Article

New Solitary and Periodic Wave Solutions of $(n + 1)$ -Dimensional Fractional Order Equations Modeling Fluid Dynamics

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Abstract: In this study, first, fractional derivative definitions in the literature are examined and their disadvantages are explained in detail. Then, it seems appropriate to apply the $(\frac{G'}{G})$ -expansion method under Atangana's definition of β -conformable fractional derivative to obtain the exact solutions of the space–time fractional differential equations, which have attracted the attention of many researchers recently. The method is applied to different versions of $(n + 1)$ -dimensional Kadomtsev–Petviashvili equations and new exact solutions of these equations depending on the β parameter are acquired. If the parameter values in the new solutions obtained are selected appropriately, 2D and 3D graphs are plotted. Thus, the decay and symmetry properties of solitary wave solutions in a nonlocal shallow water wave model are investigated. It is also shown that all such solitary wave solutions are symmetrical on both sides of the apex. In addition, a close relationship is established between symmetric and propagated wave solutions.

Keywords: β -conformable fractional derivative of Atangana; $(\frac{G'}{G})$ -expansion method; space–time fractional differential equations; wave solution

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1. Introduction

The effectiveness of integer-ordered derivatives of known mathematical models, including nonlinear models, is discussed in most cases. One of the most important shortcomings of the integer-order fractional derivative is that the results of the research and observation conducted with the integer-order differential model are not completely consistent. However, unlike other models, the fractional differential equation model produces consistent results by removing this flaw in the study findings. The fractional differential equation facilitates the process with more practical and short expressions in the phase of determining the physical and mechanical problems and makes the problems clear with the specific contents at this stage. In recent years, fractional analysis has found application in many real-life problems such as control theory, chemistry, electricity, economics, biology, mechanics, signal and image processing. There are many derivative definitions in the literature due to the wide application area and modeling achievement: Riemann–Liouville fractional integral and fractional derivative [1], Jumarie derivative [2], Grünwald–Letnikov derivative [3], Caputo fractional derivative [4], Weyl derivative [5], Marchaud derivative [6], Hadamard derivative [7], Chen left-sided and right-sided derivative [8], Davidson–Essex derivative [9], Coimbra derivative [10], Canavati derivative [11], Osler fractional derivative [12], local fractional Yang derivative [13] and Riesz derivative [14]. When trying to

model real situations with fractional differential equations, the Riemann–Liouville derivative it has some disadvantages. The Riemann–Liouville derivative of a constant is not zero. On the other hand, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag–Leffler functions. These disadvantages narrow the field of application of the Riemann–Liouville fractional derivative. Caputo derivatives can only be defined for differentiable functions, while functions without first-order derivatives can have fractional derivatives of all orders less than one in the Riemann–Liouville sense. In Jumarie fractional derivative, if the function is not at the continuous origin, there is no fractional derivative, for example, the natural logarithm function. Even though the Weyl fractional derivative has found its place in groundwater research, it still has a significant drawback; since the integral describing these Weyl derivatives is inappropriate, greater constraints need to be placed on a function. Khalil has also introduced a new definition of derivative to the literature, which is clearly compatible with integer-order derivatives [15]. This fractional derivative definition done is conformable with traditional features such as chain rule. As in all fields of science, fractional derivative is a field that progresses cumulatively and owes its development to the completion of its deficiencies. The congruent fractional derivative has the deficiency that the derivative of any differentiable function at the zero point is equal to zero [16]. A new notation has been added to the conformable operator by Atangana and this operator has been named the β -conformable fractional derivative [17]. The new version purposed, similar to the conformable derivative, not only provided the traditional properties of the integer-order derivative, but also enabled the removal of the deficiency mentioned above.

The $\left(\frac{G'}{G}\right)$ -expansion method has been introduced to the literature by Wang in 2008 [18].

The logic of the $\left(\frac{G'}{G}\right)$ -expansion method is based on obtaining the solutions of a nonlinear differential equation in the form of polynomials. The degree of this polynomial is calculated by the homogeneous equilibrium relation. The difference of this method from other methods is that a linear differential equation with a constant coefficient of the second order is used instead of Riccati equation.

Russian physicists B. Kadomtsev and V. Petviashvili published an article revealing the KP equation. They used the KP equation to study the stability of the single-soliton solution of the Korteweg–de Vries (KdV) equation under transverse perturbations. They produced this equation, named after the researchers, to investigate the evolution of small-amplitude long ion-acoustic waves propagating in plasma [19]. The KP equation is one of the most commonly used models in nonlinear wave theory. It occurs as a reduction in a quadratic nonlinear system accepting weakly dispersed waves in the non-parallel wave approach. It is currently used as a classic model for the development and checking of new mathematical techniques, e.g., in applications of dynamical system methods for water waves [20], variational theory of existence, stability of energy minimizers [21] and non-classical function spaces [22]. It has also been widely used as a model for two-dimensional shallow water waves [23–25] and ion-acoustic waves in plasmas, for example [26]. Owing to its wide range of applications of powdered plasma in industry and microelectronics, studying nonlinear structures in various plasmas is one of the important research areas that has been the focus of researchers for the past 50 years [27,28].

Examining the spectral collocation method with the help of Chebyshev polynomials under the fractional derivative of Caputo–Fabrizio space can be given as an example to investigate a final boundary value problem for pseudo-parabolic partial differential equations with nonlinear reaction terms [29,30]. In accordance with the information given, the main framework of the study was formed by combining the $\left(\frac{G'}{G}\right)$ -expansion method which has an important place in the literature and the new fractional derivative definition. Some important properties of β -conformable fractional derivative and application of $\left(\frac{G'}{G}\right)$ -expansion method are mentioned respectively in the following chapters. Later, this method is applied to different versions of the $(n + 1)$ -dimensional KP equation. In addition, some approaches have been used recently for the derivation of conservation laws and Lie

symmetry analyzes for the fractional differential equations [31–33]. At the beginning of these approaches are the famous Noether theorem [34] and then the new method developed by Ibragimov [35]. Later, generalized Noether operator and fractional generalized Noether operator have been used for fractional differential equations [36–38]. Since the main subject of this article is not conservation laws, new studies can be done in the light of cited articles above.

The Kadomtsev–Petviashvili–Boussinesq (KPB) equation, which models both right and left waves, has been introduced to the literature by two scientists named Wazwaz and El Tantawy in 2017 [39].

$${}_0^A D_{xxxy}^{4\beta} u + 3{}_0^A D_x^\beta \left({}_0^A D_x^\beta u {}_0^A D_y^\beta u \right) + {}_0^A D_{ty}^{2\beta} u + {}_0^A D_{tx}^{2\beta} u + {}_0^A D_t^{2\beta} u - {}_0^A D_z^{2\beta} u = 0 \quad (1)$$

The generalized (2 + 1)-dimensional Camassa–Holm Kadomtsev–Petviashvili (2D-gCHKP) equation is expressed as:

$${}_0^A D_x^\beta \left({}_0^A D_t^\beta u + h_1 {}_0^A D_x^\beta u + h_2 u {}_0^A D_x^\beta u + h_3 {}_0^A D_{xx}^{3\beta} u \right) + {}_0^A D_y^{2\beta} u = 0 \quad (2)$$

The (3 + 1)-dimensional equation modeling nonlinear wave in liquid including gas bubbles

$${}_0^A D_x^\beta \left({}_0^A D_t^\beta u + h_1 u {}_0^A D_x^\beta u + {}_0^A D_x^{3\beta} u \right) + h_2 {}_0^A D_x^{2\beta} u + h_3 {}_0^A D_y^{2\beta} u + h_4 {}_0^A D_z^{2\beta} u = 0 \quad (3)$$

The (3 + 1)-dimensional B-type Kadomtsev–Petviashvili (BKP) equation

$${}_0^A D_{ty}^{2\beta} u - {}_0^A D_{xxxy}^{4\beta} u - 3{}_0^A D_x^\beta \left({}_0^A D_x^\beta u {}_0^A D_y^\beta u \right) + 3{}_0^A D_{xz}^{2\beta} u + {}_0^A D_t^{2\beta} u = 0 \quad (4)$$

where h_1 , h_2 , and h_3 are arbitrary constants. To examine the solutions of the KP equation that explains the function of dispersion in the pattern formation phase in liquid drops were used methods such as sine-cosine and tanh methods [40], homoclinic breath limit approach [41], solitary wave ansatz [42], exp-function, $\left(\frac{G'}{G}\right)$ -methods [43] and Lie group method [44].

2. Mathematical Background

Definition 1. Let $f : [0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative of a order f is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \forall t > 0, \alpha \in (0, 1)$$

If this limit exists, the function f is called α -order differentiable [15].

Definition 2. Let $a \in \mathbb{R}$ ve $f : [a, \infty) \rightarrow \mathbb{R}$ be a function β -conformable derivative of f is defined

$${}_0^A D_x^\beta \{f(x)\} = \lim_{\varepsilon \rightarrow 0} \frac{f\left(x + \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\varepsilon}, \quad 0 < \beta \leq 1$$

by the expression [45].

Definition 3. f is a function with x and t variables β -conformable fractional derivative of f function with respect to x is defined as

$${}_0^A D_x^\beta \{f(x, t)\} = \lim_{\varepsilon \rightarrow 0} \frac{f\left(x + \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}, t\right) - f(x, t)}{\varepsilon}$$

Therefore, contrary to other fractional derivative definitions, the β -derivative of a function is similar to the first-order derivative as it is identified locally in a specific case [45].

Theorem 1. *Clairaut's theorem for partial β -derivatives: Assume that $f(x, y)$ is function which $\partial_x^\beta [\partial_y^\alpha (f(x, y))]$ and $\partial_y^\alpha [\partial_x^\beta (f(x, y))]$ exist and are continuous over the domain $D \subset \mathbb{R}_2$ then,*

$$\partial_x^\beta [\partial_y^\alpha (f(x, y))] = \partial_y^\alpha [\partial_x^\beta (f(x, y))]$$

Proof. See in [45]. $\square$

Assuming that $g \neq 0$ and f are two functions β -differentiable with $\beta \in (0, 1]$ then, the following relations can be satisfied

$${}_0^A D_x^\beta (\mu f(x) + \alpha g(x)) = \mu {}_0^A D_x^\beta (f(x)) + \alpha {}_0^A D_x^\beta (g(x))$$

for all μ and α real number.

$${}_0^A D_x^\beta (c) = 0$$

for c any given constant. β -conformable fractional derivative of the product of two functions is written as

$${}_0^A D_x^\beta (f(x) \cdot g(x)) = {}_0^A D_x^\beta (f(x))g(x) + f(x){}_0^A D_x^\beta (g(x))$$

Assuming $g \neq 0$, the β -conformable fractional derivative of the quotient of the two functions is written as

$${}_0^A D_x^\beta \left(\frac{f(x)}{g(x)} \right) = \frac{{}_0^A D_x^\beta (f(x))g(x) - {}_0^A D_x^\beta (g(x))f(x)}{g^2(x)}$$

3. General Description of the Proposed Method

x and t are independent variables,

$$F(u, {}_0^A D_t^\beta u, {}_0^A D_x^\beta u, {}_0^A D_t^{2\beta} u, {}_0^A D_x^{2\beta} u, {}_0^A D_t^{3\beta} u, {}_0^A D_x^{3\beta} u, \dots) = 0 \quad (5)$$

consider the nonlinear space-time fractional partial differential equation, here F is a polynomial of $u(x, t)$ is unknown function and ${}_0^A D_x^\beta u, {}_0^A D_t^\beta u$ derivatives are conformable fractional derivatives of the β order Atangana of the function $u(x, t)$ with respect to x and t .

k and m are non-zero constants for obtaining moving-wave solutions of Equation (5) given wave transformation as $u(x, t) = u(\xi)$.

$$\xi = \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta \quad (6)$$

When (6) wave transformation is used with definition (3)

$${}_0^A D_x^\beta \{u(x, t)\} = \lim_{\varepsilon \rightarrow 0} \frac{u\left(x + \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}, t\right) - u(x, t)}{\varepsilon}$$

for simplicity

$$h = \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \quad \varepsilon = h \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1}$$

by transforming

$${}_0^A D_x^\beta \{u(x, t)\} = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \lim_{h \rightarrow 0} \frac{u(x + h, t) - u(x, t)}{h}$$

if the arrangement is made

$${}_0^A D_x^\beta \{u(x, t)\} = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{\partial u(x, t)}{\partial x}$$

is obtained. Same way, it is clear that:

$${}_0^A D_t^\beta \{u(x, t)\} = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{\partial u(x, t)}{\partial t}$$

Since $u(x, t) = u(\xi)$ is assumed and using the chain rule

$$\frac{\partial u(x, t)}{\partial x} = \frac{\partial u(\xi)}{\partial \xi} = \frac{du}{d\xi} \frac{\partial \xi}{\partial x} = u' \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1}$$

is found. Conformable fractional derivatives of the β order Atangana in Equation (6) obtained in the form of

$${}_0^A D_x^\beta \{u(x, t)\} = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} u' \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} = mu'$$

$${}_0^A D_t^\beta \{u(x, t)\} = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} u' \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} = ku'$$

Using these equalities, the (5) space–time fractional differential equation is reduced to

$$F(u, ku', mu', k^2 u'', m^2 u'', k^3 u''', m^3 u''', \dots) = 0 \quad (7)$$

the ordinary differential equation. If we assume that $G = G(\xi)$

$$G'' + \lambda G' + \mu G = 0 \quad (8)$$

it is the function that provides the second order linear differential Equation (8) and the solution of the ordinary differential equation is written in terms of polynomials

$$u(\xi) = \sum_{i=0}^n a_i \left(\frac{G'}{G}\right)^i \quad (9)$$

Here λ and μ are arbitrary constants, and n is a positive integer to be obtained from the homogeneous equilibrium relation. The balancing relation is the term with the largest order derivative $\frac{d^q u}{d\xi^q}$ and the largest order nonlinear term $u^p \left(\frac{d^r u}{d\xi^r}\right)^s$

$$D \left[\frac{d^q u}{d\xi^q} \right] = n + q \quad D \left[u^p \left(\frac{d^r u}{d\xi^r} \right)^s \right] = np + s(n + r)$$

$$n + q = np + s(n + r)$$

by equalizing these expressions is finding the value of n . The expression (9) is a polynomial according to $\left(\frac{G'}{G}\right)$, and is written instead of Equation (7) by equating the coefficients of the same degree terms are equalized to zero, and the algebraic equation system consisting of unknowns $a_0, a_1, \dots, a_n, m$ and k is obtained. The unknown constants are found using the algebraic system of equations. After these constants are written in the expression (9), Equation (6) using the wave transformation and the solution cases of the second order linear differential equation obtained previously, (5) moving-wave solutions of the space–time fractional partial differential equation are obtained.

The solutions of the second order linear differential equation number (8) according to the delta cases are as follows:

$$\lambda^2 - 4\mu > 0$$

$$\left(\frac{G'}{G}\right) = \psi_1 = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{\varphi \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{\varphi \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right] - \frac{\lambda}{2}$$

$$\lambda^2 - 4\mu < 0$$

$$\left(\frac{G'}{G}\right) = \psi_2 = \frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-\varphi \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \gamma \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{\gamma \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \varphi \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right] - \frac{\lambda}{2}$$

$$\lambda^2 - 4\mu = 0$$

$$\left(\frac{G'}{G}\right) = \psi_3 = \frac{\frac{\varphi + \gamma}{2}}{\frac{\varphi + \gamma}{2} + \frac{\varphi - \gamma}{2} \xi} - \frac{\lambda}{2}$$

4. Applications

To find the wave solutions of nonlinear fractional partial differential equations in the literature, the $\left(\frac{G'}{G}\right)$ -expansion method, which is a reliable and effective method that greatly reduces the size of the transaction volume compared to existing techniques, has been applied to different versions of $(n + 1)$ -dimensional KP equations.

To seek for the traveling wave solution of Equation (1)

$${}_0^A D_{xxxy}^{4\beta} u + 3{}_0^A D_x^\beta \left({}_0^A D_x^\beta u {}_0^A D_y^\beta u \right) + {}_0^A D_{ty}^{2\beta} u + {}_0^A D_{tx}^{2\beta} u + {}_0^A D_t^{2\beta} u - {}_0^A D_z^{2\beta} u = 0$$

$$\xi = \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{o}{\beta} \left(z + \frac{1}{\Gamma(\beta)} \right)^\beta$$

if $u = u(\xi)$ is applied to the wave transform Equation (1) and then its integral is calculated, Equation (1) is reduced to

$$m^3 l u''' + 3m^2 l (u')^2 + k l u' + k m u' + k^2 u' - o^2 u' = 0$$

the ordinary differential equation. When the balancing relation is applied to this ordinary differential equation

$$n + 3 = 2(n + 1) \quad n = 1$$

is found. Considering the found value of n and substituting the expression (7) in the ordinary differential equation, the coefficients of the power of the expression $\left(\frac{G'}{G}\right)$ are obtained as follows:

$$\left(\frac{G'}{G}\right)^0 = -\mu a_1 (k^2 + k l + k m - o^2 + l m^3 \lambda^2 + 2 l m^3 \mu - 3 l m^2 \mu a_1)$$

$$\left(\frac{G'}{G}\right)^1 = -\lambda a_1 (k^2 + k l + k m - o^2 + l m^3 \lambda^2 + 8 l m^3 \mu - 6 l m^2 \mu a_1)$$

$$\left(\frac{G'}{G}\right)^2 = -a_1(k^2 + kl + km - o^2 + 7lm^3\lambda^2 + 8lm^3\mu - 3lm^2\lambda^2a_1 - 6lm^2\mu a_1)$$

$$\left(\frac{G'}{G}\right)^3 = 6lm^2\lambda a_1(-2m + a_1)$$

$$\left(\frac{G'}{G}\right)^4 = 3lm^2a_1(-2m + a_1)$$

Here, if each coefficient obtained equals zero and the resulting system of equations is solved,

$$a_1 = 2m, \quad k = -\frac{l}{2} - \frac{m}{2} - \frac{1}{2}\sqrt{l^2 + 2lm + m^2 + 4o^2 - 4lm^3\lambda^2 + 16lm^3\mu}$$

$$a_1 = 2m, \quad k = -\frac{l}{2} - \frac{m}{2} + \frac{1}{2}\sqrt{l^2 + 2lm + m^2 + 4o^2 - 4lm^3\lambda^2 + 16lm^3\mu}$$

coefficients are acquired. If the k value found is substituted in the expression ξ

$$\xi = \frac{-\frac{l}{2} - \frac{m}{2} + \frac{1}{2}\sqrt{l^2 + 2lm + m^2 + 4o^2 - 4lm^3\lambda^2 + 16lm^3\mu}}{\beta} \left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$$

$$+ \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{o}{\beta} \left(y + \frac{1}{\Gamma(\beta)}\right)^\beta$$

According to the cases of the coefficients and $\left(\frac{G'}{G}\right)$ expressions Δ , the following $u(x, t)$ solutions are acquired.

$$u_1(x, t) = a_0 + (2m) \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{\varphi \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{\varphi \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right] - \frac{\lambda}{2} \right]$$

$$u_2(x, t) = a_0 + (2m) \left[\frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-\varphi \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \gamma \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{\varphi \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \gamma \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right] - \frac{\lambda}{2} \right]$$

2D and 3D graphics for $u_2(x, t)$ of the solution functions are given in Figure 1 as an example.

$$u_3(x, t) = a_0 + (2m) \left(\frac{\frac{\varphi + \gamma}{2}}{\frac{\varphi + \gamma}{2} + \frac{\varphi - \gamma}{2} \xi} - \frac{\lambda}{2} \right)$$

Figure 1. The 3D graph of the solution function $u_2(x, t)$ for $a_0 = 0$; $l = -1$; $m = -2$; $o = 1$; $\lambda = 4$; $\mu = 8$; $c = 4$; $n = 4$; $\beta = 0.5$; $z = -2$; $y = -2$ values is given on the left graph, the 2D graph for $u_2(0, t)$ is given on the right.

If the other k value obtained is written instead of ξ , the other solutions of Equation (1) are written similarly. To look for the traveling wave solution of Equation (2)

$${}_0^A D_x^\beta \left({}_0^A D_t^\beta u + h_1 {}_0^A D_x^\beta u + h_2 u {}_0^A D_x^\beta u + h_3 {}_0^A D_{xt}^{2\beta} u \right) + {}_0^A D_y^{2\beta} u = 0$$

$$\xi = \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)} \right)^\beta$$

if $u = u(\xi)$ is applied to the wave transform Equation (2) and then its integral is calculated, Equation (2) turns into

$$m \left(ku' + h_1 mu' + h_2 muu' + h_3 m^2 ku''' \right) + l^2 u' = 0$$

the ordinary differential equation is obtained. When the balancing relation is applied to this ordinary differential equation

$$n + n + 1 = n + 3 \quad n = 2$$

is found. Considering the found value of n and substituting the expression (7) in the ordinary differential equation, the coefficients of the power of the expression $\left(\frac{G'}{G} \right)$ are obtained as follows:

$$\begin{aligned} \left(\frac{G'}{G} \right)^0 &= -\mu \left(l^2 a_1 + kma_1 + m^2 a_1 h_1 + m^2 a_0 a_1 h_2 + km^3 \lambda^2 a_1 h_3 + 2km^3 \mu a_1 h_3 + 6km^3 \lambda \mu a_2 h_3 \right) \\ \left(\frac{G'}{G} \right)^1 &= \left(-l^2 \lambda a_1 - km \lambda a_1 - 2l^2 \mu a_2 - 2km \mu a_2 - m^2 \lambda a_1 h_1 - 2m^2 \mu a_2 h_1 - m^2 \lambda a_0 a_1 h_2 \right. \\ &\quad \left. - m^2 \mu a_1^2 h_2 - 2m^2 \mu a_0 a_2 h_2 - km^3 \lambda (\lambda^2 - 2\mu) a_1 h_3 - 10km^3 \lambda \mu a_1 h_3 - 12km^3 \lambda^2 \mu a_2 h_3 \right. \\ &\quad \left. - 18km^3 \mu^2 a_2 h_3 + 2km^3 \mu (-\lambda^2 + \mu) a_2 h_3 \right) \\ \left(\frac{G'}{G} \right)^2 &= \left(-l^2 a_1 - kma_1 - 2l^2 \lambda a_2 - 2km \lambda a_2 - m^2 a_1 h_1 - 2m^2 \lambda a_2 h_1 - m^2 a_0 a_1 h_2 - m^2 \lambda a_1^2 h_2 \right. \\ &\quad \left. - 2m^2 \lambda a_0 a_2 h_2 - 3m^2 \mu a_1 a_2 h_2 - 7km^3 \lambda^2 a_1 h_3 - 8km^3 \mu a_1 h_3 - 6km^3 \lambda^3 a_2 h_3 \right. \\ &\quad \left. - 2km^3 \lambda (\lambda^2 - 2\mu) a_2 h_3 - 56km^3 \lambda \mu a_2 h_3 \right) \end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^3 &= -\left(2l^2a_2 + 2kma_2 + 2m^2a_2h_1 + m^2a_1^2h_2 + 2m^2a_0a_2h_2 + 3m^2\lambda a_1a_2h_2 + 2m^2\mu a_2^2h_2h_3\right. \\ &\quad \left.+ 12km^3\lambda a_1 + 38km^3\lambda^2a_2h_3 + 40km^3\mu a_2h_3\right) \\ \left(\frac{G'}{G}\right)^4 &= -m^2\left(3a_1a_2h_2 + 2\lambda a_2^2h_2 + 6kma_1h_3 + 54km\lambda a_2h_3\right) \\ \left(\frac{G'}{G}\right)^5 &= -2m^2a_2(a_2h_2 + 12kmh_3)\end{aligned}$$

Here, if each coefficient obtained equals zero and the resulting equation system is solved,

$$a_0 = -\frac{l^2 + km + m^2h_1}{m^2h_2} - \frac{km(\lambda^2 + 8\mu)h_3}{h_2}, a_1 = -\frac{12km\lambda h_3}{h_2}, a_2 = -\frac{12kmh_3}{h_2}$$

coefficients are obtained. According to the cases of the coefficients and $\left(\frac{G'}{G}\right)$ expressions Δ , the following $u(x, t)$ solutions are written.

$$u_1(x, t) = -\frac{l^2 + km + m^2h_1}{m^2h_2} - \frac{km(\lambda^2 + 8\mu)h_3}{h_2} + \left(-\frac{12km\lambda h_3}{h_2}\right)\psi_1 + \left(-\frac{12kmh_3}{h_2}\right)\psi_1^2$$

$$u_2(x, t) = -\frac{l^2 + km + m^2h_1}{m^2h_2} - \frac{km(\lambda^2 + 8\mu)h_3}{h_2} + \left(-\frac{12km\lambda h_3}{h_2}\right)\psi_2 + \left(-\frac{12kmh_3}{h_2}\right)\psi_2^2$$

2D and 3D graphics for $u_2(x, t)$ of the solution functions are given in Figure 2 as an example.

$$u_3(x, t) = -\frac{l^2 + km + m^2h_1}{m^2h_2} - \frac{km(\lambda^2 + 8\mu)h_3}{h_2} + \left(-\frac{12km\lambda h_3}{h_2}\right)\psi_3 + \left(-\frac{12kmh_3}{h_2}\right)\psi_3^2$$

Figure 2. The 3D graph of the solution function $u_2(x, t)$ for $k = 3; l = -1; m = 2; \lambda = 1; \mu = 3; c = -4; n = 6; \beta = 0.5; h_1 = -1; h_2 = 2; h_3 = -2; y = 6$; values is given on the left graph, the 2D graph for $u_2(0, t)$ is given on the right.

To seek for the traveling wave solution of Equation (3)

$${}_0^A D_x^\beta \left({}_0^A D_t^\beta u + h_1 u {}_0^A D_x^\beta u + {}_0^A D_x^{3\beta} u \right) + h_{20} {}_0^A D_x^{2\beta} u + h_{30} {}_0^A D_y^{2\beta} u + h_{40} {}_0^A D_z^{2\beta} u = 0$$

$$\xi = \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{o}{\beta} \left(z + \frac{1}{\Gamma(\beta)} \right)^\beta$$

if $u = u(\xi)$ is applied to the wave transform Equation (3) and then its integral is calculated, Equation (3) is reduced to

$$m(ku' + h_1 muu' + h_3 m^3 u''') + m^2 u' + l^2 u' + o^2 u' = 0$$

the ordinary differential equation is obtained. When the balancing relation is applied to this ordinary differential equation

$$n + n + 1 = n + 3 \quad n = 2$$

is found. Considering the found value of n and substituting the expression (7) in the ordinary differential equation, the coefficients of the power of the expression $\left(\frac{G'}{G}\right)$ are obtained as follows:

$$\left(\frac{G'}{G}\right)^0 = -\mu(kma_1 + m^4 \lambda^2 a_1 + 2m^4 \mu a_1 + 6m^4 \lambda \mu a_2 + m^2 a_0 a_1 h_1 + m^2 a_1 h_2 + l^2 a_1 h_3 + o^2 a_1 h_4)$$

$$\begin{aligned} \left(\frac{G'}{G}\right)^1 = & \left(-km\lambda a_1 - m^4 \lambda(\lambda^2 - 2\mu)a_1 - 10m^4 \lambda \mu a_1 - 2km\mu a_2 - 12m^4 \lambda^2 \mu a_2 - 18m^4 \mu^2 a_2 \right. \\ & + 2m^4 \mu(-\lambda^2 + \mu)a_2 - m^2 \lambda a_0 a_1 h_1 - m^2 \mu a_1^2 h_1 - 2m^2 \mu a_0 a_2 h_1 - m^2 \lambda a_1 h_2 - 2m^2 \mu a_2 h_2 \\ & \left. - l^2 \lambda a_1 h_3 - 2l^2 \mu a_2 h_3 - o^2 \lambda a_1 h_4 - 2o^2 \mu a_2 h_4\right) \end{aligned}$$

$$\begin{aligned} \left(\frac{G'}{G}\right)^2 = & \left(-kma_1 - 7m^4 \lambda^2 a_1 - 8m^4 \mu a_1 - 2km\lambda a_2 - 6m^4 \lambda^3 a_2 - 2m^4 \lambda(\lambda^2 - 2\mu)a_2 \right. \\ & - 56m^4 \lambda \mu a_2 - m^2 a_0 a_1 h_1 - m^2 \lambda a_1^2 h_1 - 2m^2 \lambda a_0 a_2 h_1 - 3m^2 \mu a_1 a_2 h_1 - m^2 a_1 h_2 - 2m^2 \lambda a_2 h_2 \\ & - m^2 \lambda a_1^2 h_1 - 2m^2 \lambda a_0 a_2 h_1 - 3m^2 \mu a_1 a_2 h_1 - m^2 a_1 h_2 - 2m^2 \lambda a_2 h_2 - l^2 a_1 h_3 - 2l^2 \lambda a_2 h_3 - o^2 a_1 h_4 \\ & \left. - 2o^2 \lambda a_2 h_4\right) \end{aligned}$$

$$\begin{aligned} \left(\frac{G'}{G}\right)^3 = & \left(12m^4 \lambda a_1 + 2kma_2 + 38m^4 \lambda^2 a_2 + 40m^4 \mu a_2 + m^2 a_1^2 h_1 + 2m^2 a_0 a_2 h_1 + 3m^2 \lambda a_1 a_2 h_1 \right. \\ & \left. + 2m^2 \mu a_2^2 h_1 + 2m^2 a_2 h_2 + 2l^2 a_2 h_3 + 2o^2 a_2 h_4\right) \end{aligned}$$

$$\left(\frac{G'}{G}\right)^4 = -m^2(6m^2 a_1 + 54m^2 \lambda a_2 + 3a_1 a_2 h_1 + 2\lambda a_2^2 h_1)$$

$$\left(\frac{G'}{G}\right)^5 = -2m^2 a_2(12m^2 + a_2 h_1)$$

Here, if each coefficient obtained equals zero and the resulting equation system is solved,

$$a_0 = -\frac{km + m^4 \lambda^2 + 8m^4 \mu + m^2 h_2 + l^2 h_3}{m^2 h_1}, a_1 = -\frac{12m^2 \lambda}{h_1}, a_2 = -\frac{12m^2}{h_1}$$

coefficients are obtained. According to the cases of the coefficients and $\left(\frac{G'}{G}\right)$ expressions Δ , the following $u(x, t)$ solutions are written.

$$u_1(x, t) = -\frac{km + m^4\lambda^2 + 8m^4\mu + m^2h_2 + l^2h_3}{m^2h_1} - \frac{o^2h_4}{m^2h_1} + \left(-\frac{12m^2\lambda}{h_1}\right)\psi_1 + \left(-\frac{12m^2}{h_1}\right)\psi_1^2$$

$$u_2(x, t) = -\frac{km + m^4\lambda^2 + 8m^4\mu + m^2h_2 + l^2h_3}{m^2h_1} - \frac{o^2h_4}{m^2h_1} + \left(-\frac{12m^2\lambda}{h_1}\right)\psi_2 + \left(-\frac{12m^2}{h_1}\right)\psi_2^2$$

2D and 3D graphics for $u_2(x, t)$ of the solution functions are given in Figure 3 as an example.

$$u_3(x, t) = -\frac{l^2 + km + m^2h_1}{m^2h_2} - \frac{km(\lambda^2 + 8\mu)h_3}{h_2} + \left(-\frac{12km\lambda h_3}{h_2}\right)\psi_3 + \left(-\frac{12kmh_3}{h_2}\right)\psi_3^2$$

Figure 3. The 3D graph of the solution function $u_2(x, t)$ for $l = 2; m = 2; o = 2; \lambda = 1; \mu = 5; c = -2; n = -2; k = 5; \beta = 0.75; h_1 = 3; h_2 = 0; h_3 = 2; h_4 = 2; y = 6; z = 3$; values is given on the left graph, the 2D graph for $u_2(x, 0)$ is given on the right.

To look for the traveling wave solution of Equation (4)

$${}_0^A D_{ty}^{2\beta} u - {}_0^A D_{xxy}^{4\beta} u - 3{}_0^A D_x^\beta \left({}_0^A D_x^\beta u {}_0^A D_y^\beta u \right) + 3{}_0^A D_{xz}^{2\beta} u + {}_0^A D_t^{2\beta} u = 0$$

$$\xi = \frac{k}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)} \right)^\beta + \frac{o}{\beta} \left(z + \frac{1}{\Gamma(\beta)} \right)^\beta$$

if $u = u(\xi)$ is applied to the wave transform Equation (4) and then its integral is calculated, Equation (4) turns into

$$klu' - m^3lu''' - 3m \left(ml(u')^2 \right) + 3mou' + k^2u' = 0$$

the ordinary differential equation is obtained. When the balancing relation is applied to this ordinary differential equation

$$n + 3 = 2(n + 1) \quad n = 1$$

is found. Considering the found value of n and substituting the expression (7) in the ordinary differential equation, the coefficients of the power of the expression $\left(\frac{G'}{G}\right)$ are obtained as follows:

$$\begin{aligned}\left(\frac{G'}{G}\right)^0 &= -\mu a_1 (k^2 + kl + 3mo - lm^3 \lambda^2 - 2lm^3 \mu + 3lm^2 \mu a_1) \\ \left(\frac{G'}{G}\right)^1 &= -\lambda a_1 (k^2 + kl + 3mo - lm^3 \lambda^2 - 8lm^3 \mu + 6lm^2 \mu a_1) \\ \left(\frac{G'}{G}\right)^2 &= -a_1 (k^2 + kl + 3mo - 7lm^3 \lambda^2 - 8lm^3 \mu + 3lm^2 \lambda^2 a_1 + 6lm^2 \mu a_1) \\ \left(\frac{G'}{G}\right)^3 &= -6lm^2 \lambda a_1 (-2m + a_1) \\ \left(\frac{G'}{G}\right)^4 &= -3lm^2 a_1 (-2m + a_1)\end{aligned}$$

Here, if each coefficient obtained equals zero and the resulting equation system is solved,

$$\begin{aligned}a_1 &= 2m, \quad k = -\frac{l}{2} + \frac{1}{2} \sqrt{l^2 - 12mo + 4lm^3 \lambda^2 - 16lm^3 \mu} \\ a_1 &= 2m, \quad k = -\frac{l}{2} - \frac{1}{2} \sqrt{l^2 - 12mo + 4lm^3 \lambda^2 - 16lm^3 \mu}\end{aligned}$$

coefficients are obtained. If the k value found is substituted in the expression ξ

$$\begin{aligned}\xi &= \frac{-\frac{l}{2} - \frac{1}{2} \sqrt{l^2 - 12mo + 4lm^3 \lambda^2 - 16lm^3 \mu}}{\beta} \left(t + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{m}{\beta} \left(x + \frac{1}{\Gamma(\beta)}\right)^\beta \\ &\quad + \frac{l}{\beta} \left(y + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{o}{\beta} \left(y + \frac{1}{\Gamma(\beta)}\right)^\beta\end{aligned}$$

According to the cases of the coefficients and $\left(\frac{G'}{G}\right)$ expressions Δ , the following $u(x, t)$ solutions are acquired.

$$\begin{aligned}u_1(x, t) &= a_0 + (2m) \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{\varphi \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{\varphi \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + \gamma \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right] - \frac{\lambda}{2} \right] \\ u_2(x, t) &= a_0 + (2m) \left[\frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-\varphi \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \gamma \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{\varphi \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + \gamma \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right] - \frac{\lambda}{2} \right]\end{aligned}$$

2D and 3D graphics for $u_2(x, t)$ of the solution functions are given in Figure 4 as an example.

$$u_3(x, t) = a_0 + (2m) \left(\frac{\frac{\varphi + \gamma}{2}}{\frac{\varphi + \gamma}{2} + \frac{\varphi - \gamma}{2} \xi} - \frac{\lambda}{2} \right)$$

Figure 4. The 3D graph of the solution function $u_2(x, t)$ for $a_0 = 2; l = -1; m = 2; o = 3; \lambda = 3; \mu = 6; c = 2; n = 4; k = -3; \beta = 0.75; z = 8; y = 4$; values is given on the left graph, the 2D graph for $u_2(x, 0)$ is given on the right.

If the other k value obtained is written instead of ζ , the other solutions of (4) are written similarly.

Hyperbolic, trigonometric and rational function solutions of the above Equations (1)–(4) with arbitrary constants according to the states of Δ are obtained. All solutions obtained in this study are validated using Mathematica 12.0 software.

5. Discussion and Conclusions

When previous studies are examined, it is observed that there is no single method for finding exact solutions of nonlinear differential equations and each method has advantages and disadvantages depending on the experience of the researchers. It can also be said that all theories are dependent on the problem, i.e., while some methods give good results for certain problems, they may not be useful for other problems. Therefore, it is important to apply some of the methods commonly used in the literature to previous unsolved nonlinear partial differential equations to search for possible new exact solutions or validate existing solutions with a different approach. In recent years, researchers have made many publications examining the effect of different derivative definitions on the application of different methods by taking these implications into consideration.

In this study, different versions of Kadomtsev–Petviashvili equations of $(n + 1)$ -dimensional β -conformable fractional derivatives, which express situations such as modeling nonlinear wave in gas bubbly liquid and explaining its role in dispersion, are discussed. Solitary and periodic wave soliton solutions have been obtained in the method that enables us to find different function solutions together. To show that the method is effective and reliable, the accuracy of the new exact solutions is validated by Mathematica 12.0 software. Since any differentiable function does not have conformable fractional derivative at the zero point, a physical interpretation of the solution cannot be made at this point. The β -conformable derivative makes physical interpretation possible. We think that the method and the definition of β -conformable fractional derivative of Atangana will yield effective results for different nonlinear differential equations as well.

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