

Article

# On the Johnson–Tzitzeica Theorem, Graph Theory, and Yang–Baxter Equations

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**Abstract:** This paper presents several types of Johnson–Tzitzeica theorems. Graph diagrams are used in this analysis. A symmetric scheme is derived, and new results are obtained and open problems stated. We also present results relating the graphs and the Yang–Baxter equation. This equation has certain symmetries, which are used in finding solutions for it. All these constructions are related to integrable systems.

**Keywords:** Johnson–Tzitzeica theorem; graph; Yang–Baxter equation

**MSC:** 05C20; 05C25; 05C62; 16T25; 51K10



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## 1. Introduction

There are several purposes of the current work: to present the Johnson–Tzitzeica theorem and related (open) problems, to relate the Johnson–Tzitzeica theorem to graph theory, and to point out possible applications of the Johnson–Tzitzeica theorem in integrable systems and in the Yang–Baxter equation theory. Historical aspects are also considered.

According to [1], a set of Johnson circles refers to three circles of equal radius and a common point of intersection. In the Johnson–Tzitzeica theorem, “the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection. Even if geometers have overlooked it, someone must have noticed it in casually drawing circles. But if this were the case, it seems a theorem of sufficient interest to receive some prominence in the literature, and therefore ought to be well known” (cf. [2]).

In the next section, we present the Johnson–Tzitzeica theorem. A general symmetric scheme is derived, some open problems proposed, and a new theorem proven. There exists a whole field of discrete integrable geometry, where concepts from elementary geometry to nonlinear continuum or discrete systems of equations are studied. The most similar ideas can be found in a series of papers by A. Doliwa, where the elementary incidence geometry is a source of interesting (usually generic) problems in the theory of completely integrable systems. The paper [3] on circular lattices and then Desargues maps entered into the scene, leading to several steps to the Yang–Baxter equations through integrable systems and the famous Hirota–Miwa equations (see, for example, the papers [4–6]).

In Section 1, we associate a graph diagram with the Johnson–Tzitzeica theorem. This point of view eventually leads to new results both in geometry and in graph theory. Section 4 is a gentle introduction to the Yang–Baxter equation (see, for example, [7–12]). This equation has certain symmetries, which are used in finding solutions for it, and it plays an important role in integrable systems. V. Drinfeld (see [13]) suggested considering “set-theoretical” solutions to the Yang–Baxter equation (see also [14,15]). We first restrict our attention to the set-theoretical solutions. We recall that graph diagrams are important in the proof of a theorem and in visualizing some results. We present the Yang–Baxter equation in the vectorial space framework, proposing other similar equations. The relationship among these equations are studied using a graph diagram. Notice that the method presented in

in the beginning of this section could be applied for finding solutions to them. The last section presents some historical aspects related to the Tzitzeica–Johnson theorem, several remarks (on integrable systems), and some research problems.

### 2. The Tzitzeica–Johnson Theorem and Pictorial Mathematics

In a talk at the 12th International Workshop on Differential Geometry and Its Applications, UPG Ploiesti 2015, Florin Caragiu explained that there exists a special mathematical discourse, called “proofs without words”. This discourse uses pictures or diagrams in order to boost the intuition of the reader (see [16,17]). The pictorial (diagrammatic) style of mathematical language is much appreciated by both educators and researchers in mathematics. Being very easy to grasp, some pictorial-style problems are similar to some nuts being small, but for which we need an entire “artillery” in order to crack them.

We now consider the Tzitzeica–Johnson problem (see, for example, [18]).

Let  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$  be three circles of radius  $r$ . Let us suppose that  $\mathcal{X} \cap \mathcal{Y} = \{O, A\}$ ,  $\mathcal{X} \cap \mathcal{Z} = \{O, B\}$ , and  $\mathcal{Y} \cap \mathcal{Z} = \{O, C\}$ . Then, there exists a circle of radius  $r$  through  $A, B,$  and  $C$  (see Figure 1).

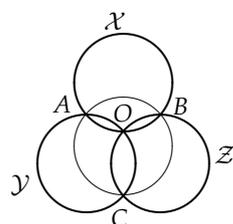


Figure 1. The three coins problem (Tzitzeica 1908; Johnson 1916).

The above theorem can be included in a more general scheme (see Figure 2).

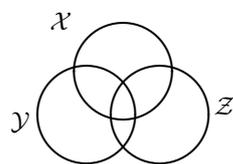


Figure 2. A mathematical scheme. If  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$  have certain properties, then there exists a related  $\mathcal{T}$ , with the same properties.

One could include in this scheme the inclusion–exclusion principle, for three sets:  
 $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.$

Let, for example, the case be as follows:  $|X| = |Y| = |Z| = 7$ ,  $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ , and  $|X \cap Y \cap Z| = 1$ . Then, there exists a set  $T$ , with  $|T| = 7$  that contains elements from all possible (sub)sets.

One could better visualize the relationship between the inclusion–exclusion principle and the mathematical scheme from Figure 2, if we represent the elements of the sets  $X, Y, Z,$  and  $T$  as circular lattice points.

Notice that Tzitzeica–Johnson’s theorem has many generalizations [18] and interpretations related to the above mathematical scheme (from Figure 2).

The dual Desargues theorem can be interpreted in the light of the above scheme by using arrangements of (three) colored lines.

The hypothesis is as follows:

$$\begin{aligned} \mathcal{X} &= \{(d, f, h) \text{ such that } d \cap f \cap h = P\} \\ \mathcal{Y} &= \{(d', f, h') \text{ such that } d' \cap f \cap h' = Q\} \\ \mathcal{Z} &= \{(d'', f, h'') \text{ such that } d'' \cap f \cap h'' = R\} \\ \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} &= \{f\} \\ \mathcal{X} \cap \mathcal{Y} &= \{A, A'\} \cup \{f\} \\ \mathcal{Y} \cap \mathcal{Z} &= \{B, B'\} \cup \{f\} \end{aligned}$$

$$\mathcal{X} \cap \mathcal{Z} = \{C, C'\} \cup \{f\}$$

The conclusion is as follows:

There exists a triple  $(l, m, n)$  through  $A, A', B, B', C,$  and  $C'$ .

Tzitzeica–Johnson’s theorem can be interpreted in terms of disks:

– “If three disks of the same radius have a common point of intersection, then they contain inside of their union a fourth disk with the same radius.”;

– “If three disks  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{Z}$  of the same radius have a common point, then there exists a fourth disk of the same radius which includes  $(\mathcal{X} \cap \mathcal{Y}) \cup (\mathcal{X} \cap \mathcal{Z}) \cup (\mathcal{Y} \cap \mathcal{Z})$ .”

It is an open problem to prove a similar statement for a domain bounded by an arbitrary closed convex curve. In other words, we conjecture that if we consider a domain  $\mathcal{X}$  bounded by an arbitrary closed convex curve and two copies of  $\mathcal{X}$ , denoted by  $\mathcal{Y}$  and  $\mathcal{Z}$ , such that  $\mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} \neq \emptyset$ , then there exists another copy of  $\mathcal{X}$ , denoted by  $\mathcal{T}$ , such that  $(\mathcal{X} \cap \mathcal{Y}) \cup (\mathcal{X} \cap \mathcal{Z}) \cup (\mathcal{Y} \cap \mathcal{Z}) \subset \mathcal{T} \subset \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$ .

Even the case when  $\mathcal{X}$  is a square is very difficult.

We now propose a new construction for a set of Johnson circles (in Figure 3).

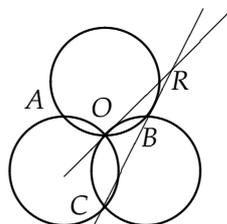


Figure 3. The tangent to the third circle in  $O$  meets the line  $BC$  in  $R$ .

At this moment, we can present a new theorem.

For a set of Johnson circles, let  $R$  be the intersection point between the tangent in  $O$  to the third circle and the line  $BC$ ,  $Q$  the intersection point between the tangent in  $O$  to the second circle and the line  $AC$ , and  $P$  the intersection point between the tangent in  $O$  to the first circle and the line  $AB$ .

Then,  $P, Q,$  and  $R$  are collinear points (see Figure 4 below).

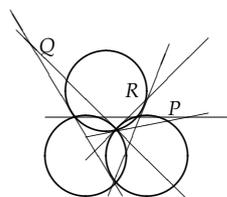


Figure 4. The points  $P, Q,$  and  $R$  are collinear.

**Proof.** The proof of the above theorem is based on the properties of the power of a circle and on Desargues’ theorem. It is the limit case from Figure 5. □

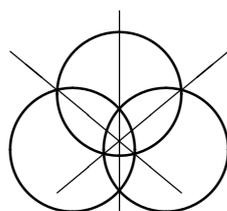
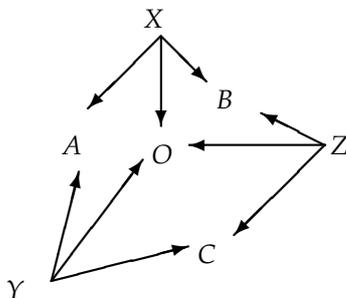


Figure 5. The limit case of the following picture implies our theorem.

It is an open problem to find the relationship between the line through  $P, Q,$  and  $R$  and the circle through  $A, B,$  and  $C$ .

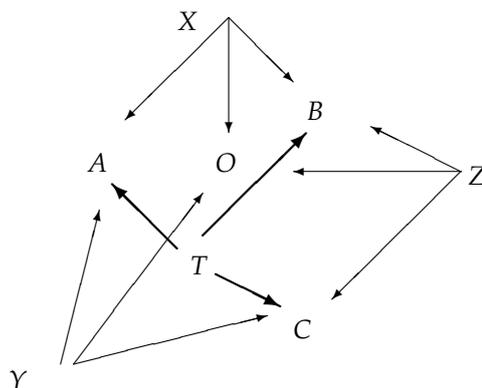
### 3. The Tzitzeica–Johnson Theorem and Graph Diagrams

We would like to transfer the Tzitzeica–Johnson theorem into the graph theory setting. We first associate a graph diagram with the Johnson–Tzitzeica theorem (recall Figure 1). If, we denote by  $X$  the center of the circle  $\mathcal{X}$ , by  $Y$  the center of the circle  $\mathcal{Y}$ , and by  $Z$  the center of the circle  $\mathcal{Z}$ , then we could consider the following graph diagram related to the Johnson–Tzitzeica theorem (see Figure 6).



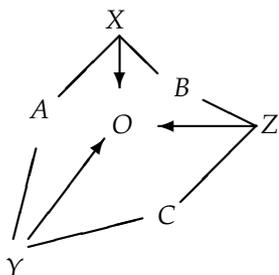
**Figure 6.** The hypothesis of a Johnson–Tzitzeica theorem for graph diagrams: we consider an oriented graph with 7 vertices and 9 oriented edges.

The conclusion of this Johnson–Tzitzeica theorem for graph diagrams is that this graph, from Figure 6, could be embedded in a regular graph. In other words, we would like the in-degree or the out-degree of each vertex of the new graph to be equal to three. The proof is given in Figure 7.



**Figure 7.** Proof of a Johnson–Tzitzeica theorem for graph diagrams: there exist a new vertex  $T$  and 3 oriented arrows making all the vertices in the graph sinks or sources of degree 3.

We consider an example of how this approach could lead to a geometry theorem. We now consider a mixed graph related to a set of Johnson circles (see Figure 8).



**Figure 8.** A mixed graph.

We embed this mixed graph into a (regular) mixed graph (see Figure 9).

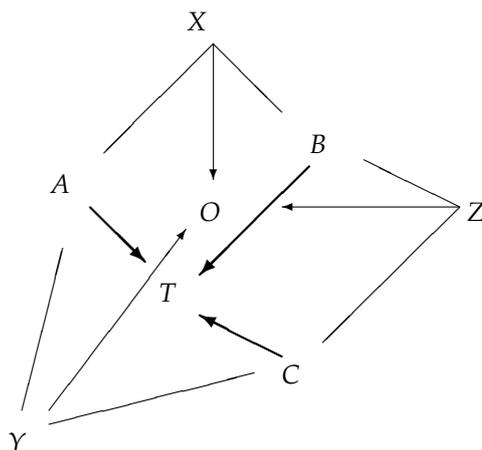


Figure 9. A mixed graph leading to a geometry theorem.

Going back to geometry, we obtain a new theorem.

For a set of Johnson circles, the circles of radius  $r$  with centers  $A, B,$  and  $C$  meet at one point (see Figure 10).

**Proof.** According to the Tzitzeica–Johnson theorem, there exists a circle of radius  $r$  through  $A, B,$  and  $C$ . Let  $T$  be the center of this circle. It follows easily that the circles of radius  $r$  with centers  $A, B,$  and  $C$  meet at the point  $T$  (see Figure 10). □

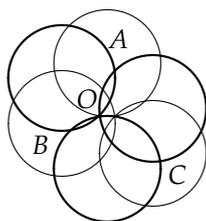


Figure 10. The circles of radius  $r$  with centers  $A, B,$  and  $C$  meet at the point  $T$ .

The question of how to embed an oriented graph in a regular graph (how many vertices need to be added, etc.) could be a research theme in graph theory (with implications for geometry).

#### 4. Yang–Baxter Equations, Relations, and Graphs

If  $X$  is a set, let  $S : X \times X \rightarrow X \times X$  be a function,  $S^{12} = S \times I$  and  $S^{23} = I \times S$ .

Using the above notation, the set-theoretical Yang–Baxter equation reads:

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23} \tag{1}$$

We use the following notation for a relation  $R$  on the set  $X$ : we denote by  $R^{op}$  the opposite relation of  $R$ ; we denote by  $\bar{R}$  the complementary relation of  $R$ .

(D. Hobby and F. F. Nichita [19]) Let  $X$  be a set and  $R \subset X \times X$  a reflexive relation on  $X$ . We define the function  $S = S_R : X \times X \rightarrow X \times X$  by  $S(u, v) = \begin{cases} (u, v), & \text{if } (u, v) \in R; \\ (v, u), & \text{otherwise.} \end{cases}$

Then,  $S$  satisfies (1) if and only if  $R \cup R^{op}$  is an equivalence relation and  $\bar{R}$  is a strict partial order relation on each class of  $R \cup R^{op}$ .

The proof from [19] relies heavily on graph diagrams. After many cases are investigated, the theorem is proven. An example using Hasse diagrams and graphs is then presented.

For each of the relations represented by the graphs from Figures 6 and 7, if we consider  $S_{\bar{R}}$  ( $S$  for the complementary relation), we obtain a solution for (1).

We continue the section working over a generic field  $k$ . The tensor products are defined over  $k$ . Let  $V$  be a vector space over  $k$ . Let  $I = I_V : V \rightarrow V$  be the identity map of the space  $V$ . We denote by  $\tau : V \otimes V \rightarrow V \otimes V$  the twist map defined by  $\tau(v \otimes_B w) = w \otimes_B v$ . For  $R : V \otimes_B V \rightarrow V \otimes_B V$ , a  $k$ -linear map, let  $R^{12} = R \otimes_B I$ ,  $R^{23} = I \otimes_B R : V \otimes_B V \otimes_B V \rightarrow V \otimes_B V \otimes_B V$ . In a similar manner, we denote by  $R^{13}$  a linear map acting on the first and third component of  $V \otimes V \otimes V$ . It turns out that  $R^{13} = (I \otimes_B \tau)(R \otimes_B I)(I \otimes_B \tau)$ .

A Yang–Baxter operator is a  $k$ -linear map  $R : V \otimes_B V \rightarrow V \otimes_B V$ , which satisfies the braid condition (the Yang–Baxter equation):

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{2}$$

Usually, we also require that the map  $R$  be invertible.

An important observation is that if  $R$  satisfies (2), then both  $R \circ \tau$  and  $\tau \circ R$  satisfy the quantum Yang–Baxter equation (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{3}$$

We now consider other (systems of) equations for a  $k$ -linear map  $R : V \otimes_B V \rightarrow V \otimes_B V$ :

$$R^{12} \circ R^{23} = R^{13} \circ R^{12} = R^{23} \circ R^{13} \tag{4}$$

$$R^{23} \circ R^{12} = R^{12} \circ R^{13} \tag{5}$$

$$R^{12} \circ R^{13} \circ R^{12} \circ R^{23} = R^{23} \circ R^{13} \circ R^{23} \circ R^{12} \tag{6}$$

$$(R^{12} \circ R^{23} \circ R^{12} - R^{23} \circ R^{12} \circ R^{23}) \circ (R^{12} \circ R^{13} \circ R^{23} - R^{23} \circ R^{13} \circ R^{12}) = 0 \tag{7}$$

$$R = XY, \quad R^{12} \circ X^{13} \circ R^{23} \circ Y^{12} = R^{23} \circ X^{13} \circ R^{12} \circ Y^{23} \tag{8}$$

$$R \circ R = I \otimes I \tag{9}$$

$$R^{12} R^{23} R^{13} R^{23} + R^{23} R^{12} R^{13} R^{12} = R^{23} R^{12} R^{23} R^{12} R^{13} R^{23} + R^{12} R^{23} R^{12} R^{23} R^{13} R^{12} \tag{10}$$

(i) If a  $k$ -map  $R : V \otimes_B V \rightarrow V \otimes_B V$  verifies (4) and (5), then  $R$  is a common solution for (2) and (3);

(ii) If a  $k$ -map  $R : V \otimes_B V \rightarrow V \otimes_B V$  verifies (2) or (3), then  $R$  is a solution for (7);

(iii) If a  $k$ -map  $R : V \otimes_B V \rightarrow V \otimes_B V$  verifies (2) or (3), then it is a solution for (8);

(iv) If a  $k$ -map  $R : V \otimes_B V \rightarrow V \otimes_B V$  verifies (2), (3) and (9), then it is a solution for (6);

(v) If a  $k$ -map  $R : V \otimes_B V \rightarrow V \otimes_B V$  verifies (7) and (9), then it is a solution for (10).

**Proof.** (i) We only sketch the proof of the first claim:  $R^{23} \circ R^{12} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} = R^{12} \circ R^{23} \circ R^{12}$ , etc (see [12]).

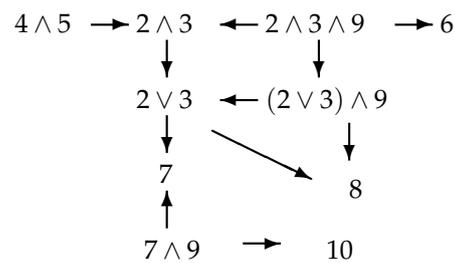
(ii) The proof is direct;

(iii) For  $X = R$  and  $Y = I \otimes_B I$ , this equation becomes Equation (3). For  $X = I \otimes_B I$  and  $Y = R$ , this equation becomes Equation (3);

(iv)  $R^{23} \circ R^{13} \circ R^{23} \circ R^{12} = R^{23} \circ R^{13} \circ R^{12} \circ R^{12} \circ R^{23} \circ R^{12} = R^{12} \circ R^{13} \circ R^{23} \circ R^{23} \circ R^{12} \circ R^{23} = R^{12} \circ R^{13} \circ R^{12} \circ R^{23}$ ;

(v) The proof is direct.  $\square$

We could present the above results using a graph diagram (see Figure 11).



**Figure 11.** A graph diagram describing the relations among the above equations.

### 5. Applications, Historical Aspects, and Final Comments

The Romanian mathematician Gheorghe Tzitzeica is also known for a result on the geometry of circles and triangles in the plane, referred to as Tzitzeica’s five lei coin problem, a problem he proposed (and solved) at the “Gazeta Matematica” contest in Galati in 1908. The problem was posed independently by Roger Arthur Johnson in 1916 [20,21].

There exists a whole field of discrete integrable geometry, where similar concepts, from elementary geometry to nonlinear continuum or discrete systems of equations, have been studied for many years. The paper [3] on circular lattices where the Miguel theorem plays the crucial role and then Desargues maps entered into the scene, leading to several steps to the Yang–Baxter equations. It can be shown that the Miguel theorem implies the dual of the Johnson–Tzitzeica theorem (in this case we take into consideration the midpoints of the edges of the given triangle).

Related to another section of the current paper, one could attempt to find solutions for Equations (4)–(10) using the function  $S_R$  (see Theorem 4).

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