 1$ if $v \in V(H)$ or $\alpha(v) = 0$ otherwise. For convenience for every $v_i \in V(P_n)$ we will write shortly α_i instead of $\alpha(v_i)$. Then instead of a graph P_n we can consider a binary n -tuple. Clearly a P_k -component of H corresponds to a subsequence β of k consecutive 1s in the n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$. To determine the number $\sigma_{P_n}(k, n, p)$ it suffices to consider all binary n -tuples $(\alpha_1, \dots, \alpha_n)$ such that $\sum_{i=1}^n \alpha_i = pk$ and satisfying additional conditions given below.

Denote by $(\beta_1, \dots, \beta_p), p \geq 1$ sequence of k -tuples such that

- (i) every $\beta_j, 1 \leq j \leq p$ contains k 1s being consecutive in the sequence α ,
- (ii) if $\alpha_i \in \beta_j, 1 \leq i \leq n, 1 \leq j \leq p$, then $\alpha_i = 1$,
- (iii) for each two β_j, β_r there is a subsequence of n -tuple $(\alpha_1, \dots, \alpha_n)$ of the form $(\beta_j, 0, \beta_r)$.

We substitute each $\beta_j, 1 \leq j \leq p$, by exactly one 1 and obtain $(n - p(k-1))$ -tuple containing p non-consecutive 1s. To determine all possible $(n - p(k-1))$ -tuples firstly we consider $(n - pk)$ -tuple of zeros and next we put p 1s using $n - pk + 1$ places, so $\sigma_{P_n}(k, n, p) = \binom{n - pk + 1}{p}$. $\square$

Let $\sigma_G(k, n) = \sum_{p \geq 0} \sigma_G(k, n, p)$.

Corollary 1. Let $k \geq 1, n \geq 0, p \geq 0$ be integers. Then

$$\sigma_{P_n}(k, n) = \sum_{p \geq 0} \binom{n - pk + 1}{p}. \quad (15)$$

From the Theorems 3 and 4 and Lemma 1 we obtain the binomial formula for $\varphi_n(k, x)$.

Corollary 2. Let $k \geq 1, n \geq 0, x \geq 1$ be integers. Then

$$\varphi_n(k, x) = \varphi_{P_n[h_n]}(k, x) = \sum_{p \geq 0} \binom{n - pk + 1}{p} x^{kp}. \quad (16)$$

3. Connections with Pascal's Triangle

It is well known that Fibonacci numbers can be obtained from Pascal's triangle. Many authors studying Fibonacci type sequences also give connections with Pascal's triangle or modified Pascal's triangle, see [23–25]. Such interpretations can be used to derive binomial formulas for Fibonacci type sequences. In [26] it was proved that all Fibonacci type sequences have binomial formulas. It is natural to find similar properties of generalized distance Fibonacci polynomials. In this section we inspect infinite matrices generated by coefficients of generalized distance Fibonacci polynomials. These matrices we can obtain by simple modifications of Pascal's triangle.

For a convenience we will write Pascal's triangle in a matrix form.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 & \dots \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 & \dots \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 & \dots \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Let us consider for example generalized distance Fibonacci polynomials for $k = 1$, $k = 2$ and $k = 3$. Initial polynomials of these sequences are:

$$\varphi_n(1, x) : 1, x + 1, 2x + 1, x^2 + 3x + 1, 3x^2 + 4x + 1, x^3 + 6x^2 + 5x + 1, 4x^3 + 10x^2 + 6x + 1, x^4 + 10x^3 + 15x^2 + 7x + 1, 5x^4 + 20x^3 + 21x^2 + 8x + 1, x^5 + 15x^4 + 35x^3 + 28x^2 + 9x + 1, \dots$$

$$\varphi_n(2, x) : 1, 1, x^2 + 1, 2x^2 + 1, 3x^2 + 1, x^4 + 4x^2 + 1, 3x^4 + 5x^2 + 1, 6x^4 + 6x^2 + 1, x^6 + 10x^4 + 7x^2 + 1, 4x^6 + 15x^4 + 8x^2 + 1, 10x^6 + 21x^4 + 9x^2 + 1, x^8 + 20x^6 + 28x^4 + 10x^2 + 1, 5x^8 + 35x^6 + 36x^4 + 11x^2 + 1, \dots$$

$$\varphi_n(3, x) : 1, 1, 1, x^3 + 1, 2x^3 + 1, 3x^3 + 1, 4x^3 + 1, x^6 + 5x^3 + 1, 3x^6 + 6x^3 + 1, 6x^6 + 7x^3 + 1, 10x^6 + 8x^3 + 1, x^9 + 15x^6 + 9x^3 + 1, 4x^9 + 21x^6 + 10x^3 + 1, 10x^9 + 28x^6 + 11x^3 + 1, \dots$$

If we write coefficients of consecutive polynomials in rows of a matrix, then an element $a_{i,j}$ is the coefficient at $x^{(j-1)k}$. The matrices A_1 and A_2 presented below correspond to sequences for $k = 1$ and 2, respectively.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 10 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 15 & 10 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 21 & 20 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 28 & 35 & 15 & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 36 & 56 & 35 & 6 & 0 & 0 & 0 & 0 \\ 1 & 11 & 45 & 84 & 70 & 21 & 1 & 0 & 0 & 0 \\ 1 & 12 & 55 & 120 & 126 & 56 & 7 & 0 & 0 & 0 \\ 1 & 13 & 66 & 165 & 210 & 126 & 28 & 1 & 0 & 0 \\ 1 & 14 & 78 & 220 & 330 & 252 & 84 & 8 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 3 & 0 & 0 & 0 \\ 1 & 6 & 6 & 0 & 0 & 0 \\ 1 & 7 & 10 & 1 & 0 & 0 \\ 1 & 8 & 15 & 4 & 0 & 0 \\ 1 & 9 & 21 & 10 & 0 & 0 \\ 1 & 10 & 28 & 20 & 1 & 0 \\ 1 & 11 & 36 & 35 & 5 & 0 \\ 1 & 12 & 45 & 56 & 15 & 0 \end{bmatrix}$$

Let C_i denote the i th column of the matrix P . We obtain a matrix A_k , for an arbitrary $k \geq 1$, from the matrix P by the following transformation: The column C_2 we shift $k - 1$ rows downward and columns C_i , $i \geq 3$, we shift $(i - 1)k - 1$ rows downward. See, for example, matrices A_1 and A_2 .

We can observe that sums in rows or in diagonals form known sequences, see [27], dependences are presented in Table 1.

Table 1. Sums in rows and sums on diagonals of A_k matrices.

k	A_k	Sums in Rows of A_k	Sums on Diagonals of A_k
1	A_1	Fibonacci sequence	Narayana sequence
2	A_2	Narayana sequence	A003269 [27]
3	A_3	A003269 [27]	

Analogously to classical Pascal's triangle we can give a general rule of calculating entries of a matrix A_k . For a convenience, in the next theorem rows and columns of a matrix are numbered starting from the number zero.

Theorem 5. Let $k \geq 1$, $n \geq 0$, $i \geq 0$ be integers. Then $A_k = [a_{n,i}]$ where $a_{n,i} = a_{n-1,i} + a_{n-(k+1),i-1}$ with $a_{n,0} = 1$ for $n \in \{0, \dots, k-1\}$ and $a_{n,i} = 0$ for $n < 0$ or $i < 0$.

Proof. For $i \in \{1, \dots, k\}$ the result is obvious.

Let $i > k$. Then $\varphi_n(k, x) = \sum_{i=0}^{\infty} a_{n,i} x^{ik}$ where coefficients $a_{n,i}$ are elements of the matrix A_k . Then

$$\begin{aligned}
 \varphi_n(x) &= \varphi_{n-1}(k, x) + x^k \varphi_{n-(k+1)}(k, x) = \sum_{i=0}^{\infty} a_{n-1,i} x^{ik} + \sum_{i=0}^{\infty} a_{n-(k+1),i} x^{ik+k} \\
 &= \sum_{i=0}^{\infty} (a_{n-1,i} x^{ik} + a_{n-(k+1),i} x^{ik+k}) = \sum_{i=0}^{\infty} (a_{n-1,i} x^{ik} + a_{n-(k+1),i-1} x^{ik}) \\
 &= \sum_{i=0}^{\infty} (a_{n-1,i} + a_{n-(k+1),i-1}) x^{ik}.
 \end{aligned}$$

Comparing coefficients at x^n we have $a_{n,i} = a_{n-1,i} + a_{n-(k+1),i-1}$, which ends the proof. $\square$

We can also calculate polynomials $\varphi_n(k, x)$ directly from Pascal's triangle P . This method we will call a staircase method.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}$$

For example, for $k = 2$ we have stairs with steps of the height equal to 2 and the length equal to 1. We extend the stairs presented above up to infinity in both directions. Moving such infinite stairs one row downward we obtain coefficients of consecutive polynomials. The blue colour indicates coefficients of $\varphi_9(2, x)$ and the red colour coefficients of $\varphi_8(3, x)$.

Generally for an arbitrary $k \geq 1$ the length of the step is 1 and the height of the step is k .

The above considerations lead to the same binomial formula for $\varphi_n(k, x)$ as in Corollary 2.

Theorem 6. Let $k \geq 1$, $n \geq 0$ and $x \geq 1$ be integers. Then

$$\varphi_n(k, x) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n+1-ik}{i} x^{ik}. \quad (17)$$

Proof. Let $a_{n,i}^k$ denote an element of a matrix A_k . We know that $\varphi_n(k, x) = \sum_{i=0}^n a_{n,i}^k x^{ik}$. To prove this theorem it is enough to perform transformations A_k to Pascal's triangle P . A reverse operation to shifting columns downwards gives $a_{n,i}^k = p_{n-ik,i}$ and we obtain a corresponding entry in Pascal's triangle P . In the sum we omit zero terms, because all entries above the main diagonal of Pascal's triangle are zeros. Thus we have

$$\varphi_n(k, x) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n+1-ik}{i} x^{ik},$$

which ends the proof. $\square$

4. Matrix Generators

Linear recurrence equations have interesting relations with matrices. Matrix generators of Fibonacci type sequences were used to describe properties of these sequences, see, for example, [4,28,29]. We show that matrix generators of $\varphi_n(k, x)$ can be found using the method given in [21,28].

Let $k \geq 1$ be an integer and $Q_k = [q_{ij}]_{(k+1) \times (k+1)}$ be a square matrix. For a fixed $1 \leq i \leq k+1$ an element $q_{i,1}$ is equal to the coefficient at $\varphi_{n-i}(x)$ of the right hand side of the formula (5). If $j > 1$, then $q_{i,j} = \begin{cases} 1 & j = i+1 \\ 0 & \text{otherwise} \end{cases}$. The above definition gives matrices

$$Q_1 = \begin{bmatrix} 1 & 1 \\ x & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ x^2 & 0 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x^3 & 0 & 0 & 0 \end{bmatrix}, \dots,$$

$$Q_k = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ x^k & 0 & 0 & \dots & 0 \end{bmatrix}.$$

To obtain a matrix generator of $\varphi_n(k, x)$, firstly, we define a square matrix R_k of order $k+1$ as the matrix of initial conditions

$$R_k = \begin{bmatrix} \varphi_{2k}(k, x) & \varphi_{2k-1}(k, x) & \dots & \varphi_{k+1}(k, x) & \varphi_k(k, x) \\ \varphi_{2k-1}(k, x) & \varphi_{2k-2}(k, x) & \dots & \varphi_k(k, x) & \varphi_{k-1}(k, x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{k-1}(k, x) & \varphi_k(k, x) & \dots & \varphi_2(k, x) & \varphi_1(k, x) \\ \varphi_k(k, x) & \varphi_{k-1}(k, x) & \dots & \varphi_1(k, x) & \varphi_0(k, x) \end{bmatrix}.$$

Using basic properties of determinants, we get the following results.

Theorem 7. Let $k \geq 1$ be an integer. Then

$$\det Q_k = (-1)^k x^k \text{ and } \det R_k = (-1)^{\lfloor \frac{k+1}{2} \rfloor} x^{k^2+k}. \quad (18)$$

Theorem 8. Let $k \geq 1, n \geq 1$ be integers. Then

$$R_k Q_k^n = \begin{bmatrix} \varphi_{n+2k}(k, x) & \varphi_{n+2k-1}(k, x) & \cdots & \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) \\ \varphi_{n+2k-1}(k, x) & \varphi_{n+2k-2}(k, x) & \cdots & \varphi_{n+k}(k, x) & \varphi_{n+k-1}(k, x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) & \cdots & \varphi_{n+2}(k, x) & \varphi_{n+1}(k, x) \\ \varphi_{n+k}(k, x) & \varphi_{n+k-1}(k, x) & \cdots & \varphi_{n+1}(k, x) & \varphi_n(k, x) \end{bmatrix}. \quad (19)$$

Proof. (by induction on n) If $n = 1$, then by (5) the result immediately follows. Assume that formula (19) holds for n ; we will prove it for $n + 1$. Since $R_k Q_k^{n+1} = (R_k Q_k^n) Q_k$, by our assumption and by the recurrence (5) we obtain $R_k Q_k^{n+1} =$

$$= \begin{bmatrix} \varphi_{n+2k}(k, x) & \varphi_{n+2k-1}(k, x) & \cdots & \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) \\ \varphi_{n+2k-1}(k, x) & \varphi_{n+2k-2}(k, x) & \cdots & \varphi_{n+k}(k, x) & \varphi_{n+k-1}(k, x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) & \cdots & \varphi_{n+2}(k, x) & \varphi_{n+1}(k, x) \\ \varphi_{n+k}(k, x) & \varphi_{n+k-1}(k, x) & \cdots & \varphi_{n+1}(k, x) & \varphi_n(k, x) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x^k & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \varphi_{n+2k+1}(k, x) & \varphi_{n+2k}(k, x) & \cdots & \varphi_{n+k+2}(k, x) & \varphi_{n+k+1}(k, x) \\ \varphi_{n+2k}(k, x) & \varphi_{n+2k-1}(k, x) & \cdots & \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n+k+2}(k, x) & \varphi_{n+k+1}(k, x) & \cdots & \varphi_{n+3}(k, x) & \varphi_{n+2}(k, x) \\ \varphi_{n+k+1}(k, x) & \varphi_{n+k}(k, x) & \cdots & \varphi_{n+2}(k, x) & \varphi_{n+1}(k, x) \end{bmatrix},$$

which ends the proof. $\square$

By Theorem 7 we can obtain the following result, it being the Cassini formula for Fibonacci polynomials $\varphi_n(k, x)$.

Corollary 3. Let $k \geq 3, n \geq 2$ be integers. Then

$$\det R_k Q_k^n = (-1)^{nk + \lfloor \frac{k+1}{2} \rfloor} x^{k^2 + (n+1)k}. \quad (20)$$

5. Conclusions

Sequences of the Fibonacci type defined by the homogenous linear recurrence relations can be extended to polynomials. In the literature we have found distinct definitions of Fibonacci type polynomials. Most of them are obtained by putting the variable x as a coefficient in the recurrence equation of the Fibonacci type sequences. It is necessary to pay attention to the connection of the Fibonacci type polynomials with counting problems in graphs. Graph interpretations of Fibonacci type polynomials give useful tools in studying properties of that sequence, in particular in finding direct formulas.

As introduced in this paper, distance Fibonacci polynomials by graph interpretation in the lexicographic product of graphs could initiate the study of other Fibonacci type polynomials with respect to graph methods. It seems to be interesting to study other Fibonacci type polynomials which generalize, for example, Lucas, Pell, Padovan numbers with respect to graph interpretations.

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