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Weighted Sobolev–Morrey Estimates for Nondivergence Degenerate Operators with Drift on Homogeneous Groups

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Abstract: Let $X_0, X_1, \dots, X_q$ ($q < N$) be real vector fields, which are left invariant on homogeneous group G , provided that X_0 is homogeneous of degree two and $X_1, \dots, X_q$ are homogeneous of degree one. We consider the following nondivergence degenerate operator with drift $L = \sum_{i,j=1}^q a_{ij}(x) X_i X_j + a_0(x) X_0$, where the coefficients $a_{ij}(x), a_0(x)$ belonging to vanishing mean oscillation space are bounded measurable functions. Furthermore, $a_{ij}(x)$ satisfies the uniform ellipticity condition on $\mathbb{R}^q$ and $a_0(x) \neq 0$. We obtain the local weighted Sobolev–Morrey estimates by applying the boundedness of commutators and interpolation inequalities on weighted Morrey spaces.

Keywords: nondivergence operator; weighted Sobolev–Morrey estimates; homogeneous group; singular integral; commutators



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1. Introduction

Agmon, Douglis, and Nirenberg [1] investigated the uniformly elliptic operator in nondivergence form with continuous coefficients and derived the classical L^p estimates. Then, fairly natural questions are whether the conclusion in [1] remains valid when the assumption on the coefficients becomes weaker or the uniformly elliptic operator changes to various degenerate operators including degenerate elliptic operators and parabolic operators formed by vector fields. To the first question, based on the theory of singular integrals by Calderón and Zygmund [2] and the commutators of singular integrals with BMO functions (see Definition 5) by Coifman, Rochberg, and Weiss [3], the study of equations with VMO (see Definition 5) coefficients was initiated by Chiarenza–Frasca–Longo [4] and Bramanti–Cerutti [5], who derived L^p estimates for elliptic equations and parabolic equations, respectively. As a generalization of the space L^p , Morrey spaces, originally introduced in [6], play an important role in the local behaviors of solutions to second-order elliptic equations; see [7]. The boundedness in Morrey spaces for some important operators in harmonic analysis, for instance the fractional integral operators, the Hardy–Littlewood maximal operators, and the singular integral operators, was treated [8,9]. Komori and Shirai [10] defined the weighted Morrey spaces, which extend the weighted Lebesgue spaces, and obtained estimates for the operators mentioned above in the weighted Morrey spaces. Furthermore, Di Fazio and Ragusa [11] investigated the local regularity in Morrey spaces of solutions to elliptic operators in nondivergence form with VMO coefficients; see [12–17] for more related results.

The second question owns a wide range of interest, since there exist many degenerate operators. The object of the paper is a class of nondivergence degenerate operators with drift on homogeneous groups. Motivated by [10], and so on, we introduce the boundedness of commutators of Calderón–Zygmund operators with BMO functions on weighted Morrey spaces in the framework of homogeneous groups. As an application, we derive weighted Sobolev–Morrey estimates for nondivergence degenerate operators with drift.

Let $X_0, X_1, \dots, X_q$ ($q < N$) be real vector fields, which are left invariant on homogeneous group G . Provided that X_0 is homogeneous of degree two and $X_1, \dots, X_q$ are homogeneous of degree one, they satisfy the Hörmander condition:

$$\text{rank} \mathcal{L}(X_0, X_1, \dots, X_q)(z) = N, \quad z \in G,$$

where the notation $\mathcal{L}(X_0, X_1, \dots, X_q)$ is the Lie algebra formed with $X_0, X_1, \dots, X_q$. We consider the following nondivergence degenerate operators with drift:

$$L = \sum_{i,j=1}^q a_{ij}(x) X_i X_j + a_0(x) X_0, \quad (1)$$

where the coefficients $a_{ij}(x), a_0(x) \in VMO$ are bounded measurable functions. Moreover, there exists a constant $\mu > 0$ such that for all $\eta \in \mathbb{R}^q$ and *a.e.* $x \in \Omega$, where Ω is a bounded domain in G ,

$$\mu^{-1} |\eta|^2 \leq \sum_{i,j=1}^q a_{ij}(x) \eta_i \eta_j \leq \mu |\eta|^2, \quad (2)$$

$$\mu^{-1} \leq a_0(x) \leq \mu. \quad (3)$$

Hörmander put forward the operators of the sum of squares in his classic thesis [18] and gave a sufficient condition for the hypoellipticity, which is called the Hörmander condition. Since then, many authors paid attention to regularity to degenerate elliptic operators of Hörmander's vector fields (see [19]). Folland [20] investigated that any Hörmander-type operator similar to (1) has a fundamental solution, which is unique and homogeneous. Based on the singular integrals theory, Bramanti and Brandolini [21] derived L^p estimates of L in (1). Recently, many authors have also been devoted to studying a priori estimates of L (see [22–24]). We quote that L arises in the geometry of several complex variables and in models of human vision, which can be seen as a natural generalization of the Laplace operator and parabolic operator in Euclidean spaces. For example, if $a_{ij}(x) = \delta_{ij}, X_i = \partial_{x_i}, X_0 = 0$, then L is the common Laplace operator Δ ; if $a_{ij}(x) = \delta_{ij}, a_0 = 1, X_i = \partial_{x_i}, X_0 = \partial_t$, then L is the parabolic operator $\Delta + \partial_t$. Another special and important example of L is the Kolmogorov–Fokker–Planck ultraparabolic operator of the form:

$$L_1 u = \sum_{i,j=1}^q a_{ij} \partial_{x_i x_j}^2 u + \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} u - \partial_t u,$$

where $(x, t) \in \mathbb{R}^{n+1}, X_i = \partial_{x_i}, a_0(x) X_0 = \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t$, the matrix $(a_{ij})_{i,j=1}^q$ is symmetric and uniformly elliptic, and (b_{ij}) is a constant coefficients matrix with a suitable upper triangular structure. The operator L_1 involves kinetics and finance, which is still extensively studied, referring to [25–27].

In this article, we deal with weighted Sobolev–Morrey estimates of the operator L in the bounded domain $\Omega \subset G$. More precisely, we have the following main results.

Before we state our results, we give some notions: C_0^∞ means infinitely continuously differentiable functions, which have compact support. $\text{supp } u = \overline{\{x \in \Omega : u(x) \neq 0\}}$.

Theorem 1. Let $p \in (1, \infty)$. For every $u \in C_0^\infty(G)$ with $\text{supp } u \subseteq B_r$, there exist $c > 0$ and $r > 0$ such that:

$$\|D^2 u\|_{L^{p,\kappa}(w, B_r)} \leq c \|Lu\|_{L^{p,\kappa}(w, B_r)}, \quad (4)$$

where $\|D^2 u\|_{L^{p,\kappa}(w, G)} = \sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w, G)} + \|X_0 u\|_{L^{p,\kappa}(w, G)}.$

Theorem 2. Let $\Omega' \subset \subset \Omega$ (Ω' is a compact subset of Ω); if $1 < p < \infty$, $0 < \kappa < 1$, and $w \in A_p$, then for any $u \in S^{2,p,\kappa}(w, \Omega)$, there exists $c > 0$ such that:

$$\|u\|_{S^{2,p,\kappa}(w, \Omega')} \leq c \left(\|Lu\|_{L^{p,\kappa}(w, \Omega)} + \|u\|_{L^{p,\kappa}(w, \Omega)} \right), \quad (5)$$

where:

$$\|u\|_{S^{2,p,\kappa}(w,G)} = \|u\|_{L^{p,\kappa}(w,G)} + \|X_i u\|_{L^{p,\kappa}(w,G)} + \sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w,G)} + \|X_0 u\|_{L^{p,\kappa}(w,G)}.$$

A remarkable fact is that since the coefficients of L are nonsmooth (the coefficients are not infinitely continuously differentiable functions), the hypoellipticity makes no sense for L in general. However, the constant coefficients' operator L_0 induced by freezing the coefficients $a_0(x)$ and $a_{ij}(x)$ at some fixed point $x_0 \in G$ is always hypoelliptic and has a homogeneous fundamental solution. Furthermore, the second-order derivatives of vector fields for every test function can be expressed as singular integrals with respect to the homogeneous fundamental solution and corresponding commutators (see [21]).

The plan of this paper is as follows. We give the preliminary knowledge on homogeneous groups, weight functions, and the homogeneous differential operator in Section 2. In Section 3, we introduce the boundedness of commutators and establish a Morrey interpolation lemma. The proofs of the main results are given in Section 4 based on the results in previous sections.

2. Preliminaries

Let us recall some knowledge on homogeneous groups; see [28–30] for more details. The space $\mathbb{R}^N$ with the following two smooth mappings:

$$x \mapsto x^{-1} : \mathbb{R}^N \mapsto \mathbb{R}^N;$$

$$(x, y) \mapsto x \circ y : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N;$$

becomes a group. What is more, the origin is the identity. Let $0 < \omega_1 \leq \omega_2 \leq \dots \leq \omega_N$, if the dilations:

$$\tau(\lambda) : (x_1, \dots, x_N) \mapsto (\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_N} x_N), \text{ for any } \lambda > 0$$

are group automorphisms. Thus, $\mathbb{R}^N$ with the dilations is said to be a homogeneous group, which contains the Euclidean space, homogeneous Carnot group, homogeneous Heisenberg group etc. The homogeneous group is symmetric, and we denote it by G .

The positive integer $Q = \omega_1 + \dots + \omega_N$ is said to be the homogeneous dimension of G . The restricted condition is $Q > 4$ in this paper (for the rationality of $Q > 4$, see Example 2 in [21]). Then, (G, dx, d) is a homogeneous space.

The homogeneous norm $\|\cdot\|$ of G is defined as follows: if $x \in G$ and $x \neq 0$, then it holds:

$$\|x\| = \rho \Leftrightarrow |\tau(1/\rho)x| = 1,$$

with the notation $|\cdot|$ being the Euclidean norm; in addition, we set $\|0\| = 0$.

Furthermore, we give the following definition of quasidistance d naturally,

$$d(x, y) = \|y^{-1} \circ x\|;$$

the quasidistance d is quasisymmetric. Setting $d'(x, y) = d(x, y) + d(y, x)$, we obtain a symmetric quasidistance d' , which is equivalent to d .

In order to investigate the boundedness of Hardy–Littlewood maximal operators in weighted L^p spaces, Muckenhoupt [31] introduced A_p classes. Analogously, we define A_p classes on homogeneous groups as follows:

Definition 1 (A_p class). Assume that $1 < p < \infty$; we say a weight w belongs to the class A_p , if there exists a constant $c > 1$ such that:

$$\sup_{B \subset G} \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq c,$$

with p' the conjugate exponent to p (i.e., $1/p + 1/p' = 1$).

In the case of $p = 1$, let M be the Hardy–Littlewood maximal operator; a weight w is said to be in the class A_1 , if there exists a constant $c > 1$ such that:

$$Mw(x) \leq cw(x).$$

In the case of $p = \infty$, it is defined by $A_\infty = \cup_{1 < p < \infty} A_p$.

Definition 2. Let Y be a differential operator on G ; we say that Y is homogeneous of degree b ($b > 0$), if it holds for any test function u , $\lambda > 0$, and $x \in G$,

$$Y(u(\tau(\lambda)x)) = \lambda^b (Yu)(\tau(\lambda)x);$$

we say that g is homogeneous of degree a , if it holds for any $\lambda > 0$, $x \in G$,

$$g((\tau(\lambda)x)) = \lambda^a g(x).$$

Let g be a homogeneous function of degree a and Y be a homogeneous differential operator of degree b . It is not difficult to verify that Yg is homogeneous of degree $a - b$.

Definition 3. Assume that w is a weight; for $1 \leq p < \infty$ and $0 < \kappa < 1$, the weighted Morrey space $L^{p,\kappa}(w, G)$ consists of all functions $f \in L^p_{loc}(w, G)$ for which:

$$\|f\|_{L^{p,\kappa}(w,G)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(y)|^p w(y) dy \right)^{1/p}$$

is finite.

Notice that if $0 < \lambda < Q$, $\kappa = \lambda/Q$ and $w \equiv 1$, then $L^{p,\kappa}(w, G)$ is the common Morrey spaces $L^{p,\lambda}(G)$. To be more convenient, we give the following notations.

$$\|Df\|_{L^{p,\kappa}(w,G)} = \sum_{i=1}^q \|X_i f\|_{L^{p,\kappa}(w,G)},$$

$$\|D^2 f\|_{L^{p,\kappa}(w,G)} = \sum_{i,j=1}^q \|X_i X_j f\|_{L^{p,\kappa}(w,G)} + \|X_0 f\|_{L^{p,\kappa}(w,G)}.$$

Without loss of generality, we set:

$$\|D^h f\|_{L^{p,\kappa}(w,G)} = \sum \|X_{k_1} \cdots X_{k_l} f\|_{L^{p,\kappa}(w,G)}, \quad (6)$$

with $X_{k_1} \cdots X_{k_l}$ homogeneous of degree h , which is denoted by P^h (notice that X_0 is homogeneous of degree two, whereas the others are homogeneous of degree one).

Definition 4. Let k be a nonnegative integer and w be a weight; if $1 \leq p < \infty$, $0 < \kappa < 1$, the weighted Sobolev–Morrey space $S^{k,p,\kappa}(w, G)$ is defined as follows:

$$S^{k,p,\kappa}(w, G) = \left\{ f \in L^{p,\kappa}(w, G) : \|f\|_{S^{k,p,\kappa}(w,G)} < \infty \right\},$$

where:

$$\|f\|_{S^{k,p,\kappa}(w,G)} = \sum_{i=0}^k \|D^i f\|_{L^{p,\kappa}(w,G)}.$$

Definition 5. For a locally integrable function f , set:

$$\eta_f(R) = \sup_{r \leq R} \frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}| dx,$$

where B_r is the ball of radius r and f_{B_r} is the average of f over B_r .

Let $\|f\|_* = \sup_R \eta_f(R)$; if $\|f\|_* < \infty$, we say $f \in BMO$ (bounded mean oscillation; see [32]).

If $f \in BMO$ and $\lim_{R \rightarrow 0} \eta_f(R) = 0$, we say $f \in VMO$ (vanishing mean oscillation; see [33]).

For a domain $\Omega \subset G$, we can similarly define the spaces $BMO(\Omega)$ and $VMO(\Omega)$, just replacing B_r with $B_r \cap \Omega$.

3. Commutators and Weighted Interpolation Inequalities

Definition 6. Let b be a locally integrable function and T be a linear operator; the commutator $[b, T]$ is defined by:

$$[b, T]g(x) = b(x)Tg(x) - T(bg)(x).$$

It is easy to verify that these commutators are antisymmetric.

For any $x_0 \in G$, let us freeze the coefficients $a_0(x)$, $a_{ij}(x)$ at x_0 , and then, the frozen operator:

$$L_0 = \sum_{i,j=1}^q a_{ij}(x_0) X_i X_j + a_0(x_0) X_0$$

is always hypoelliptic. The following results can be found in [20,21]:

Lemma 1. The left invariant differential operator L_0 is a homogeneous operator with degree two, which has a fundamental solution $\Gamma(x_0; \cdot)$; moreover, the following hold for every $x \in G$ and every test function u :

- (a) $\Gamma(x_0; \cdot)$ is a homogeneous function with degree $2 - Q$;
- (b) $\Gamma(x_0; \cdot)$ is an infinitely continuously differentiable function outside the origin;
- (c) $u(x) = (L_0 u * \Gamma)(x) = \int_G \Gamma(x_0; y^{-1} \circ x) L_0 u(y) dy$.

$$\begin{aligned} X_i X_j u(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\|y^{-1} \circ x\| > \varepsilon} \Gamma_{ij}(x_0; y^{-1} \circ x) L_0 u(y) dy + \beta_{ij}(x_0) L_0 u(x) \\ &= P.V. \int_G \Gamma_{ij}(x_0; y^{-1} \circ x) L_0 u(y) dy + \beta_{ij}(x_0) L_0 u(x), \end{aligned} \quad (7)$$

where $i, j = 1, \dots, q$, β_{ij} are constants that satisfy uniformly bounded and $\Gamma_{ij} = X_i X_j \Gamma$;

(d) $\Gamma_{ij}(x_0; \cdot)$ is a homogeneous function with degree $-Q$;

(e) $\Gamma_{ij}(x_0; \cdot)$ is an infinitely continuously differentiable function outside the origin.

Let $L_0 = L + (L_0 - L)$ and $x = x_0$, $u \in C_0^\infty(G)$. We obtain the representation formula from (7) that:

$$\begin{aligned} X_i X_j u(x) &= P.V. \int_G \Gamma_{ij}(x; y^{-1} \circ x) \left\{ \sum_{h,k=1}^q [a_{hk}(x) - a_{hk}(y)] X_h X_k u(y) \right. \\ &\quad \left. + [a_0(x) - a_0(y)] X_0 u(y) + Lu(y) \right\} dy + \beta_{ij}(x) Lu(x). \end{aligned} \quad (8)$$

For the sake of simplicity, we define a singular integral operator as follows:

$$K_{ij}g(x) = P.V. \int_G \Gamma_{ij}(x; y^{-1} \circ x) g(y) dy, \quad (9)$$

and the commutator of singular integral operator K with a BMO function a :

$$[a, K]g(x) = a(x)Kg(x) - K(ag)(x). \quad (10)$$

Then, (8) can be rewritten in a simpler form:

$$X_i X_j u = \sum_{h,k=1}^q [a_{hk}, K_{ij}](X_h X_k u) + [a_0, K_{ij}](X_0 u) + K_{ij}(Lu) + \beta_{ij}(x)Lu. \quad (11)$$

Definition 7. If $g \in L^1_{loc}(G)$, then the Calderón–Zygmund operator T on G is defined as:

$$Tg(z) = P.V. \int_G K(y^{-1} \circ z)g(y)dy,$$

with K satisfying the following two constraints:

$$|K(z)| \leq \frac{c}{\|z\|^Q}; \quad |\nabla K(z)| \leq \frac{c}{\|z\|^{Q+1}}, \quad z \neq 0.$$

The boundedness of the Calderón–Zygmund operator on weighted Morrey spaces was derived.

Lemma 2. (Ref. [34]) Assume T is a Calderón–Zygmund operator and $w \in A_p$, if $1 < p < Q$, $0 < \kappa < 1$, then T is bounded on the weighted Morrey space $L^{p,\kappa}(w, G)$.

Komori and Shirai in [10] derived the boundedness of commutators of Calderón–Zygmund operators with BMO functions on Euclidean spaces. It is not difficult to verify that this result remains valid on homogeneous groups. Here, we list it as a lemma for later use.

Lemma 3. Let T be a Calderón–Zygmund operator and $b \in BMO(G)$. If $1 < p < Q$, $w \in A_p$ and $0 < \kappa < 1$, then for every $f \in L^{p,\kappa}(w, G)$, the following estimate holds:

$$\|[b, T]f\|_{L^{p,\kappa}(w, G)} \leq c\|b\|_* \|f\|_{L^{p,\kappa}(w, G)}, \quad (12)$$

where c is a positive constant.

Using Lemma 3 and the way of proving Theorem 2.13 in [4], we directly conclude about the following localized result.

Lemma 4. Under the assumptions of Lemma 3, if $b \in VMO(G)$, then for any $\varepsilon > 0$, there exists positive constants r and c such that for every $f \in L^{p,\kappa}(w, G)$ with $\text{supp } f \subseteq B_r$,

$$\|[b, T]f\|_{L^{p,\kappa}(w, B_r)} \leq c\varepsilon \|f\|_{L^{p,\kappa}(w, B_r)}.$$

Given two balls B_{r_1}, B_{r_2} and a function $\varphi \in C_0^\infty(G)$, then the notation $B_{r_1} \prec \varphi \prec B_{r_2}$ indicates that $0 \leq \varphi(x) \leq 1$, $\varphi(x) \equiv 1$ ($x \in B_{r_1}$) and $\text{supp } \varphi \subseteq B_{r_2}$.

Lemma 5 (Ref. [21]). If k is a positive integer, for every $\theta \in (0, 1)$ and every $r > 0$, there exists $\varphi \in C_0^\infty(G)$ satisfying the following properties:

$$B_{\theta r} \prec \varphi \prec B_{\theta' r}, \quad \text{for } \theta' = (1 + \theta)/2,$$

$$|P^i \varphi| \leq \frac{c}{\theta^{i-1}(1 - \theta)^i r^i}, \quad 1 \leq i \leq k,$$

where P^i is any left invariant differential monomial with homogeneous degree of i .

The following interpolation inequality has been proven.

Lemma 6 (Ref. [34]). If $w \in A_p$, $1 < p < Q$, and $0 < \kappa < 1$, then for every $\varepsilon > 0$ and every test function u , there is a constant $c > 0$ such that:

$$\|Du\|_{L^{p,\kappa}(w,G)} \leq \delta \|D^2u\|_{L^{p,\kappa}(w,G)} + \frac{c}{\delta} \|u\|_{L^{p,\kappa}(w,G)}.$$

For every $u \in S^{2,p,\kappa}(w, B_r)$, the seminorm is defined as follows:

$$\Phi_i = \sup_{1/2 < \theta < 1} \left((1-\theta)^i r^i \|D^i u\|_{L^{p,\kappa}(w, B_{r\theta})} \right), \quad i = 0, 1, 2.$$

Further, the following weighted interpolation lemma can be derived.

Lemma 7. There exists $c > 0$ such that for any $\varepsilon > 0$,

$$\Phi_1 \leq \varepsilon \Phi_2 + c(\varepsilon) \Phi_0. \quad (13)$$

Proof. Let φ be a test function in Lemma 5, then $u\varphi \in S_0^{2,p,\kappa}(w, B_{r\theta'})$; we have by using Lemmas 5 and 6,

$$\begin{aligned} & \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \leq \|D(u\varphi)\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \leq \delta \|D^2(u\varphi)\|_{L^{p,\kappa}(w, B_{\theta'r})} + \frac{c}{\delta} \|(u\varphi)\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \leq \delta \left(\varphi \|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + 2\|DuD\varphi\|_{L^{p,\kappa}(w, B_{\theta'r})} + \|uD^2\varphi\|_{L^{p,\kappa}(w, B_{\theta'r})} \right) \\ & \quad + \frac{c}{\delta} \|u\varphi\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \leq \delta \left(\|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + \frac{c}{(1-\theta)r} \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} + \frac{c}{\theta(1-\theta)^2 r^2} \|u\|_{L^{p,\kappa}(w, B_{\theta'r})} \right) \\ & \quad + \frac{c}{\delta} \|u\|_{L^{p,\kappa}(w, B_{\theta'r})}. \end{aligned} \quad (14)$$

Then, we multiply both sides in (14) by $(1-\theta)r$,

$$\begin{aligned} (1-\theta)r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} & \leq \delta(1-\theta)r \|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + c\delta \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \quad + c \left(\frac{\delta}{\theta(1-\theta)r} + \frac{(1-\theta)r}{\delta} \right) \|u\|_{L^{p,\kappa}(w, B_{\theta'r})}. \end{aligned}$$

Choosing $\delta = \varepsilon\theta(1-\theta)r$, we immediately derive:

$$\begin{aligned} (1-\theta)r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} & \leq \varepsilon\theta(1-\theta)^2 r^2 \|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + c\varepsilon\theta(1-\theta)r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \quad + c \left(\varepsilon + \frac{1}{\varepsilon\theta} \right) \|u\|_{L^{p,\kappa}(w, B_{\theta'r})}. \end{aligned} \quad (15)$$

It is easy to see that $1-\theta=2(1-\theta')$, then:

$$\begin{aligned} (1-\theta)r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} & \leq 4\varepsilon\theta(1-\theta')^2 r^2 \|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + c\varepsilon\theta(1-\theta')r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \quad + c \left(\varepsilon + \frac{1}{\varepsilon\theta} \right) \|u\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \leq 4\varepsilon\theta(1-\theta')^2 r^2 \|D^2u\|_{L^{p,\kappa}(w, B_{\theta'r})} + c\varepsilon\theta(1-\theta')r \|Du\|_{L^{p,\kappa}(w, B_{\theta'r})} \\ & \quad + c \left(\varepsilon + \frac{1}{\varepsilon\theta} \right) \|u\|_{L^{p,\kappa}(w, B_{\theta'r})}. \end{aligned}$$

This implies:

$$\Phi_1 \leq 4\varepsilon\Phi_2 + c\varepsilon\Phi_1 + c\left(\varepsilon + \frac{2}{\varepsilon}\right)\Phi_0,$$

and consequently,

$$\Phi_1 \leq \frac{4\varepsilon}{1 - c\varepsilon}\Phi_2 + \frac{c}{1 - c\varepsilon}\left(\varepsilon + \frac{2}{\varepsilon}\right)\Phi_0.$$

Since ε is small enough, this shows:

$$\Phi_1 \leq \varepsilon\Phi_2 + \frac{c}{\varepsilon}\Phi_0, \quad (16)$$

and the proof is complete. $\square$

4. Proofs of the Main Theorems

Lemma 8 (Ref. [22]). If $K \in C(G \setminus \{0\})$ is a homogeneous function with degree β and $\beta \in \mathbb{R}$, thus:

$$|K(x)| \leq c\|x\|^\beta,$$

where $c = \sup_{x \in \Sigma} K(x)$ and the notation Σ is the unit sphere of homogeneous groups.

Assume that $K(\cdot)$ is a homogeneous function with degrees $-Q$; it is obvious that

$$Tg(z) = P.V. \int_G K(z, y^{-1} \circ z)g(y)dy$$

is a Calderón–Zygmund operator.

Lemma 9 (Refs. [20,21]). Suppose that $i \in (0, Q)$ is an integer and $K_i \in C^\infty(G \setminus \{0\})$ is homogeneous of degree $i - Q$; define an operator T_i by:

$$T_i g = g * K_i.$$

If P^i is a left invariant differential operator, which is homogeneous of degree i , it holds:

$$P^i T_i g = P.V. (g * P^i K_i) + cg,$$

where the constant c depends on K_i and P^i .

Proof of Theorem 1. Let us start from (11),

$$X_i X_j u = \sum_{h,k=1}^q [a_{hk}, K_{ij}](X_h X_k u) + [a_0, K_{ij}](X_0 u) + K_{ij}(Lu) + \beta_{ij}(x)Lu.$$

Since K_{ij} is a Calderón–Zygmund operator, we use Lemma 2 for K_{ij} and Lemma 4 for $[a_{hk}, K_{ij}]$ and $[a_0, K_{ij}]$, respectively, and obtain that there exists $c > 0$ such that for arbitrary $\varepsilon > 0$ and arbitrary $u \in C_0^\infty(G)$ with: $\text{supp } u \subseteq B_r$,

$$\|[a_{hk}, K_{ij}](X_h X_k u)\|_{L^{p,\kappa}(w, B_r)} \leq c\varepsilon \|X_h X_k u\|_{L^{p,\kappa}(w, B_r)},$$

$$\|[a_0, K_{ij}](X_0 u)\|_{L^{p,\kappa}(w, B_r)} \leq c\varepsilon \|X_0 u\|_{L^{p,\kappa}(w, B_r)},$$

$$\|K_{ij}(Lu)\|_{L^{p,\kappa}(w, B_r)} \leq c \|Lu\|_{L^{p,\kappa}(w, B_r)}.$$

Referring to the results of [21],

$$\sup_{x \in G} |\beta_{ij}(x)| \leq c,$$

then:

$$\|\beta_{ij}(Lu)\|_{L^{p,\kappa}(w, B_r)} \leq c\|Lu\|_{L^{p,\kappa}(w, B_r)},$$

and as a consequence,

$$\begin{aligned} \|X_i X_j u\|_{L^{p,\kappa}(w, B_r)} &\leq c\varepsilon \left(\sum_{h,k=1}^q \|X_h X_k u\|_{L^{p,\kappa}(w, B_r)} + \|X_0 u\|_{L^{p,\kappa}(w, B_r)} \right) \\ &\quad + c\|Lu\|_{L^{p,\kappa}(w, B_r)}. \end{aligned} \quad (17)$$

By the difference, we obtain:

$$X_0 u = \frac{1}{a_0(x)} \left(Lu - \sum_{i,j=1}^q a_{ij}(x) X_i X_j u \right).$$

Note that $a_{ij}(x)$ are bounded and:

$$\mu^{-1} \leq a_0(x) \leq \mu;$$

this shows that:

$$\|X_0 u\|_{L^{p,\kappa}(w, B_r)} \leq c \left(\|Lu\|_{L^{p,\kappa}(w, B_r)} + \sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w, B_r)} \right). \quad (18)$$

Putting (18) into (17), this implies:

$$\sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w, B_r)} \leq c\varepsilon \sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w, B_r)} + c\|Lu\|_{L^{p,\kappa}(w, B_r)},$$

and for some proper small $\varepsilon > 0$,

$$\sum_{i,j=1}^q \|X_i X_j u\|_{L^{p,\kappa}(w, B_r)} \leq c\|Lu\|_{L^{p,\kappa}(w, B_r)}. \quad (19)$$

Hence, it follows from (18) and (19) that:

$$\|X_0 u\|_{L^{p,\kappa}(w, B_r)} \leq c\|Lu\|_{L^{p,\kappa}(w, B_r)}, \quad (20)$$

and Theorem 1 is proven by (19) and (20). $\square$

Proof of Theorem 2. Let $u \in S^{2,p,\kappa}(w, \Omega)$. There exist $r_0 > 0$ and $c > 0$ such that for any $r < r_0$, $B_r \subset \Omega$, Theorem 1 holds. By density, we can apply Theorem 1 to $u\varphi \in S_0^{2,p,\kappa}(w, B_r)$ with φ being a test function in Lemma 5,

$$\begin{aligned} &\|D^2 u\|_{L^{p,\kappa}(w, B_{\theta r})} \\ &\leq c\|L(u\varphi)\|_{L^{p,\kappa}(w, B_{\theta' r})} \\ &\leq c \left(\|\varphi Lu\|_{L^{p,\kappa}(w, B_{\theta' r})} + \|Du D\varphi\|_{L^{p,\kappa}(w, B_{\theta' r})} + \|u L\varphi\|_{L^{p,\kappa}(w, B_{\theta' r})} \right) \\ &\leq c \left(\|Lu\|_{L^{p,\kappa}(w, B_{\theta' r})} + \frac{1}{(1-\theta)r} \|Du\|_{L^{p,\kappa}(w, B_{\theta' r})} + \frac{1}{\theta(1-\theta)^2 r^2} \|u\|_{L^{p,\kappa}(w, B_{\theta' r})} \right), \end{aligned}$$

with $\theta \in (\frac{1}{2}, 1)$, $\theta' = \frac{1+\theta}{2}$.

Multiplying both sides by $(1 - \theta)^2 r^2$, we obtain:

$$(1 - \theta)^2 r^2 \|D^2 u\|_{L^{p,\kappa}(w, B_{\theta r})} \leq c \left((1 - \theta)^2 r^2 \|Lu\|_{L^{p,\kappa}(w, B_{\theta' r})} + (1 - \theta) r \|Du\|_{L^{p,\kappa}(w, B_{\theta' r})} + \|u\|_{L^{p,\kappa}(w, B_{\theta' r})} \right). \quad (21)$$

Observe that $1 - \theta = 2(1 - \theta')$; this yields:

$$(1 - \theta)^2 r^2 \|D^2 u\|_{L^{p,\kappa}(w, B_{\theta r})} \leq c \left((1 - \theta')^2 r^2 \|Lu\|_{L^{p,\kappa}(w, B_{\theta' r})} + (1 - \theta') r \|Du\|_{L^{p,\kappa}(w, B_{\theta' r})} + \|u\|_{L^{p,\kappa}(w, B_{\theta' r})} \right);$$

hence,

$$\Phi_2 \leq c \left(r^2 \|Lu\|_{L^{p,\kappa}(w, B_{\theta' r})} + \Phi_1 + \|u\|_{L^{p,\kappa}(w, B_{\theta' r})} \right),$$

and using $\Phi_1 \leq \delta \Phi_2 + \frac{c}{\delta} \Phi_0$,

$$\Phi_1 + \Phi_2 \leq c \left(r^2 \|Lu\|_{L^{p,\kappa}(w, B_r)} + \|u\|_{L^{p,\kappa}(w, B_r)} \right).$$

Then:

$$r \|Du\|_{L^{p,\kappa}(w, B_{r/2})} + r^2 \|D^2 u\|_{L^{p,\kappa}(w, B_{r/2})} \leq c \left(r^2 \|Lu\|_{L^{p,\kappa}(w, B_r)} + \|u\|_{L^{p,\kappa}(w, B_r)} \right),$$

and:

$$\|u\|_{S^{2,p,\kappa}(w, B_{r/2})} \leq c \left(\|Lu\|_{L^{p,\kappa}(w, B_r)} + \|u\|_{L^{p,\kappa}(w, B_r)} \right).$$

Finally, we conclude Theorem 2 by the usual covering argument. $\square$

5. Conclusions

Recently, the regularity of nondivergence degenerate operators with drift has been considered by some scholars. However, there is no result on a priori estimates in weighted Sobolev–Morrey spaces for this form of operator. In the framework of homogeneous groups, firstly, we introduced the boundedness of commutators of Calderón–Zygmund operators with functions in weighted Morrey spaces. Next, the corresponding interpolation inequalities were deduced. As an application, we obtained weighted Sobolev–Morrey estimates. The Hörmander-type operators that we considered can be seen as a natural generalization of the Laplace operator and parabolic operator in Euclidean spaces. This kind of operator arises in many research fields, such as mathematical finance theory, models of human vision, and kinetic models. Via the obtained prior estimates, we can discuss the existence and uniqueness of solutions to the corresponding equations; furthermore, this is helpful to solve problems in practical applications.

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