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Symmetry Solutions and Conservation Laws for the 3D Generalized Potential Yu-Toda-Sasa-Fukuyama Equation of Mathematical Physics

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Abstract: In this paper we study the fourth-order three-dimensional generalized potential Yu-Toda-Sasa-Fukuyama (gpYTSF) equation by first computing its Lie point symmetries and then performing symmetry reductions. The resulting ordinary differential equations are then solved using direct integration, and exact solutions of gpYTSF equation are obtained. The obtained group invariant solutions include the solution in terms of incomplete elliptic integral. Furthermore, conservation laws for the gpYTSF equation are derived using both the multiplier and Noether's methods. The multiplier method provides eight conservation laws, while the Noether's theorem supplies seven conservation laws. These conservation laws include the conservation of energy and mass.

Keywords: three-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation; Lie symmetries; exact solution; conservation laws; Noether's theorem

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1. Introduction

Many natural phenomena of the real world are modelled using nonlinear partial differential equations (NPDEs). It is therefore important to find their exact solutions in order to understand the real world better. There have been several studies done on NPDEs and many researchers have suggested various techniques for finding exact solutions for such equations, since there is no general theory that can be applied to find exact solutions. These techniques include the Jacobi elliptic function expansion method [1], the homogeneous balance method [2], the Kudryashov's method [3], the ansatz method [4], the inverse scattering transform method [5], the Bäcklund transformation [6], the Darboux transformation [7], the Hirota bilinear method [8], the $(G'/G)-$expansion method [9], and the Lie symmetry method [10–15], just to mention a few.

In the late 19th century, a powerful symmetry-based technique for solving differential equations (DEs), known today as Lie group analysis, was developed by the Norwegian mathematician Marius Sophus Lie (1844–1899). This technique is an efficient technique that can be used to compute exact solutions of DEs. It only became well-known in the early 1960s when the Russian mathematician L. V. Ovsyannikov (1919–2014) demonstrated the power of these methods for computing explicit solutions of complicated partial differential equations (PDEs) arising in mathematical physics. Since then, a robust amount of research based on Lie's work has been published by various researchers.

The German mathematician Emmy Noether (1882–1935) in 1918 presented a procedure for deriving conservation laws for systems of DEs that are derived from the variational principle, and this procedure is referred to as Noether's theorem [16]. A given DE that is derived from the variational principle should have a Lagrangian. However, there are

DEs that are not derived from a variational principle and as a result Noether's theorem cannot be invoked to determine conservation laws for such DEs. In such a case, multiplier method [12] or Ibragimov's theorem on conservation laws [17] can be employed to construct conservation laws. The computation of conservation laws are very important as they play a vital role in the study of DEs. They describe physical conserved quantities, e.g., conservation of mass, energy, momentum, charge, and other constants of motion. They are also important for the investigation of integrability and uniqueness of solutions. See for example [16–25]. For the connection between the Lie and Noether symmetries, the reader is referred to [26,27].

The three-dimensional potential Yu-Toda-Sasa-Fukuyama (3DYTSF) equation given by

$$u_{xxxz} - 4u_{tx} + 3u_{yy} + 4u_x u_{xz} + 2u_z u_{xx} = 0 \tag{1}$$

was introduced in [28] using the strong symmetry, and its travelling solitary wave solutions were presented. Yan [29] studied Equation (1) and obtained auto-Bäcklund transformations. Using auto-Bäcklund transformations, some exact solutions of (1) were found. These included the non-travelling wave solutions, soliton-like solutions, and rational solutions. The authors of [30] investigated Equation (1) using homoclinic and extended homoclinic test techniques, the two-soliton method along with bilinear form method, and obtained some new exact wave solutions that included periodic kink-wave, periodic soliton, cross kink wave, and doubly periodic wave solutions. In [31], the exp-function method, with the aid of symbolic computation, was employed, and new generalized solitary solutions and periodic solutions with free parameters were obtained. Using a modification of extended homoclinic test approach, the authors of [32] obtained some analytic solutions of 3DYTSF Equation (1). In [33], using some 1D subalgebras, group invariant solutions were constructed for (1) that involve arbitrary functions. Additionally, some particular solutions were sketched. Exact solutions that included lump solutions and interaction solutions of (1) were obtained using the generalized Hirota bilinear method [34]. In [35] analytical solutions and conservation laws for the 2D form of (1) were presented. Also, 2D and 3D graphical representations of the some solutions were given. N-soliton solutions were derived for (1) by using bilinear transformation that included period soliton, line soliton, lump soliton, and their interaction. Moreover, for some solutions their images were drawn and their dynamic behavior was discussed in [36]. The authors of [37] invoked the extended homoclininc test and Hirota bilinear method and constructed a class of lump solutions of (1). Additionally, periodic lump-type solutions were obtained in [37]. In [38], Equation (1) was reduced to the potential YTSF equation, which is a 2D equation (see also the ref [35]). General lump solutions of this equation were established and its propagation path was discovered. By letting $u = w_x$, the authors of [39] increased the order of the (3+1) YTSF equation to five and applied Lie symmetry methods and constructed dark, bright, topological, Peregrine, and multi-soliton.

In this paper, we shall work with the three-dimensional generalized potential Yu-Toda-Sasa-Fukuyama (3DgYTSF) equation, namely

$$u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx} = 0, \tag{2}$$

where α, β, and γ are real constants. We seek to derive its exact solutions by using symmetry analysis, along with various other methods. Moreover, conserved quantities of Equation (2) are established using two approaches: multiplier approach and Noether's approach.

2. Solutions of the 3DgYTSF Equation

In this section, we firstly present Lie point symmetries and symmetry reductions of 3DgYTSF Equation (2). Moreover, we obtain travelling wave solution of (2) by employing Kudryashov's method.

2.1. Lie Point Symmetries

Here, we compute Lie point symmetries for the 3DgYTSF Equation (2). The vector field for this Equation (2) is written as

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \tag{3}$$

where ξ^1, ξ^2, ξ^3, ξ^4, and η are functions of the variables t, x, y, z, and u. We recall that (3) is a Lie point symmetry of (2) if

$$X^{[4]} E|_{E=0} = 0, \tag{4}$$

where $E \equiv u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx}$. Here $X^{[4]}$ is the fourth prolongation [12] of (3) that is defined by

$$\begin{aligned} X^{[4]} = X &+ \zeta_2 \frac{\partial}{\partial u_x} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{33} \frac{\partial}{\partial u_{yy}} + \zeta_{24} \frac{\partial}{\partial u_{xz}} \\ &+ \zeta_{2224} \frac{\partial}{\partial u_{xxxz}} \end{aligned} \tag{5}$$

and the coefficients $\zeta' s$ are given by [12]

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3) - u_z D_t(\xi^4), \\
\zeta_2 &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4), \\
\zeta_3 &= D_y(\eta) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3) - u_z D_y(\xi^4), \\
\zeta_4 &= D_z(\eta) - u_t D_z(\xi^1) - u_x D_z(\xi^2) - u_y D_z(\xi^3) - u_z D_z(\xi^4), \\
\zeta_{12} &= D_z(\zeta_1) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - u_{ty} D_x(\xi^3) - u_{tz} D_x(\xi^4), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4), \\
\zeta_{33} &= D_y(\zeta_3) - u_{ty} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{yy} D_y(\xi^3) - u_{yz} D_y(\xi^4), \\
\zeta_{24} &= D_z(\zeta_2) - u_{tx} D_z(\xi^1) - u_{xx} D_z(\xi^2) - u_{xy} D_z(\xi^3) - u_{xz} D_z(\xi^4), \\
\zeta_{2224} &= D_z(\zeta_{222}) - u_{txxx} D_z(\xi^1) - u_{xxxx} D_z(\xi^2) - u_{xxxy} D_z(\xi^3) - u_{xxxz} D_z(\xi^4).
\end{aligned}$$

From Equation (4) we get

$$\begin{aligned} &\{ X + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{33} \frac{\partial}{\partial u_{yy}} + \zeta_{24} \frac{\partial}{\partial u_{xz}} \\ &+ \zeta_{2224} \frac{\partial}{\partial u_{xxxz}} \Big\} \Big(u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx} \Big) \Big|_{(2)} = 0. \end{aligned} \tag{6}$$

Upon expanding the determining Equation (6), we attain

$$\alpha \zeta_2 u_{xz} + \alpha \zeta_4 u_{xx} - 2\alpha \zeta_{12} + \alpha \zeta_{22} u_z + \beta \zeta_{33} + 2\alpha \zeta_{24} u_x + \zeta_{2224} \Big|_{(2)} = 0.$$

Substituting values of $\zeta_2, \zeta_4, \zeta_{12}, \zeta_{22}, \zeta_{33}, \zeta_{24}$, and ζ_{2224} in the above equation, replacing u_{xxxz} by $2\alpha u_{tx} - \beta u_{yy} - 2\alpha u_x u_{xz} - \alpha u_z u_{xx}$ and splitting on the derivatives of u, we get the over-determined system of twenty-two linear PDEs

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_z = 0,\ \xi^1_u = 0,\ \xi^1_{tt} = 0,\ 2\xi^3_y - \xi^1_t - \xi^2_x = 0,\ \beta \xi^2_y - \alpha \xi^3_t = 0,\ \xi^2_z = 0, \\
&\xi^2_u = 0,\ \xi^3_x = 0,\ \xi^3_z = 0,\ \xi^3_u = 0,\ \xi^3_{ty} = 0,\ \xi^3_{yy} = 0,\ \xi^3_u = 0,\ \xi^4_x = 0,\ \xi^4_y = 0, \\
&\xi^4_z + 4\xi^3_y - 3\xi^1_t = 0,\ \eta_u + 2\xi^3_y - \xi^1_t = 0,\ \eta_x + \xi^4_t = 0,\ \eta_z + 2\xi^2_t = 0, \\
&\beta \eta_{yy} + 2\alpha \xi^4_{tt} = 0.
\end{aligned} \tag{7}$$

Solving the above equations, we acquire the infinitesimals

$$
\begin{aligned}
\xi^1 &= C_1 t + C_2,\\
\xi^2 &= -C_1 x + 2C_3 x + \frac{\alpha}{\beta} y F_1'(t) + F_3(t),\\
\xi^3 &= C_3 y + F_1(t),\\
\xi^4 &= 3C_1 z - 4C_3 z + F_2(t),\\
\eta &= C_1 u - 2C_3 u - \frac{2\alpha}{\beta} yz F_1''(t) - \frac{\alpha}{\beta} y^2 F_2''(t) - xF_2'(t) - 2zF_3'(t) + yF_5(t) + F_4(t),
\end{aligned}
$$

where $C_1, \ldots, C_3$ are arbitrary constants and F_j, $j = 1, 2, \ldots, 5$ are arbitrary functions of t. By taking $F_j(t) = 1$, $j = 1, 2, \ldots, 5$, we obtain the following Lie point symmetries of the 3DgYTSF Equation (2):

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial z},\ X_5 = \frac{\partial}{\partial u},\ X_6 = y\frac{\partial}{\partial u},\\
&X_7 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 3z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u},\ X_8 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 4z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}.
\end{aligned}
$$

By solving the Lie equations together with initial conditions

$$
\begin{aligned}
&\frac{d\bar{t}}{da} = \xi^1(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}),\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \xi^2(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}),\ \bar{x}|_{a=0} = x,\\
&\frac{d\bar{y}}{da} = \xi^3(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}),\ \bar{y}|_{a=0} = y, \quad \frac{d\bar{z}}{da} = \xi^4(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}),\ \bar{z}|_{a=0} = z,\\
&\frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}),\ \bar{u}|_{a=0} = u,
\end{aligned}
$$

we obtain the following group transformations that are generated by the Lie symmetries $X_i\ (i = 1, 2, \ldots, 8)$:

$$
\begin{aligned}
T_1 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t+a, x, y, z, u),\\
T_2 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, x+a, y, z, u),\\
T_3 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, x, y+a, z, u),\\
T_4 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, x, y, z+a, u),\\
T_5 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, x, y, z, u+a),\\
T_6 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, x, y, z, ay+u),\\
T_7 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (te^{a}, xe^{-a}, y, ze^{3a}, ue^{a}),\\
T_8 &: (\bar{t},\bar{x},\bar{y},\bar{z},\bar{u}) \longrightarrow (t, xe^{-2a}, ye^{-a}, ze^{4a}, ue^{2a}).
\end{aligned}
$$

Consequently, if $u = M(t, x, y, z)$ is a known solution of 3DgYTSF Equation (2), then by using the group of transformations $T_i\ (i = 1, 2, \ldots, 8)$, so are the functions

$$
\begin{aligned}
u_1 &= M(t+a, x, y, z),\\
u_2 &= M(t, x+a, y, z),\\
u_3 &= M(t, x, y+a, z),\\
u_4 &= M(t, x, y, z+a),\\
u_5 &= M(t, x, y, z) + a,\\
u_6 &= M(t, x, y, z) - ay,\\
u_7 &= M(te^{a}, xe^{-a}, y, ze^{3a})e^{a},\\
u_8 &= M(t, xe^{-2a}, ye^{-a}, ze^{4a})e^{2a}.
\end{aligned}
$$

2.2. Group Invariant Solutions under $X_1, \ldots, X_4$

We now consider the linear combination of the four translation symmetries, namely $X = X_1 + aX_2 + bX_3 + cX_4$, where a, b, c are arbitrary constants. Solving the associated characteristics equations of X, we acquire the invariants

$$f = x - at,\ \ g = y - bt,\ \ h = z - ct,\ \ \ u = \theta(f, g, h)$$

and using these invariants, the 3DgYTSF Equation (2) reduces to the NPDE

$$\theta_{fffh} + 2\alpha\Big(a\theta_{ff} + b\theta_{fg} + c\theta_{fh}\Big) + \beta\theta_{gg} + 2\alpha\theta_f\theta_{fh} + \alpha\theta_h\theta_{ff} = 0. \tag{8}$$

The Lie symmetries of (8) are

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial f},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = \frac{\partial}{\partial h},\ \Gamma_4 = \frac{\partial}{\partial \theta},\ \Gamma_5 = g\frac{\partial}{\partial \theta},\\
&\Gamma_6 = \left(\frac{2}{3}f + \frac{\alpha bg}{3\beta}\right)\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + \left(\frac{1}{3}\frac{(-4ah - 4cf - 2\theta)\beta + 2\alpha b^2 h}{\beta}\right)\frac{\partial}{\partial \theta},\\
&\Gamma_7 = \left(\frac{\alpha bg}{3\beta} - \frac{1}{3}f\right)\frac{\partial}{\partial f} + h\frac{\partial}{\partial h} + \left(\frac{1}{3}\frac{(-4ah + 2cf + \theta)\beta + 2\alpha b^2 h}{\beta}\right)\frac{\partial}{\partial \theta}.
\end{aligned}$$

Now, utilizing the symmetry $\Gamma_1 + d\Gamma_2 + e\Gamma_3$ with d, e being arbitrary constants, we obtain the three invariants

$$r = g - df,\ s = h - ef,\ \theta = \phi(r, s),$$

which reduces Equation (8) to the NPDE

$$\begin{aligned}
&-d^3\phi_{rrrs} - 3d^2e\phi_{rrss} - 3de^2\phi_{rsss} - e^3\phi_{ssss} + 2\alpha ad^2\phi_{rr} + 4\alpha ade\phi_{rs} + 2\alpha ae^2\phi_{ss}\\
&-2\alpha bd\phi_{rr} - 2\alpha be\phi_{rs} - 2\alpha cd\phi_{rs} - 2\alpha ce\phi_{ss} + \beta\phi_{rr} + 2\alpha d^2\phi_r\phi_{rs} + 2\alpha de\phi_r\phi_{ss}\\
&+4\alpha de\phi_s\phi_{rs} + 3\alpha e^2\phi_s\phi_{ss} + \alpha d^2\phi_s\phi_{rr} = 0,
\end{aligned} \tag{9}$$

The symmetries of (9) include

$$\Sigma_1 = \frac{\partial}{\partial r},\ \Sigma_2 = \frac{\partial}{\partial s},\ \Sigma_3 = \frac{\partial}{\partial \phi}$$

and utilizing the symmetry $\Sigma = \Sigma_1 + \omega\Sigma_2$, where ω is a constant, we obtain two invariants

$$\xi = s - \omega r,\ \ \phi = \psi(\xi).$$

Using these invariants, Equation (9) reduces to the nonlinear ordinary differential equation (NODE)

$$\begin{aligned}
&\Big(2\alpha ad^2\omega^2 + 2\alpha ae^2 + 2\alpha be\omega + 2\alpha cd\omega + \beta\omega^2 - 4\alpha ade\omega - 2\alpha bd\omega^2 - 2\alpha ce\Big)\psi''\\
&-\Big(6\alpha de\omega - 3\alpha d^2\omega^2 - 3\alpha e^2\Big)\psi'\psi'' + \Big(3de^2\omega - e^3 + \omega^3 d^3 - 3d^2e\omega^2\Big)\psi'''' = 0,
\end{aligned}$$

which we rewrite as

$$A\psi'' - B\psi'\psi'' + C\psi'''' = 0, \tag{10}$$

where $A = 2\alpha ad^2\omega^2 + 2\alpha ae^2 + 2\alpha be\omega + 2\alpha cd\omega + \beta\omega^2 - 4\alpha ade\omega - 2\alpha bd\omega^2 - 2\alpha ce$, $B = 6\alpha de\omega - 3\alpha d^2\omega^2 - 3\alpha e^2$, and $C = 3de^2\omega - e^3 + \omega^3 d^3 - 3d^2e\omega^2$.

2.3. Solution via the Incomplete Elliptic Integral

We now obtain the solution of 3DgYTSF Equation (2) in terms of the incomplete elliptic integral. Twice integration of Equation (10) gives

$$C\psi''^2 - \frac{1}{3}B\psi'^3 + A\psi'^2 + 2c_1\psi' + 2c_2 = 0, \tag{11}$$

where c_1, c_2 are integration constants. By letting $\psi'(\xi) = \Psi(\xi)$, Equation (11) transforms to

$$\Psi'^2 - \frac{B}{3C}\Psi^3 + \frac{A}{C}\Psi^2 + \frac{2c_1}{C}\Psi + \frac{2c_2}{C} = 0, \tag{12}$$

which has the well-known general solution [40,41]

$$\Psi(\xi) = \gamma_2 + (\gamma_1 - \gamma_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{B(\gamma_1-\gamma_3)}{12C}}\;(\xi - \xi_0), X^2\right\},\; X^2 = \frac{\gamma_1 - \gamma_2}{\gamma_1 - \gamma_3}, \tag{13}$$

where $\gamma_1 > \gamma_2 > \gamma_3$ are the real roots of the cubic polynomial $\Psi^3 - (3A/B)\Psi^2 - (6c_1/B)\Psi - (6c_2)/B = 0$, (cn) is the Jacobi cosine function and ξ_0 is arbitrary constant. Integration of Equation (13) yields the expression for $\psi(\xi)$, and thus reverting to the variables t, x, y, z, u, we gain the solution of 3DgYTSF Equation (2) as

$$\begin{aligned} u(t,x,y,z) = \sqrt{\frac{12C(\gamma_1-\gamma_2)^2}{B(\gamma_1-\gamma_3)X^8}}\left[\mathrm{EllipticE}\left\{\mathrm{sn}\left(\sqrt{\frac{B(\gamma_1-\gamma_3)}{12C}}(\xi-\xi_0), X^2\right), X^2\right\}\right] \\ + \left\{\gamma_2 - (\gamma_1-\gamma_2)\frac{1-X^4}{X^4}\right\}(\xi - \xi_0) + k_0, \end{aligned} \tag{14}$$

where k_0, ξ_0 are constants, $\xi = (ae - c + b\omega - a\omega d)t - (e + \omega d)x - \omega y + z$ and EllipticE$[g,k]$ being the incomplete elliptic integral given as [42,43]

$$\mathrm{EllipticE}[v,m] = \int_0^v \sqrt{\frac{1-m^2s^2}{1-s^2}}ds.$$

The wave profile of the periodic solution (14), for parametric values $\gamma_1 = 102$, $\gamma_2 = 53$, $\gamma_3 = -58$, $C = -0.7$, $B = 0.25$, $c_1 = 1.2$, $z = 0$, $e = 0.5$, $d = 0.25$, $omega = -1.01$, $a = -4.5$, $c = -0.6$ at $t = -15$ is given in Figure 1. We note that this solution gives a periodic wave graphics, since this general solution is periodic

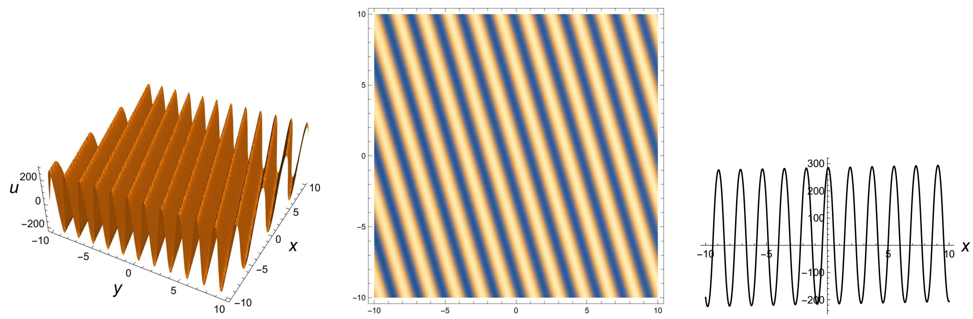

Figure 1. The 3D and 2D solution profile of (14).

2.4. Group Invariant Solution under X_7

Here we consider the Lie symmetry X_7, which is given by

$$X_7 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 3z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}.$$

Solving the associated characteristic equations for X_7, we accomplish the invariants

$$f = tx,\ g = y,\ h = \frac{z}{t^3},\ \Phi = \frac{u}{t}.$$

The use of these invariants reduces the 3DgYTSF Equation (2) to

$$\Phi_{fffh} - 4\alpha\Phi_f - 2\alpha f\Phi_{ff} + 6\alpha h\Phi_{fh} + \beta\Phi_{gg} + 2\alpha\Phi_f\Phi_{fh} + \alpha\Phi_h\Phi_{ff} = 0. \tag{15}$$

Equation (15) possesses six Lie symmetries

$$\Gamma_1 = \frac{\partial}{\partial g},\ \Gamma_2 = g\frac{\partial}{\partial \Phi},\ \Gamma_3 = \frac{\partial}{\partial \Phi},\ \Gamma_4 = \frac{\partial}{\partial f} + 2h\frac{\partial}{\partial \Phi},$$
$$\Gamma_5 = 2f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} - 4h\frac{\partial}{\partial h} - 2\phi\frac{\partial}{\partial \Phi},\ \Gamma_6 = \frac{\partial}{\partial h} - \left(3f + \frac{6\alpha g^2}{\beta}\right)\frac{\partial}{\partial \Phi}.$$

The symmetry Γ_4 gives three invariants $g, h, U = \Phi - 2fh$ and consequently, the group-invariant solution is

$$\Phi(f, g, h) = U(g, h) + 2fh,$$

which reduces Equation (15) to

$$U_{gg} + \frac{12\alpha}{\beta}h = 0,$$

whose solution is given by

$$U = -\frac{6\alpha}{\beta}g^2 h + gM(h) + K(h),$$

where M and K are functions of h. Thus, the invariant solution of the 3DYTSF Equation (2) under the symmetry X_7 is

$$u(t, x, y, z) = \frac{2xz}{t} - \frac{6\alpha y^2 z}{\beta t^2} + tyM\left(\frac{z}{t^3}\right) + tK\left(\frac{z}{t^3}\right). \tag{16}$$

In Figure 2, we depict the solution (16) with $M(z/t^3) = \cos(z/t^3)$, $K(z/t^3) = \operatorname{sech}(z/t^3)$ and parametric values $\beta = 2$, $\alpha = 2$, $x = 0$, $t = 1$, $z = 15$.

In Figure 3, we demonstrate the solution (16) with different choices of arbitrary functions, i.e., $M(z/t^3) = \operatorname{sech}(z/t^3)$, $K(z/t^3) = \cos(z/t^3)$ and the parameters $\beta = 2$, $\alpha = 2$, $x = 0$, $t = 1$, $z = 15$.

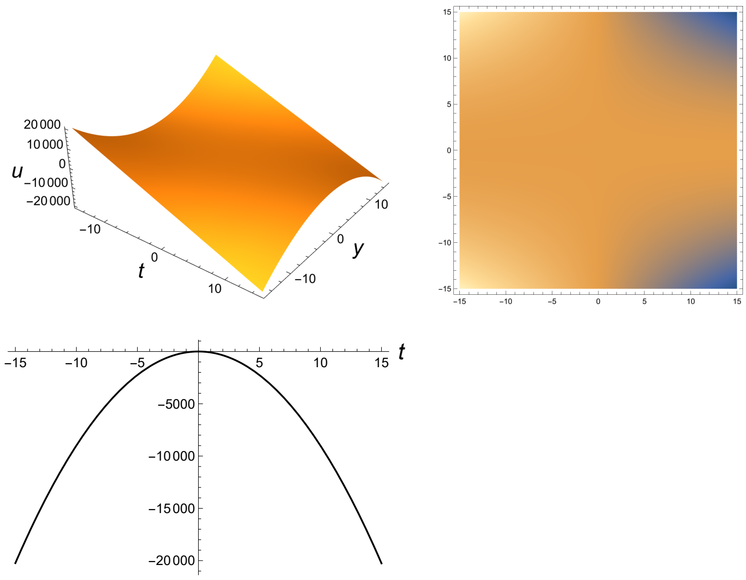

Figure 2. The 3D and 2D solution profile of (16) with $M = \cos(z/t^3)$, $K = \mathrm{sech}(z/t^3)$.

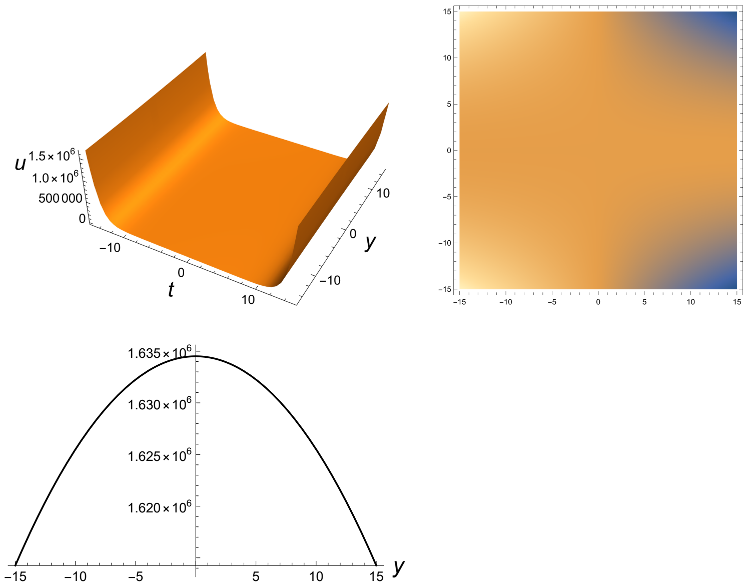

Figure 3. The 3D and 2D solution profile of (16) with $M = \mathrm{sech}(z/t^3)$, $K = \cos(z/t^3)$.

2.5. Group Invariant Solution under X_8

For the symmetry X_8, we get the group-invariant solution as

$$u(t,x,y,z) = \frac{1}{x} F(f,g,h), \tag{17}$$

where f, g, h, F are the invariants given by

$$f = t,\ g = \frac{y}{\sqrt{x}},\ h = x^2 z,\ F = xu.$$

Substituting (17) into the 3DgYTSF Equation (2) gives the NODE

$$\begin{aligned}&36ghF_{ghh}-12g^2hF_{gghh}+48gh^2F_{ghhh}-8\alpha gF_{fg}+32\alpha hFF_{hh}+g^3F_{gggh}-64h^3F_{hhhh}\\&-16\alpha F_f+32\alpha hF_{fh}-16\alpha hF_h^2-192h^2F_{hhh}+3g^2F_{ggh}-3gF_{gh}-8\beta F_{gg}-48hF_{hh}\\&-8\alpha gFF_{gh}-6\alpha gF_gF_h-4\alpha g^2F_gF_{gh}-96\alpha h^2F_hF_{hh}-2\alpha g^2F_hF_{gg}+16\alpha ghF_gF_{hh}\\&+32\alpha ghF_hF_{gh}=0\end{aligned}\tag{18}$$

whose symmetry includes $\Gamma_1=\partial/\partial f$. Using the symmetry Γ_1, we get the invariants $j_1=g$, $j_2=h$, $F=\Psi$, which reduces Equation (18) to the NPDE

$$\begin{aligned}&36gh\Psi_{ghh}-12g^2h\Psi_{gghh}+48gh^2\Psi_{ghhh}+32\alpha h\Psi\Psi_{hh}+g^3\Psi_{gggh}-64h^3\Psi_{hhhh}-16\alpha\Psi_h^2\\&-192h^2\Psi_{hhh}+3g^2\Psi_{ggh}-3g\Psi_{gh}-8\beta\Psi_{gg}-48h\Psi_{hh}-8\alpha g\Psi\Psi_{gh}-6\alpha g\Psi_g\Psi_h\\&-4\alpha g^2\Psi_g\Psi_{gh}-96\alpha h^2\Psi_h\Psi_{hh}-2\alpha g^2\Psi_h\Psi_{gg}+16\alpha gh\Psi_g\Psi_{hh}+32\alpha gh\Psi_h\Psi_{gh}=0\end{aligned}$$

with two independent variables. Thus, we have reduced the number of independent variables of 3DgYTSF Equation (2) by two.

3. Conservation Laws of (2)

In this section we construct conservation laws of the 3DgYTSF Equation (2) by using two different approaches, namely, the multiplier method and Noether's approach.

3.1. Conservation Laws Using the Multiplier Approach

We seek first-order multiplier $Q=Q(t,x,y,z,u,u_t,u_x,u_y)$ by applying the determining equation for the multipliers

$$\frac{\delta}{\delta u}\left[Q\{u_{xxxz}-2\alpha u_{tx}+\beta u_{yy}+2\alpha u_xu_{xz}+\alpha u_zu_{xx}\}\right]=0,\tag{19}$$

where the Euler-Lagrange operator $\delta/\delta u$ in our case is defined as

$$\begin{aligned}\frac{\delta}{\delta u}=&\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_z\frac{\partial}{\partial u_z}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}\\&+D_xD_z\frac{\partial}{\partial u_{xz}}+D_x^3D_z\frac{\partial}{\partial u_{xxxz}}+\cdots.\end{aligned}\tag{20}$$

Expansion of Equation (19) yields

$$\begin{aligned}&u_{xxxz}Q_u-2\alpha u_{tx}Q_u+\beta u_{yy}Q_u+2\alpha u_xu_{xz}Q_u+\alpha u_{xx}u_zQ_u-D_t\Big(u_{xxxz}Q_{u_t}-2\alpha u_{tx}Q_{u_t}\\&+\beta u_{yy}Q_{u_t}+2\alpha u_xu_{xz}Q_{u_t}+\alpha u_{xx}u_zQ_{u_t}\Big)-D_x\Big(u_{xxxz}Q_{u_x}-2\alpha u_{tx}Q_{u_x}+\beta u_{yy}Q_{u_x}\\&+2\alpha u_xu_{xz}Q_{u_x}+\alpha u_{xx}u_zQ_{u_x}\Big)-D_y\Big(u_{xxxz}Q_{u_y}-2\alpha u_{tx}Q_{u_y}+\beta u_{yy}Q_{u_y}+2\alpha u_xu_{xz}Q_{u_y}\\&+\alpha u_{xx}u_zQ_{u_y}\Big)-D_x(2\alpha u_{xz}Q)-D_tD_x(-2\alpha Q)-D_y^2(\beta Q)-D_xD_z(2\alpha u_xQ)-D_x^2(\alpha u_zQ)\\&-D_z(\alpha u_{xx}Q)-D_x^3D_z(Q)=0,\end{aligned}$$

which, on applying the total derivatives D_t, D_x, D_y, D_z and splitting over the derivatives of u, yields the following simplified determining equations:

$$\begin{aligned}&Q_x=0,\ Q_u=0,\ Q_{yy}=0,\ Q_{zz}=0,\ Q_{tu_t}=0,\ Q_{yu_t}=0,\ Q_{yu_y}=0,\ Q_{zu_t}=0,\ Q_{zu_x}=0,\\&Q_{zu_y}=0,\ Q_{u_tu_t}=0,\ Q_{u_tu_x}=0,\ Q_{u_tu_y}=0,\ 2Q_{tu_x}-Q_z=0,\ Q_{u_xu_x}=0,\ Q_{u_yu_y}=0,\\&\alpha Q_{tu_y}-\beta Q_{yu_x}=0,\ Q_{u_xu_y}=0.\end{aligned}\tag{21}$$

The solution of the above system of overdetermined equations is

$$Q = yzF''(t) + \frac{1}{2}yu_x F'(t) + C_1 yu_x + yG(t) + H'(t)z + C_2 u_t + \frac{1}{2}u_x H(t) + C_3 u_x + \frac{\beta u_y}{2\alpha}F(t) + C_1\frac{\beta t u_y}{\alpha} + C_4 u_y + J(t),$$

where $F(t), G(t), H(t)$, and $J(t)$ are functions of t, whereas C_1, C_2, C_3, and C_4 are arbitrary constants. The conservation laws are now obtained by using the divergence identity

$$D_t T^t + D_x T^x + D_y T^y + D_z T^z = Q(u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx}),$$

where T^t is the conserved density, and T^x, T^y, T^z are spatial fluxes. Thus, after some calculations, conservation laws corresponding to the eight multipliers are given below.

Case 1. For the first multiplier $Q_1 = yu_x + (\beta t u_y)/\alpha$, the corresponding conservation law is given by

$$\begin{aligned}
T_1^t =& \frac{1}{2}\alpha y u u_{xx} - \frac{1}{2}\alpha y u_x^2 + \frac{1}{2}\beta t u u_{xy} - \frac{1}{2}\beta t u_x u_y,\\
T_1^x =& \frac{1}{3}\alpha y u u_x u_{xz} - \frac{1}{3}\beta t u u_x u_{yz} - \frac{1}{3}\beta t u u_z u_{xy} + \frac{2}{3}\alpha y u_x^2 u_z + \frac{2}{3}\beta t u_x u_y u_z - \frac{1}{2}\alpha y u u_{tx}\\
&+ \frac{1}{2}\beta t u u_{ty} + \frac{1}{2}\beta y u u_{yy} - \frac{1}{2}\alpha y u_t u_x - \frac{1}{2}\beta t u_t u_y + \frac{1}{8}y u u_{xxxz} - \frac{3}{8\alpha}\beta t u u_{xxyz}\\
&+ \frac{1}{2}\beta u u_y + \frac{5}{8}y u_x u_{xxz} + \frac{1}{4\alpha}\beta t u_x u_{xyz} - \frac{3}{8}y u_{xx} u_{xz} - \frac{1}{8\alpha}\beta t u_{xx} u_{yz} + \frac{1}{8}y u_z u_{xxx}\\
&+ \frac{1}{8\alpha}\beta t u_z u_{xxy} + \frac{3}{8\alpha}\beta t u_y u_{xxz} - \frac{1}{4\alpha}\beta t u_{xy} u_{xz},\\
T_1^y =& \frac{2}{3}\beta u_t u_x u_{xz} + \frac{1}{3}\beta u_t u_z u_{xx} - \beta t u u_{tx} - \frac{1}{2}\beta y u u_{xy} - \frac{1}{2}\beta u u_x + \frac{1}{2}\beta y u_x u_y + \frac{1}{2\alpha}\beta^2 t u_y^2\\
&+ \frac{1}{2\alpha}\beta t u u_{xxxz},\\
T_1^z =& -\frac{1}{3}\alpha y u u_x u_{xx} - \frac{1}{3}\beta t u u_x u_{xy} + \frac{1}{3}\alpha y u_x^3 + \frac{1}{3}\beta t u_x^2 u_y - \frac{1}{8}y u u_{xxxx} - \frac{1}{8\alpha}\beta t u u_{xxxy}\\
&+ \frac{1}{4}y u_x u_{xxx} + \frac{1}{8\alpha}\beta t u_x u_{xxy} - \frac{1}{8}y u_{xx}^2 - \frac{1}{8\alpha}\beta t u_{xx} u_{xy} + \frac{1}{8\alpha}\beta t u_y u_{xxx}.
\end{aligned}$$

Case 2. For the second multiplier $Q_2 = u_t$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_2^t =& \frac{2}{3}\alpha u u_x u_{xz} + \frac{1}{3}\alpha u u_z u_{xx} - \frac{1}{2}\alpha u u_{tx} + \frac{1}{2}\beta u u_{yy} + \frac{1}{2}u u_{xxxz} - \frac{1}{2}\alpha u_t u_x,\\
T_2^x =& \frac{1}{3}\alpha u u_z u_{tx} - \frac{1}{3}\alpha u u_x u_{tz} + \frac{2}{3}\alpha u_t u_x u_z + \frac{1}{2}\alpha u u_{tt} - \frac{1}{2}\alpha u_t^2 - \frac{3}{8}u u_{txxz} + \frac{1}{8}u_z u_{txx}\\
&+ \frac{1}{4}u_x u_{txz} + \frac{3}{8}u_t u_{xxz} - \frac{1}{4}u_{tx} u_{xz} - \frac{1}{8}u_{tz} u_{xx},\\
T_2^y =& -\frac{1}{2}\beta u u_{ty} + \frac{1}{2}\beta u_t u_y,\\
T_2^z =& \frac{1}{3}\alpha u_t u_x^2 - \frac{1}{3}\alpha u u_{tx} + \frac{1}{8}u_x u_{txx} - \frac{1}{8}u u_{txxx} + \frac{1}{8}u_t u_{xxx} - \frac{1}{8}u_{tx} u_{xx}.
\end{aligned}$$

Case 3. For the second multiplier $Q_3 = u_x$, we obtain the corresponding conservation law as

$$
\begin{aligned}
T_3^t =& \frac{1}{2}\alpha u u_{xy} - \frac{1}{2}\alpha u_x u_y, \\
T_3^x =& \frac{1}{3}\alpha u u_x u_{xz} + \frac{2}{3}\alpha u_x^2 u_z - \frac{1}{2}\alpha u u_{tx} - \frac{1}{2}\alpha u_t u_x + \frac{5}{8}u_x u_{xxz} + \frac{1}{2}\beta u u_{yy} + \frac{1}{8}u u_{xxxz} \\
& + \frac{1}{8}u_z u_{xxx} - \frac{3}{8}u_{xx}u_{xz}, \\
T_3^y =& -\frac{1}{2}\beta u u_{xy} + \frac{1}{2}\beta u_x u_y, \\
T_3^z =& \frac{1}{3}\alpha u_x^3 - \frac{1}{3}\alpha u u_x u_{xx} + \frac{1}{4}u_x u_{xxx} - \frac{1}{8}u u_{xxxx} - \frac{1}{8}u_{xx}^2.
\end{aligned}
$$

Case 4. For the multiplier $Q_4 = u_y$, we obtain the corresponding conservation law as

$$
\begin{aligned}
T_4^t =& \frac{1}{2}\alpha u u_{xy} - \frac{1}{2}\alpha u_x u_y, \\
T_4^x =& -\frac{1}{3}\alpha u u_x u_{yz} - \frac{1}{3}\alpha u u_z u_{xy} + \frac{2}{3}\alpha u_x u_y u_z + \frac{1}{2}\alpha u u_{ty} - \frac{1}{2}\alpha u_t u_y - \frac{3}{8}u u_{xxyz} \\
& - \frac{1}{8}u_{xx}u_{yz} + \frac{1}{8}u_z u_{xxy} + \frac{3}{8}u_y u_{xxz} + \frac{1}{4}u_x u_{xyz} - \frac{1}{4}u_{xy}u_{xz}, \\
T_4^y =& \frac{2}{3}\alpha u u_x u_{xz} + \frac{1}{3}\alpha u u_z u_{xx} - \alpha u u_{tx} + \frac{1}{2}u u_{xxxz} + \frac{1}{2}\beta u_y^2, \\
T_4^z =& \frac{1}{3}\alpha u_x^2 u_y - \frac{1}{3}\alpha u u_x u_{xy} + \frac{1}{8}u_x u_{xxy} - \frac{1}{8}u u_{xxxy} + \frac{1}{8}u_y u_{xxx} - \frac{1}{8}u_{xx}u_{xy}.
\end{aligned}
$$

Case 5. For multiplier $Q_5 = \Big(F''(t)z + \frac{1}{2}F'(t)\Big)y + \frac{1}{2\alpha}\beta F(t)u_y$, we obtain the corresponding conservation law as

$$
\begin{aligned}
T_5^t =& -\alpha F''(t)yzu_x + \frac{1}{4}\alpha F(t)yuu_{xx} - \frac{1}{4}\alpha F'(t)yu_x^2 + \frac{1}{4}\beta F(t)uu_{xy} - \frac{1}{4}\beta F(t)u_x u_y, \\
T_5^x =& \frac{1}{4}F'(t)\beta yuu_{yy} - \frac{1}{6}\beta F(t)uu_z u_{xy} - \frac{1}{6}\beta F(t)uu_x u_{yz} + \frac{1}{3}\beta F(t)u_x u_y u_z \\
& - \frac{3}{16\alpha}\beta F(t)uu_{xxyz} - \frac{1}{16\alpha}\beta F(t)u_{xx}u_{yz} - \frac{1}{8\alpha}\beta F(t)u_{xy}u_{xz} + \frac{1}{16\alpha}\beta F(t)u_z u_{xxy} \\
& + \frac{3}{16\alpha}\beta F(t)u_y u_{xxz} + \frac{1}{8\alpha}\beta F(t)u_x u_{xyz} + \alpha F'''(t)yzu - \frac{1}{4}\alpha F''(t)yuu_x - \alpha F''(t)yzu_t \\
& - \frac{1}{4}\alpha F'(t)yuu_{tx} + \frac{1}{3}\alpha F'(t)yu_x^2 u_z - \frac{1}{4}\alpha F'(t)yu_t u_x - \frac{1}{4}F''(t)yu_{xx} + \alpha F''(t)yzu_x u_z \\
& + \frac{1}{6}\alpha F'(t)yuu_x u_{xz} + \frac{1}{16}F'(t)yu_{xxxz} - \frac{3}{16}F'(t)yu_{xx}u_{xz} + \frac{1}{16}F'(t)yu_z u_{xxx} \\
& + \frac{5}{16}F'(t)yu_x u_{xxz} + \frac{1}{4}\beta F'(t)uu_y + \frac{1}{4}\beta F(t)uu_{ty} - \frac{1}{4}\beta F(t)u_t u_y + \frac{3}{4}F''(t)yzu_{xxz}, \\
T_5^y =& \beta F''(t)yzu_y - \beta F''(t)zu - \frac{1}{4}\beta F'(t)yuu_{xy} + \frac{1}{4}\beta F'(t)yu_x u_y - \frac{1}{4}\beta F'(t)uu_x \\
& + \frac{1}{3}\beta F(t)uu_x u_{xz} + \frac{1}{6}\beta F(t)uu_z u_{xx} - \frac{1}{2}\beta F(t)uu_{tx} + \frac{1}{4\alpha}\beta^2 F(t)u_y^2 + \frac{1}{4\alpha}\beta F(t)uu_{xxxz}, \\
T_5^z =& \frac{1}{2}\alpha F''(t)yzu_x^2 + \frac{1}{4}F''(t)yzu_{xxx} - \frac{1}{6}\alpha F'(t)yuu_x u_{xx} + \frac{1}{6}\alpha F'(t)yu_x^3 - \frac{1}{6}F'(t)yuu_{xxxx} \\
& + \frac{1}{8}F'(t)yu_x u_{xxx} - \frac{1}{16}F'(t)yu_{xx}^2 - \frac{1}{6}\beta F(t)uu_x u_{xy} + \frac{1}{6}\beta F(t)u_x^2 u_y - \frac{1}{16\alpha}\beta F(t)uu_{xxxy} \\
& + \frac{1}{16\alpha}\beta F(t)u_x u_{xxy} + \frac{1}{16\alpha}\beta F(t)u_y u_{xxx} - \frac{1}{16\alpha}\beta F(t)u_{xx}u_{xy}.
\end{aligned}
$$

Case 6. For the multiplier $Q_6 = G(t)y$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_6^t =& -\alpha G(t)yu_x,\\
T_6^x =& \alpha G'(t)yu + \alpha G(t)yu_xu_z - \alpha G(t)yu_t + \frac{3}{4}G(t)yu_{xxz},\\
T_6^y =& \beta(G'(t)zu_y - \frac{1}{4}\beta G(t)uu_{xy} + \frac{1}{4}\beta G(t)u_xu_y,\\
T_6^z =& \alpha G'(t)yu + \alpha G(t)yu_xu_z - \alpha G(t)yu_t + \frac{3}{4}G(t)yu_{xxz}.
\end{aligned}$$

Case 7. For the multiplier $Q_7 = H'(t)z + \frac{1}{2}H(t)u_x$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_7^t =& -\alpha zu_xH'(t) + \left(\frac{1}{4}\alpha uu_{xx} - \frac{1}{4}\alpha u_x^2\right)H(t),\\
T_7^x =& \alpha zuH''(t) + \left(\alpha zu_xu_z - \frac{1}{4}\alpha uu_x - \alpha zu_t + \frac{3}{4}zu_{xxz} - \frac{1}{4}u_{xx}\right)H'(t) + \left(\frac{1}{3}\alpha u_x^2u_z\right.\\
&+\frac{1}{6}\alpha uu_xu_{xz} - \frac{1}{4}\alpha u_tu_x - \frac{1}{4}\alpha uu_{tx} + \frac{5}{16}u_xu_{xxz} + \frac{1}{4}\beta uu_{yy} + \frac{1}{16}uu_{xxxz} + \frac{1}{16}u_zu_{xxx}\\
&\left.-\frac{3}{16}u_{xx}u_{xz}\right)H(t),\\
T_7^y =& \beta H'(t)zu_y + \left(\frac{1}{4}\beta u_xu_y - \frac{1}{4}\beta uu_{xy}\right)H(t),\\
T_7^z =& \left(\frac{1}{2}\alpha zu_x^2 + \frac{1}{4}zu_{xxx}\right)H'(t) + \left(\frac{\alpha}{6}u_x^3 - \frac{\alpha}{6}u_xu_{xx} + \frac{1}{8}u_xu_{xxx} - \frac{1}{6}uu_{xxxx} - \frac{1}{6}u_{xx}^2\right)H(t).
\end{aligned}$$

Case 8. For the last multiplier $Q_8 = J(t)$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_8^t =& -\alpha u_xJ(t),\\
T_8^x =& \alpha J'(t)u + \left(\alpha u_xu_z - \alpha u_t + \frac{3}{4}u_{xxz}\right)J(t),\\
T_8^y =& \beta u_yJ(t),\\
T_8^z =& \left(\frac{1}{2}\alpha u_x^2 + \frac{1}{4}u_{xxx}\right)J(t).
\end{aligned}$$

3.2. Conservation Laws Using Noether's Approach

In this subsection, we utilize Noether's approach to derive consevation laws for the 3DgYTSF Equation (2). This equation is of fourth-order and it has a Lagrangian. It can be verified that Equation (2) has a second-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2}\left(u_{xx}u_{xz} - \beta u_y^2 - \alpha u_x^2u_z\right) + \alpha u_tu_x$$

as $\delta\mathcal{L}/\delta u = 0$, on the Equation (2). Here, the Euler-Lagrange operator $\delta/\delta u$ is given as in (20). The determining equation for Noether point symmetries is

$$\begin{aligned}
&X^{[2]}(\mathcal{L}) + \mathcal{L}\Big[D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3) + D_z(\xi^4)\Big] - D_t(B^1) - D_x(B^2)\\
&-D_y(B^3) - D_z(B^4) = 0,
\end{aligned} \tag{22}$$

where gauge terms B^1, B^2, B^3, and B^4 depend on t, x, y, z, and u. Here, $X^{[2]}$ is the second prolongation of the infinitesimal generator X and is defined by

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_3\frac{\partial}{\partial u_y} + \zeta_4\frac{\partial}{\partial u_z} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{24}\frac{\partial}{\partial u_{xz}}.$$

Hence expanding Equation (22) with the Lagrangian $\mathcal{L}$ gives

$$
\begin{aligned}
&\alpha u_x\Big\{\eta_t + u_t\eta_u - u_t\xi^1_t - u_t^2\xi^1_u - u_x\xi^2_t - u_t u_x\xi^2_u - u_y\xi^3_t - u_t u_y\xi^3_u - u_z\xi^4_t - u_t u_z\xi^4_u\Big\}\\
&+\alpha u_t\Big\{\eta_x + u_x\eta_u - u_t\xi^1_x - u_t u_x\xi^1_u - u_x\xi^2_x - u_x^2\xi^2_u - u_y\xi^3_x - u_x u_y\xi^3_u - u_z\xi^4_x - u_x u_z\xi^4_u\Big\}\\
&-\alpha u_x u_z\Big\{\eta_x + u_x\eta_u - u_t\xi^1_x - u_t u_x\xi^1_u - u_x\xi^2_x - u_x^2\xi^2_u - u_y\xi^3_x - u_x u_y\xi^3_u - u_z\xi^4_x - u_x u_z\xi^4_u\Big\}\\
&-\beta u_y\Big\{\eta_y + u_y\eta_u - u_t\xi^1_y - u_t u_y\xi^1_u - u_x\xi^2_y - u_x u_y\xi^2_u - u_y\xi^3_y - u_y^2\xi^3_u - u_z\xi^4_y - u_y u_z\xi^4_u\Big\}\\
&-\frac{1}{2}\alpha u_x^2\Big\{\eta_z + u_z\eta_u - u_t\xi^1_z - u_t u_z\xi^1_u - u_x\xi^2_z - u_x u_z\xi^2_u - u_y\xi^3_z - u_y u_z\xi^3_u - u_z\xi^4_z - u_z^2\xi^4_u\Big\}\\
&+\frac{1}{2}u_{xz}\Big\{\eta_{xx} + 2u_x\eta_{xu} + u_{xx}\xi^1_u - u_x^2\xi^1_{uu} - 2u_{xx}\xi^2_x - 2u_x^2\xi^2_{xu} - 3u_x u_{xx}\xi^2_u - u_x^3\xi^2_{uu} - 2u_{tx}\xi^1_x\\
&-u_t\xi^1_{xx} - 2u_t u_x\xi^1_{xu} - u_t u_{xx}\xi^1_u - 2u_x u_{tx}\xi^1_u - u_t u_x^2\xi^1_{uu} - u_{xy}\xi^3_x - u_x u_{xy}\xi^3_u - u_{xz}\xi^4_x\\
&-u_x u_{xz}\xi^4_u\Big\} + \frac{1}{2}u_{xx}\Big\{\eta_{xz} + u_z\eta_{xu} + u_x\eta_{zu} + u_x u_z\eta_{uu} - u_t\xi^1_{xz} - u_t u_z\xi^1_{xu} - u_t u_x\xi^1_{zu}\\
&-u_t u_x u_z\xi^1_{uu} - u_x\xi^2_{xz} - u_x u_z\xi^2_{xu} - u_x^2\xi^2_{zu} - u_x^2 u_z\xi^2_{uu} - u_y\xi^3_{xz} - u_y u_z\xi^3_{xu} - u_x u_y\xi^3_{zu}\\
&-u_x u_y u_z\xi^3_{uu} - u_z\xi^4_{xz} - u_z^2\xi^4_{xu} - u_x u_z\xi^4_{zu} - u_x u_z^2\xi^4_{uu} + u_{xz}\eta_u - u_{tz}\xi^1_x - u_x u_{tz}\xi^1_u - u_t u_{xz}\xi^1_u\\
&-u_{xz}\xi^2_x - 2u_x u_{xz}\xi^2_u - u_{yz}\xi^3_x - u_x u_{yz}\xi^3_u - u_y u_{xz}\xi^3_u - u_{zz}\xi^4_x - u_z u_{xz}\xi^4_u - u_x u_{zz}\xi^4_u\Big\}\\
&-B^1_t - u_t B^1_u - B^2_x - u_x B^2_u - B^3_y - u_y B^3_u - B^4_z - u_z B^4_u = 0.
\end{aligned}
$$

Splitting the above equation on derivatives of u yields the following system of overdetermined PDEs:

$$
\begin{aligned}
&\xi^1_u = 0,\ \xi^2_u = 0,\ \xi^3_u = 0,\ \xi^4_u = 0,\ \xi^4_y = 0,\ B^4_u = 0,\ \eta_{xx} = 0,\ \eta_{xz} = 0,\ \xi^1_x = 0,\\
&\xi^3_x = 0,\ \xi^3_z = 0,\ \xi^4_x = 0,\ \xi^1_z = 0,\ \xi^2_z = 0,\ \xi^4_x = 0,\ \xi^1_y = 0,\ \eta_{uu} = 0, \xi^2_{xx} = 0,\ \eta_{xu} = 0,\\
&\eta_{zu} = 0,\ 3\alpha\xi^2_y - 2\xi^3_t = 0,\ B^2_u - 2\eta_t = 0,\ \eta_x + \xi^4_t = 0,\ B^1_u - 2\eta_x = 0,\ \eta_z + 2\xi^2_t = 0,\\
&B^3_u + 3\alpha\eta_y = 0,\ 2\eta_u + \xi^3_y + \xi^4_z = 0,\ B^1_t + B^2_x + B^3_y + B^4_z = 0,\ 3\eta_u + \xi^1_t - \xi^2_x + \xi^3_y = 0,\\
&2\eta_u + \xi^1_t + 2\xi^2_x + \xi^3_y = 0,\ 2\eta_u + \xi^1_t + \xi^2_x - \xi^3_y + \xi^4_z = 0
\end{aligned}
\tag{23}
$$

Solving the above overdetermined system of PDEs, we obtain the Noether symmetries and their gauge functions as follows:

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_2 = \frac{\partial}{\partial x},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_3 = \frac{\partial}{\partial y},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_4 = \frac{\partial}{\partial z},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_5 = \frac{\partial}{\partial u},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_6 = y\frac{\partial}{\partial u},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0,\\
&X_7 = 7t\frac{\partial}{\partial t} + 3x\frac{\partial}{\partial x} + 5y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 3u\frac{\partial}{\partial u},\ B^t = 0,\ B^x = 0,\ B^y = 0,\ B^z = 0.
\end{aligned}
$$

We now construct the conservation laws corresponding to each Noether point symmetry using [44]

$$
\begin{aligned}
T^t &= \mathcal{L}\xi^1 + K_1 \frac{\partial \mathcal{L}}{\partial u_t} - B^t, \\
T^x &= \mathcal{L}\xi^2 + K_1 \left\{ \frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right\} + K_2 \frac{\partial \mathcal{L}}{\partial u_{xx}} + K_3 \frac{\partial \mathcal{L}}{\partial u_{xz}} - B^x, \\
T^y &= \mathcal{L}\xi^3 + K_1 \frac{\partial \mathcal{L}}{\partial u_y} - B^y, \\
T^z &= \mathcal{L}\xi^4 + K_1 \left\{ \frac{\partial \mathcal{L}}{\partial u_z} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xz}}\right)\right\} - B^z,
\end{aligned} \tag{24}
$$

where $K_1 = \eta - u_t\xi^1 - u_x\xi^2 - u_y\xi^3 - u_z\xi^4$, $K_2 = \zeta_2 - u_{xt}\xi^1 - u_{xx}\xi^2 - u_{xy}\xi^3 - u_{xz}\xi^4$ and $K_3 = \zeta_4 - u_{tz}\xi^1 - u_{xz}\xi^2 - u_{yz}\xi^3 - u_{zz}\xi^4$. Hence, using Formulae (24), we obtain the conserved vectors whose components are

$$
\begin{aligned}
T_1^t &= \frac{1}{2}u_{xx}u_{xz} - \frac{3}{2}\alpha u_y^2 - u_x^2 u_z, \\
T_1^x &= 2u_t u_x u_z + \frac{3}{4}u_t u_{xxz} - \frac{1}{4}u_{xx}u_{tz} - \frac{1}{2}u_{tx}u_{xz} - 2u_t^2, \\
T_1^y &= 3\alpha u_t u_y, \\
T_1^z &= u_t u_x^2 + \frac{1}{4}u_t u_{xxx} - \frac{1}{4}u_{xx}u_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_2^t &= -2u_x^2, \\
T_2^x &= u_x^2 u_z - \frac{1}{4}u_{xx}u_{xz} + \frac{3}{4}u_x u_{xxz} - \frac{3}{2}\alpha u_y^2, \\
T_2^y &= 3\alpha u_x u_y, \\
T_2^z &= u_x^3 + \frac{1}{4}u_{xxx}u_x - \frac{1}{4}u_{xx}^2;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= -2u_x u_y, \\
T_3^x &= -2u_t u_y + 2u_x u_y u_z - \frac{1}{2}u_{xy}u_{xz} - \frac{1}{4}u_{xx}u_{yz} + \frac{3}{4}u_y u_{xxz}, \\
T_3^y &= 2u_t u_x - u_x^2 u_z + \frac{1}{2}u_{xx}u_{xz} + \frac{3}{2}\alpha u_y^2, \\
T_3^z &= u_x^2 u_y - \frac{1}{4}u_{xx}u_{xy} + \frac{1}{4}u_{xxx}u_y
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= -2u_x u_z, \\
T_4^x &= -2u_t u_z + 2u_x u_z^2 + \frac{3}{4}u_z u_{xxz} - \frac{1}{2}u_{xz}^2 - \frac{1}{4}u_{xx}u_{zz}, \\
T_4^y &= 3\alpha u_y u_z, \\
T_4^z &= 2u_t u_x + \frac{1}{4}u_{xx}u_{xz} + \frac{1}{4}u_{xxx}u_z - \frac{3}{2}\alpha u_y^2
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= 2u_x, \\
T_5^x &= 2u_t - 2u_x u_z - \frac{3}{4}u_{xxz}, \\
T_5^y &= -3\alpha u_y, \\
T_5^z &= -u_x^2 - \frac{1}{4}u_{xxx}
\end{aligned}
$$

$$
\begin{aligned}
T_6^t &= 2yu_x, \\
T_6^x &= 2yu_t - 2yu_x u_z - \frac{3}{4}yu_{xxz}, \\
T_6^y &= -3\alpha y u_y, \\
T_6^z &= -yu_x^2 - \frac{1}{4}yu_{xxx};
\end{aligned}
$$

$$
\begin{aligned}
T_7^t &= \frac{7}{2}tu_{xx}u_{xz} - \frac{21}{2}\alpha t u_y^2 - 7tu_x^2 u_{-}6uu_x - 6xu_x^2 - 10yu_x u_y - 2zu_x u_z, \\
T_7^x &= -\frac{9}{2}\alpha x u_y^2 + 3xu_x^2 u_z - 6uu_t - 14tu_t^2 - 10yu_t u_y - 2zu_t u_z + 6uu_t u_z + 14tu_t u_x u_z \\
&\quad + 10yu_x u_y u_z + 2zu_x u_z^2 + \frac{3}{2}uu_{xxxz} + \frac{7}{2}tu_t u_{xxz} + \frac{3}{2}xu_x u_{xxz} + \frac{5}{2}yu_y u_{xxz} \\
&\quad + \frac{1}{2}zu_z u_{xxz} - 3u_x u_{xz} - \frac{7}{2}tu_{tx}u_{xz} - \frac{5}{2}yu_{xy}u_{xz} - 2u_{xx}u_z - \frac{7}{2}tu_{tz}u_{xx} - \frac{3}{2}xu_{xx}u_{xz} \\
&\quad - \frac{5}{2}u_{xx}u_{[}yz] - \frac{1}{2}zu_{xx}u_{zz}, \\
T_7^y &= 10yu_t u_x + \frac{5}{2}yu_{xx}u_{xz} + \frac{15}{2}\alpha y u_y^2 - 5yu_x^2 u_z + 9\alpha u u_y + 21\alpha t u_t u_y + 9\alpha x u_x u_y \\
&\quad + 3\alpha z u_y u_z, \\
T_7^z &= 2zu_t u_x + \frac{1}{2}zu_{xx}u_{xz} - \frac{3}{2}\alpha z u_y^2 + 3uu_x^2 + 7tu_t u_x^2 + 3xu_x^3 + 5yu_x^2 u_y + \frac{3}{2}uu_{xxx} \\
&\quad + \frac{7}{2}tu_t u_{xxx} + \frac{3}{2}xu_x u_{xxx} + \frac{5}{2}yu_y u_{xxx} + \frac{1}{2}zu_z u_{xxx}.
\end{aligned}
$$

4. Conclusions

In this paper, we studied the fourth-order three-dimensional generalized potential Yu-Toda-Sasa-Fukuyama (gpYTSF) Equation (2). We first determined its Lie point symmetries and then presented the corresponding group of transformations. These Lie symmetries were then used to perform symmetry reductions, and as a result reduced ordinary differential equations were obtained. The ordinary differential equations were then solved using direct integration and exact solutions of gpYTSF equation were obtained. The group invariant solutions obtained included a solution in terms of incomplete elliptic integral. It should be noted that the solutions obtained in this paper are new and different from the ones presented in the literature. Moreover, conservation laws for the gpYTSF equation were constructed by employing two different approaches; the multiplier approach and Noether's approach. The multiplier approach provided us with eight conservation law multipliers which resulted in eight conservation laws, whereas the Noether's theorem yielded seven conservation laws. This is the first time that conservation laws have been derived for Equation (2). These conservation laws included the conservation of mass and energy.

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