Article

k -Zero-Divisor and Ideal-Based k -Zero-Divisor Hypergraphs of Some Commutative Rings

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Abstract: Let $\mathcal{R}$ be a commutative ring with nonzero identity and $k \geq 2$ be a fixed integer. The k -zero-divisor hypergraph $\mathcal{H}_k(\mathcal{R})$ of $\mathcal{R}$ consists of the vertex set $Z(\mathcal{R}, k)$, the set of all k -zero-divisors of $\mathcal{R}$, and the hyperedges of the form $\{a_1, a_2, a_3, \dots, a_k\}$, where $a_1, a_2, a_3, \dots, a_k$ are k distinct elements in $Z(\mathcal{R}, k)$, which means (i) $a_1 a_2 a_3 \cdots a_k = 0$ and (ii) the products of all elements of any $(k - 1)$ subsets of $\{a_1, a_2, a_3, \dots, a_k\}$ are nonzero. This paper provides two commutative rings so that one of them induces a family of complete k -zero-divisor hypergraphs, while another induces a family of k -partite σ -zero-divisor hypergraphs, which illustrates unbalanced or asymmetric structure. Moreover, the diameter and the minimum length of all cycles or girth of the family of k -partite σ -zero-divisor hypergraphs are determined. In addition to a k -zero-divisor hypergraph, we provide the definition of an ideal-based k -zero-divisor hypergraph and some basic results on these hypergraphs concerning a complete k -partite k -uniform hypergraph, a complete k -uniform hypergraph, and a clique.

Keywords: k -zero-divisor; k -zero-divisor hypergraph; ideal-based k -zero-divisor; ideal-based k -zero-divisor hypergraph; complete k -uniform hypergraph; k -partite k -uniform hypergraph; k -partite σ -uniform hypergraph; clique

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1. Introduction

Graph structures and algebraic structures are closely related. For example, if $\mathcal{R}$ is a commutative ring and $Z(\mathcal{R})$ is the finite set of all zero-divisors of $\mathcal{R}$, then $Z(\mathcal{R})$ can be regarded as a set of vertices of a graph G , and two elements in $Z(\mathcal{R})$ can have an edge connecting between them whenever their product is zero. This graph G is called a zero-divisor graph of commutative ring $\mathcal{R}$, denoted by $\Gamma(\mathcal{R})$, which was first introduced by Beck [1]. Beck studied the coloring and clique of a zero-divisor graph.

Afterward, the definition of a zero-divisor graph was slightly changed by Anderson and Livingston [2]. The vertex set consists of all nonzero zero-divisors of commutative rings. They investigated the completeness and automorphisms, denoted by $\text{Aut}(\Gamma(\mathcal{R}))$, of a zero-divisor graph.

Several algebraic structures have been considered, for instance, the zero-divisor graph of non-commutative rings [3,4] and the zero-divisor graph of semigroups [5,6].

Later, the graph was generalized into a hypergraph and the zero-divisor was generalized into a k -zero-divisor. Eslahchi and Rahimi [7] were the first who introduced k -zero-divisor and its k -zero-divisor hypergraph.

In [7], $\mathcal{R}$ is a commutative ring with nonzero identity. A nonzero and nonunit element z_1 of $\mathcal{R}$ is called a k -zero-divisor of $\mathcal{R}$ if $k - 1$ distinct nonunit elements $z_2, z_3, z_4, \dots, z_k$ differ from z_1 and satisfy the following statements:

- (i) $z_1 z_2 z_3 \cdots z_k = 0$;
- (ii) the products of all elements of any $(k - 1)$ subsets of $\{z_1, z_2, z_3, \dots, z_k\}$ are nonzero.

We use $Z(\mathcal{R}, k)$ to denote the set of all k -zero-divisors of $\mathcal{R}$. Note that the elements $z_2, z_3, z_4, \dots, z_k$ above must be nonzero elements. Thus, if $Z(\mathcal{R}, k)$ is finite, then it is natural to define the k -zero-divisor hypergraph [7] as follows:

Let $k \geq 2$ be a fixed positive integer. A k -zero-divisor hypergraph of a commutative ring $\mathcal{R}$ with nonzero identity, denoted by $\mathcal{H}_k(\mathcal{R})$, is defined as a k -uniform hypergraph $\mathcal{H}$ whose vertex set is $Z(\mathcal{R}, k)$, and the set $\{a_1, a_2, a_3, \dots, a_k\} \subseteq Z(\mathcal{R}, k)$ is a hyperedge if it satisfies the following statements:

- (i) $a_1 a_2 a_3 \cdots a_k = 0$;
- (ii) the products of all elements of any $(k - 1)$ subsets of $\{a_1, a_2, a_3, \dots, a_k\}$ are nonzero.

k -zero-divisor hypergraphs have been extensively studied. Eslahchi and Rahimi [7] studied a two-colorable $\mathcal{H}_k(\mathcal{R})$, the connectedness and the completeness of three-zero-divisor hypergraphs; then Chelvam et al. [8] studied the planarity of k -zero-divisor hypergraphs.

After, Siri Wong et al. [9] used a k -partite k -uniform hypergraph modified from the definition of Kuhl and Schroeder [10] to construct a complete k -partite k -zero-divisor hypergraph. For ease of reference, they found that there exists a complete k -partite k -zero-divisor hypergraph whose vertex set is $Z(D/D\gamma, k)$, where $\gamma = \prod_{i=1}^k p_i$, $k \geq 2$, and D is a principal ideal domain (PID) with nonzero identity containing at least distinct k prime elements, say $p_1, p_2, p_3, \dots$, and p_k , such that $D/D\gamma$ is finite.

In addition to a zero-divisor graph, Redmond [11] provided the definition of an ideal-based zero-divisor graph as follows: Let I be an ideal of a commutative ring $\mathcal{R}$. An ideal-based zero-divisor graph of a commutative ring $\mathcal{R}$, denoted by $\Gamma_I(\mathcal{R})$, is defined as an undirected graph whose vertex set is $\{x \in \mathcal{R} \setminus I \mid xy \in I \text{ for some } y \in \mathcal{R} \setminus I\}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. Several authors investigated some results of the ideal-based zero-divisor graph of a commutative ring $\mathcal{R}$ [12–14].

Elele [15] generalized the idea of an ideal-based zero-divisor graph of a commutative ring to a three-zero-divisor hypergraph of a commutative ring $\mathcal{R}$ regarding I , denoted by $\mathcal{H}_3(\mathcal{R}, I)$. A three-zero-divisor hypergraph of a commutative ring $\mathcal{R}$ regarding I is a three-uniform hypergraph whose vertex set is $\{x \in \mathcal{R} \setminus I \mid x_1 x_2 x_3 \in I \text{ for some } x_2, x_3 \in \mathcal{R} \setminus I \text{ such that } x_1 x_2 \notin I, x_1 x_3 \notin I, x_2 x_3 \notin I\}$.

In this paper, we let $k \geq 2$ be a positive integer and $U(\mathcal{R})$ be the set of all units of a ring $\mathcal{R}$. If D is a PID and I is its ideal, then D/I is a commutative ring. Instead of considering the k -zero divisor hypergraph of a commutative ring, we consider the k -zero-divisor hypergraph of D/I . The following are some useful tools equipped with a PID [16].

Let a and b be nonzero elements of D , and let d be a generator of the principal ideal containing a and b . Then,

- (A) $d = \gcd(a, b)$ can be written as a D -linear combination of a and b , and d is unique up to multiplication by a unit of D .
- (B) If $\gcd(a, b) = 1$, then $a + Db$ is a unit of the quotient ring D/Db .

For other mathematical tools in abstract algebra, we refer the reader to the book by Dummit and Foote [16].

Together with a commutative ring D/I , we are interested in two hypergraph structures:

- (i) For $k \geq 2$, the complete k -uniform hypergraph on n vertices is a k -uniform hypergraph, which has all k subsets of the n set of vertices as hyperedges [17].
- (ii) For a fixed integer $\sigma \geq k$, the k -partite σ -uniform hypergraph consists of the vertex set V partitioned into k subsets $V_1, V_2, V_3, \dots, V_k$ and E is a hyperedge if $|E| = \sigma$ and $E \not\subseteq V_i$ for all $1 \leq i \leq k$. A k -partite σ -uniform hypergraph is said to be complete if the hyperedge set contains all hyperedges satisfying the above property (modified from [18]).

In Section 2, by choosing an appropriate ideal I of PID D , we obtain the completeness of $\mathcal{H}_k(D/I)$.

In Section 3, through combination with the idea of Siri Wong et al. [9] and the idea presented in Section 2, we choose another ideal I of D that enables us to construct the

k -partite σ -zero-divisor hypergraph of D/I with the integer $\sigma \geq k$. Moreover, we determine its diameter and minimum length of all cycles or girth. For the definition of diameter and cycles in a hypergraph, we refer to [19,20], respectively.

In Section 4, by considering an appropriate ideal I of D in the previous sections, we generalize the concept of an ideal-based zero-divisor graph and a three-zero-divisor hypergraph regarding ideal to an ideal-based k -zero-divisor hypergraph and provide some results on this hypergraph.

In the last section, conclusions and some observations are provided.

2. Complete k -Zero-Divisor Hypergraphs

Assume that D has a prime element, say p . Then, we can obtain the ideal Dp^k consequently, D/Dp^k is a commutative ring with nonzero identity and we can easily see that

$$(Dp/Dp^k) \setminus (Dp^2/Dp^k) = \{a + Dp^k \mid a \in Dp \setminus Dp^2\}. \quad (1)$$

Next, under some conditions, we can determine $Z(D/Dp^k, k)$, the set of all k -zero-divisors of the ring D/Dp^k , and show the completeness of the k -zero-divisor hypergraph $\mathcal{H}_k(D/Dp^k)$.

Theorem 1. Assume that D/Dp^k is finite and $|(Dp/Dp^k) \setminus (Dp^2/Dp^k)| \geq k$. Then, $Z(D/Dp^k, k) = (Dp/Dp^k) \setminus (Dp^2/Dp^k)$. Consequently, $\mathcal{H}_k(D/Dp^k)$ is a complete k -zero-divisor hypergraph.

Proof. Let $x \in (Dp/Dp^k) \setminus (Dp^2/Dp^k)$. Then, x can be written in the form of $a + Dp^k$, where $a \in Dp \setminus Dp^2$. We see that a is a multiples of a prime p but not of p^2 . Then, $x = a + Dp^k$ can find $k - 1$ elements in $(Dp/Dp^k) \setminus (Dp^2/Dp^k)$ so that their product is $0 + Dp^k$ and any $k - 1$ elements of them do not have product $0 + Dp^k$. Thus, $(Dp/Dp^k) \setminus (Dp^2/Dp^k) \subseteq Z(D/Dp^k, k)$.

Conversely, let $x \in Z(D/Dp^k, k)$. Then, there exist $k - 1$ distinct nonzero nonunit elements that differ from x such that their product is $0 + Dp^k$ and the products of all elements of any $(k - 1)$ elements of such k elements are not $0 + Dp^k$. Since $x \in D/Dp^k$, x can be written in the form of $a + Dp^k$. Since $a + Dp^k \notin U(D/Dp^k)$, by the property B, $\gcd(a, p^k) \neq 1$, states d_1 .

Since $a + Dp^k$ is a nonzero element of D/Dp^k , we obtain $d_1 \neq p^k$. Therefore, $d_1 = p$. Then, $a \in Dp \setminus Dp^2$. By (1), $a + Dp^k \in (Dp/Dp^k) \setminus (Dp^2/Dp^k)$. Otherwise, some products of $k - 1$ elements of such k elements are $0 + Dp^k$, which is a contradiction. Thus, $x \in (Dp/Dp^k) \setminus (Dp^2/Dp^k)$. Therefore, $Z(D/Dp^k, k) \subseteq (Dp/Dp^k) \setminus (Dp^2/Dp^k)$.

Hence, we can determine $Z(D/Dp^k, k)$ by $(Dp/Dp^k) \setminus (Dp^2/Dp^k)$.

By the above consideration, the product of any k element is $0 + Dp^k$, but all products of any $k - 1$ element of such k elements are not $0 + Dp^k$. Therefore, we have all k subsets forming the hyperedges of $\mathcal{H}_k(D/Dp^k)$. Hence, $\mathcal{H}_k(D/Dp^k)$ is a complete k -uniform hypergraph. $\square$

Since $Z(D/Dp^k, k) = (Dp/Dp^k) \setminus (Dp^2/Dp^k)$, we obtain the following corollary:

Corollary 1. D/Dp^k can be separated into four disjoint subsets: $U(D/Dp^k)$, $\{0 + Dp^k\}$, $Z(D/Dp^k, k)$, and $(Dp^2/Dp^k) \setminus \{0 + Dp^k\}$.

Proof. Let $a + Dp^k \in D/Dp^k$. Assume that $a + Dp^k \neq 0 + Dp^k$. If $\gcd(a, p^k) \neq 1$, then $a + Dp^k \in Z(D/Dp^k, k)$ when $p^2 \nmid a$ or $a + Dp^k \in (Dp^2/Dp^k) \setminus \{0 + Dp^k\}$ when $p^2 \mid a$. Otherwise, by the property B, we have that $a + Dp^k$ is a unit of D/Dp^k . $\square$

Corollary 1 has a computational consequence. In particular, if $D = \mathbb{Z}$ and φ is the Euler φ -function, then $|U(\mathbb{Z}/p^k\mathbb{Z})| = \varphi(p^k)$ and $|(p^2\mathbb{Z}/p^k\mathbb{Z}) \setminus \{0 + p^k\mathbb{Z}\}| = p^{k-2} -$

1. Thus, the numbers of vertices and hyperedges of $\mathcal{H}_k(D/Dp^k)$ are $p^{k-1} - p^{k-2}$ and $\binom{p^{k-1} - p^{k-2}}{k}$, respectively.

Example 1. Consider $\mathbb{Z}_{81} \cong \mathbb{Z}/3^4\mathbb{Z}$. By Corollary 1, $|Z(\mathbb{Z}/3^4\mathbb{Z}, 4)| = 18$. Furthermore, $Z(\mathbb{Z}/3^4\mathbb{Z}, 4) = \{\bar{3}, \bar{6}, \bar{12}, \bar{15}, \bar{21}, \bar{24}, \bar{30}, \bar{33}, \bar{39}, \bar{42}, \bar{48}, \bar{51}, \bar{57}, \bar{60}, \bar{66}, \bar{69}, \bar{72}, \bar{78}\}$. The cardinality of the hyperedge set, $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{81}))$, is $\binom{18}{4} = 3060$. These are some examples of elements of $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{81}))$,

$$\begin{aligned} &\{\bar{3}, \bar{6}, \bar{12}, \bar{15}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{21}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{24}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{30}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{33}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{39}\}, \\ &\{\bar{3}, \bar{6}, \bar{12}, \bar{42}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{48}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{51}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{57}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{60}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{66}\}, \\ &\{\bar{3}, \bar{6}, \bar{12}, \bar{69}\}, \{\bar{3}, \bar{6}, \bar{12}, \bar{72}\}, \text{ and } \{\bar{3}, \bar{6}, \bar{12}, \bar{78}\}. \end{aligned}$$

3. k -Partite σ -Zero-Divisor Hypergraphs

Assume further in this section that D has at least k nonassociative distinct prime elements, say $p_1, p_2, p_3, \dots, p_k$, and let $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Then, $D/Dp_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ and its $(\sum_{i=1}^k \alpha_i)$ -zero-divisors are considered. Throughout this section, let $\sigma = \sum_{i=1}^k \alpha_i$ and $\pi = \prod_{i=1}^k p_i^{\alpha_i}$.

We see that each hyperedge of such k -partite σ -uniform hypergraph can have one or more elements from some partite sets, but no more than σ elements from such partite sets.

It is tedious to check but worth to mention the following facts:

$$(Dp_i/D\pi) \setminus \bigcup_{j=1, j \neq i}^k (Dp_j/D\pi) = \left\{ a + D\pi \mid a \in Dp_i \setminus \bigcup_{j=1, j \neq i}^k Dp_j \right\} \quad (2)$$

and

$$\begin{aligned} (Dp_i/D\pi) \setminus \left((Dp_i^2/D\pi) \cup \bigcup_{j=1, j \neq i}^k (Dp_j/D\pi) \right) \\ = \left\{ a + D\pi \mid a \in Dp_i \setminus \left(Dp_i^2 \cup \bigcup_{j=1, j \neq i}^k Dp_j \right) \right\}. \quad (3) \end{aligned}$$

Next, under some conditions, we can separate the set of all σ -zero-divisors of $D/D\pi$ into k -partite sets.

Theorem 2. Assume that $D/D\pi$ is finite, and for all $1 \leq i \leq k$,

$\left| (Dp_i/D\pi) \setminus \left((Dp_i^2/D\pi) \cup \bigcup_{j=1, j \neq i}^k (Dp_j/D\pi) \right) \right| \geq \alpha_i$. Then, $Z(D/D\pi, \sigma) = \bigcup_{j=1}^k V_j$, where

$$V_j = \begin{cases} (Dp_j/D\pi) \setminus \bigcup_{l=1, l \neq j}^k (Dp_l/D\pi), & \text{if } \alpha_j = 1, \\ (Dp_j/D\pi) \setminus \left((Dp_j^2/D\pi) \cup \bigcup_{l=1, l \neq j}^k (Dp_l/D\pi) \right), & \text{if } \alpha_j \geq 2, \end{cases}$$

and $\{V_j \mid 1 \leq j \leq k\}$ is mutually disjoint. Consequently, we construct a k -partite σ -zero-divisor hypergraph whose vertex set is $Z(D/D\pi, \sigma)$.

Proof. Let $x \in Z(D/D\pi, \sigma)$. Then, x is written in the form of $a + D\pi$, where a is neither divisible by two distinct primes p_i and p_j where $j \neq i$, nor multiples of p_i^2 . Otherwise, some $k - 1$ elements in $Z(D/D\pi, \sigma)$ have product $0 + D\pi$. Therefore, $Z(D/D\pi, \sigma) \subseteq \bigcup_{j=1}^k V_j$.

Conversely, let $x \in \bigcup_{j=1}^k V_j$. Then, $x \in V_l$ for some $1 \leq l \leq k$. Assume that $\alpha_l \geq 2$. Thus, x is written in the form of $a + D\pi$, where $a \in Dp_l - (Dp_l^2 \cup \bigcup_{j=1, j \neq l}^k Dp_j)$. Therefore, there exist $\sigma - 1$ elements, consisting of $\alpha_l - 1$ elements from V_l and α_s elements from V_s , where $s \neq l$, together with x , so that their product is $0 + D\pi$ and the products of all elements of any $(\sigma - 1)$ subsets of the set containing x and such $\sigma - 1$ chosen elements are not $0 + D\pi$. Otherwise, $a \in Dp_l^2 \cup \bigcup_{j=1, j \neq l}^k Dp_j$. We can prove the case $\alpha_l = 1$ using a similar idea for the case $\alpha_l \geq 2$. Therefore, $\bigcup_{j=1}^k V_j \subseteq Z(D/D\pi, \sigma)$.

By the above consideration, we find σ elements whose product is $0 + D\pi$, but any $\sigma - 1$ element of such σ elements such that their product is not $0 + D\pi$. We see that at least one element of such σ elements is from one of the k -partite sets. Therefore, the σ subset forms a hyperedge of $\mathcal{H}_\sigma(D/D\pi)$. Hence, $\mathcal{H}_\sigma(D/D\pi)$ is a k -partite σ -uniform hypergraph. $\square$

From the structure of the k -partite σ -zero-divisor hypergraph obtained from Theorem 2, we can see that each hyperedge contains different numbers of vertices in each partite set. This illustrates the unbalanced or asymmetric structure of our hypergraph.

In this section, the k -partite σ -zero-divisor hypergraph is not complete because we have only the hyperedge of the form $\{a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,\alpha_1}, a_{2,1}, a_{2,2}, a_{2,3}, \dots, a_{2,\alpha_2}, a_{3,1}, a_{3,2}, a_{3,3}, \dots, a_{3,\alpha_3}, \dots, a_{k,1}, a_{k,2}, a_{k,3}, \dots, a_{k,\alpha_k}\}$, where $a_{i,j} \in V_i$ for all $1 \leq i \leq k$ and $1 \leq j \leq \alpha_i$. However, we cannot find the hyperedge of the form $\{a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,\alpha_1+1}, a_{2,1}, a_{2,2}, a_{2,3}, \dots, a_{2,\alpha_2}, a_{3,1}, a_{3,2}, a_{3,3}, \dots, a_{3,\alpha_3}, \dots, a_{k,1}, a_{k,2}, a_{k,3}, \dots, a_{k,\alpha_k-1}\}$, where $a_{i,j} \in V_i$ for all $1 \leq i \leq k$ and $1 \leq j \leq \alpha_i + 1$ from $\mathcal{H}_\sigma(D/D\pi)$; that is, if this hypergraph is complete, we need to construct a hyperedge by α_{m_1} vertices from V_1 , α_{m_2} vertices from V_2 , α_{m_3} vertices from V_3 , $\dots$, and α_{m_k} vertices from V_k for all $1 \leq i \leq k$, $1 \leq m_i \leq k$.

Remark 1. For computational purpose, let $1 \leq i \neq j \leq k$, p_i , and p_j be nonassociate distinct prime elements. Since $\gcd(p_i, p_j) = 1$, by property A, there exist nonzero elements $a, b \in D$ such that $1 = ap_i + bp_j \in Dp_i + Dp_j$. Then, $Dp_i + Dp_j = D$, that is, Dp_i and Dp_j are comaximal, see [16]. Per Chinese Remainder Theorem [16], $D/Dp_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \cong (D/Dp_1^{\alpha_1}) \times (D/Dp_2^{\alpha_2}) \times (D/Dp_3^{\alpha_3}) \times \dots \times (D/Dp_k^{\alpha_k})$. Then, V_j in Theorem 2 can actually be defined by:

$$V_j = \{(u_1 + Dp_1^{\alpha_1}, u_2 + Dp_2^{\alpha_2}, u_3 + Dp_3^{\alpha_3}, \dots, 0 + Dp_j^{\alpha_j}, \dots, u_k + Dp_k^{\alpha_k}) \mid u_m + Dp_m^{\alpha_m} \in U(D/Dp_m^{\alpha_m}) \text{ for all } m \neq j, \text{ if } \alpha_j = 1\}$$

and

$$V_j = \{(u_1 + Dp_1^{\alpha_1}, u_2 + Dp_2^{\alpha_2}, u_3 + Dp_3^{\alpha_3}, \dots, a_j + Dp_j^{\alpha_j}, \dots, u_k + Dp_k^{\alpha_k}) \mid u_m + Dp_m^{\alpha_m} \in U(D/Dp_m^{\alpha_m}) \text{ for all } m \neq j \text{ and } a_j + Dp_j^{\alpha_j} \in Z(D/Dp_j^{\alpha_j}, \alpha_j)\}, \text{ if } \alpha_j \geq 2.$$

Thus, if $D = \mathbb{Z}$, then Corollary 1 implies:

1. $|U(\mathbb{Z}/p_i\mathbb{Z})| = p_i - 1$ and $|Z(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}, k)| = 1$ for $\alpha_i = 1$;
2. $|U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})| = \varphi(p_i^{\alpha_i})$ and $|Z(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}, k)| = p_i^{\alpha_i-1} - p_i^{\alpha_i-2}$ for $\alpha_i \geq 2$.

Remark 2. We know that $\mathbb{Z}/2^2\mathbb{Z}$ has no 2-zero-divisor. If one component of $(D/Dp_1^{\alpha_1}) \times (D/Dp_2^{\alpha_2}) \times (D/Dp_3^{\alpha_3}) \times \dots \times (D/Dp_k^{\alpha_k})$ is $\mathbb{Z}/2^2\mathbb{Z} \cong \mathbb{Z}_4$, we define the partite set corresponding to $\mathbb{Z}_4$ by

$$V_j = \{(u_1 + Dp_1^{\alpha_1}, u_2 + Dp_2^{\alpha_2}, u_3 + Dp_3^{\alpha_3}, \dots, \bar{2}_j, \dots, u_k + Dp_k^{\alpha_k}) \mid u_m + Dp_m^{\alpha_m} \in U(D/Dp_m^{\alpha_m}) \text{ for all } m \neq j \text{ and } \bar{2} \in \mathbb{Z}_4\}.$$

Example 2. Considering $\mathbb{Z}/60\mathbb{Z} \cong \mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}$, we have $|V_1| = |Z(\mathbb{Z}/2^2\mathbb{Z}, 2)| \cdot |U(\mathbb{Z}/3\mathbb{Z})| \cdot |U(\mathbb{Z}/5\mathbb{Z})| = 8$, $|V_2| = |U(\mathbb{Z}/2^2\mathbb{Z})| \cdot 1 \cdot |U(\mathbb{Z}/5\mathbb{Z})| = 8$, and $|V_3| = |U(\mathbb{Z}/2^2\mathbb{Z})| \cdot |U(\mathbb{Z}/3\mathbb{Z})| \cdot 1 = 4$. Then, $|Z(\mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}, 4)| = 8 + 8 + 4 = 20$, and $|\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))| = \binom{8}{2} \cdot 8 \cdot 4 = 896$. These are some examples of elements of $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))$,

$$\{\bar{2}, \bar{14}, \bar{3}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{3}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{3}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{3}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{9}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{9}, \bar{25}\},$$

$$\begin{aligned} & \{\bar{2}, \bar{14}, \bar{9}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{9}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{21}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{21}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{21}, \bar{35}\}, \\ & \{\bar{2}, \bar{14}, \bar{21}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{27}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{27}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{27}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{27}, \bar{55}\}, \\ & \{\bar{2}, \bar{14}, \bar{33}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{33}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{33}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{33}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{39}, \bar{5}\}, \\ & \{\bar{2}, \bar{14}, \bar{39}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{39}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{39}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{51}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{51}, \bar{25}\}, \\ & \{\bar{2}, \bar{14}, \bar{51}, \bar{35}\}, \{\bar{2}, \bar{14}, \bar{51}, \bar{55}\}, \{\bar{2}, \bar{14}, \bar{57}, \bar{5}\}, \{\bar{2}, \bar{14}, \bar{57}, \bar{25}\}, \{\bar{2}, \bar{14}, \bar{57}, \bar{35}\} \text{ and } \\ & \{\bar{2}, \bar{14}, \bar{57}, \bar{55}\}. \end{aligned}$$

Now, we consider $\mathcal{H}_\sigma(D/D\pi)$ with its diameter and girth.

Corollary 2. *The diameter of $\mathcal{H}_\sigma(D/D\pi)$ is 2.*

Proof. Let x and y be distinct vertices of $\mathcal{H}_\sigma(D/D\pi)$. If x and y are in the same partite set V_i with $\alpha_i = 1$, then there is no hyperedge E connecting x and y . Assume that $x \in E_1$ and $y \in E_2$ for some hyperedges E_1 and E_2 . We see that there exists a vertex v in the other partite sets such that $v \in E_1 \cap E_2$. Then, we obtain a path x, E_1, v, E_2, y of length 2. Otherwise, we obtain that the distance between x and y is 1. $\square$

To consider the girth, we need to split into two cases of $|Z(D/D\pi, \sigma)|$, that is, $|Z(D/D\pi, \sigma)| = \sigma$ and $|Z(D/D\pi, \sigma)| \geq \sigma + 1$, which can be separated into two sub-cases with cases of k .

Since $\mathcal{H}_\sigma(D/D\pi)$ has only one hyperedge, this hypergraph has no cycle. If a hypergraph contains no cycle, its girth is defined as ∞ . We obtain the following corollary:

Corollary 3. *The girth of $\mathcal{H}_\sigma(D/D\pi)$ is ∞ when $|Z(D/D\pi, \sigma)| = \sigma$.*

Next, we consider the case $|Z(D/D\pi, \sigma)| \geq \sigma + 1$. By applying Theorem 2, we obtain the girth of $\mathcal{H}_\sigma(D/D\pi)$ in the other cases as follows.

Corollary 4. *The girth of $\mathcal{H}_\sigma(D/D\pi)$ is 2 when $k \geq 3$ and $|Z(D/D\pi, \sigma)| \geq \sigma + 1$.*

Proof. Assume that $k \geq 3$ and $|Z(D/D\pi, \sigma)| \geq \sigma + 1$. Since $|Z(D/D\pi, \sigma)| \geq \sigma + 1$, there exist two distinct σ subsets, which form two distinct hyperedges E_1 and E_2 having one common element, say x_1 . Then, we have a cycle $C = x_1, E_1, x_2, E_2, x_1$ of length 2 for some vertex $x_2 \neq x_1$, which is a possible smallest cycle. Thus, the girth is 2. $\square$

Corollary 5. *Assume that $k = 2$ and $|Z(D/D\pi, \alpha_1 + \alpha_2)| \geq \alpha_1 + \alpha_2 + 1$. The girth is*

$$\begin{cases} \infty, & \text{if } |V_1| = 1 \text{ and } \alpha_2 = 1, \\ 2, & \text{if } |V_1| = 1 \text{ and } \alpha_2 \geq 2, \\ 2, & \text{if } |V_i| \geq 2 \text{ for all } i \in \{1, 2\} \text{ and there exists } i \in \{1, 2\} \text{ such that } \alpha_i \geq 2, \\ 4, & \text{if } |V_i| \geq 2 \text{ with } \alpha_i = 1 \text{ for all } i \in \{1, 2\}. \end{cases}$$

Proof. By the condition of $|V_1|$ and α_2 , $\mathcal{H}_k(D/D\pi)$ has no cycle. Therefore, the girth is ∞ in the first case.

In cases 2 and 3, we see that $|Z(D/D\pi, \alpha_1 + \alpha_2)| \geq \alpha_1 + \alpha_2 + 1$ implies that two distinct $(\alpha_1 + \alpha_2)$ subsets form two distinct hyperedges E_1 and E_2 having one common element, say x_1 . Then, we have a cycle $C = x_1, E_1, x_2, E_2, x_1$ of length 2 for some vertex $x_2 \neq x_1$, which is a possible smallest cycle. Thus, the girth is 2.

Finally, we see that $\mathcal{H}_\sigma(D/D\pi)$ is a bipartite graph that has no multiple edge. Since bipartite graphs have no odd cycle [21], the girth is 4. $\square$

4. Ideal-Based k -Zero-Divisor Hypergraphs

In the previous consideration, we considered a commutative ring D/I , where D is a PID and I is the appropriate ideal of D , which enabled us to define the relationship between an algebraic structure and a hypergraph structure related to an ideal-based zero-divisor graph.

By extending the definition of a 3-zero-divisor hypergraph of a commutative ring $\mathcal{R}$ regarding I [15], we have the following definitions:

Definition 1. Let $\mathcal{R}$ be a commutative ring with nonzero identity and I be a nonzero proper ideal of $\mathcal{R}$. An element x_1 of $\mathcal{R} \setminus I$ is called an ideal-based k -zero-divisor if there exist $k - 1$ distinct elements $x_2, x_3, x_4, \dots, x_k$ differing from x_1 and satisfying the following statements:

- (i) $\prod_{i=1}^k x_i \in I$;
- (ii) the products of all elements of any $(k - 1)$ subsets of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in I .

Let $Z_I(\mathcal{R}, k)$ denote the set of all ideal-based k -zero-divisors of $\mathcal{R}$. Note that the elements $x_2, x_3, x_4, \dots, x_k$ above are also not in I . Let us define the ideal-based k -zero-divisor hypergraph as follows:

Definition 2. Let $\mathcal{R}$ be a commutative ring with nonzero identity and $k \geq 2$ be a fixed positive integer. An ideal-based k -zero-divisor hypergraph of a commutative ring $\mathcal{R}$, denoted by $\mathcal{H}_k^I(\mathcal{R})$, is defined as a k -uniform hypergraph $\mathcal{H}$ whose vertex set is $Z_I(\mathcal{R}, k)$; the set $\{y_1, y_2, y_3, \dots, y_k\} \subseteq Z_I(\mathcal{R}, k)$ is a hyperedge if it satisfies the following statements:

- (i) $\prod_{i=1}^k y_i \in I$;
- (ii) the products of all elements of any $(k - 1)$ subsets of $\{y_1, y_2, y_3, \dots, y_k\}$ are not in I .

Next, we provide the definition of a useful ideal to investigate some results involving an ideal-based k -zero-divisor hypergraph.

Definition 3 ([22]). Let $\mathcal{R}$ be a commutative ring with nonzero identity. A proper ideal I of $\mathcal{R}$ is called an n -absorbing ideal if whenever $x_1 x_2 x_3 \cdots x_{n+1} \in I$ for $x_1, x_2, x_3, \dots, x_{n+1} \in \mathcal{R}$, then there are n elements of x_i whose product is in I .

By generalizing the idea of Proposition 1 in [15], we obtain the following theorem:

Theorem 3. Let $\mathcal{R}$ be a commutative ring with nonzero identity and $k \geq 2$ be a fixed positive integer. The following statements hold:

1. If I is a zero ideal, then $\mathcal{H}_k^I(\mathcal{R}) = \mathcal{H}_k(\mathcal{R})$.
2. Let I be a nonzero ideal of $\mathcal{R}$. Then, $\mathcal{H}_k^I(\mathcal{R}) = \emptyset$ if and only if I is a $(k - 1)$ -absorbing ideal of $\mathcal{R}$.

Proof.

1. Assume that I is a zero ideal. Then,

- (i) $\prod_{i=1}^k x_i \in I$ if and only if $\prod_{i=1}^k x_i = 0$;
- (ii) the products of all elements of any $(k - 1)$ subsets of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in I if and only if the products of all elements of any $(k - 1)$ subset of $\{x_1, x_2, x_3, \dots, x_k\}$ are nonzero.

We obtain $Z_I(\mathcal{R}, k) = Z(\mathcal{R}, k)$, as two hyperedge sets are also the same set.

2. Let I be a nonzero ideal of $\mathcal{R}$. Assume that I is a $(k - 1)$ -absorbing ideal of $\mathcal{R}$. Suppose that $\mathcal{H}_k^I(\mathcal{R}) \neq \emptyset$. Then, there exists $x_1 \in Z_I(\mathcal{R}, k)$ such that $\prod_{i=1}^k x_i \in I$ and the products of all elements of any $(k - 1)$ subset of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in I .

Since I is a $(k-1)$ -absorbing ideal, a product of k elements of $\{x_1, x_2, x_3, \dots, x_k\}$ is in I , which is a contradiction.

Conversely, assume that $\mathcal{H}_k^I(\mathcal{R}) = \emptyset$. If $x_1 \in \mathcal{R} \setminus I$ and $\prod_{i=1}^k x_i \in I$ for some $x_2, x_3, x_4, \dots, x_k \in \mathcal{R} \setminus I$, then the products of some $(k-1)$ subsets of $\{x_1, x_2, x_3, \dots, x_k\}$ are in I . Therefore, I is a $(k-1)$ -absorbing ideal.

□

Next, we determine the relationship between a k -zero-divisor hypergraph of a commutative ring $\mathcal{R}/I$, denoted by $\mathcal{H}_k(\mathcal{R}/I)$, and an ideal-based k -zero-divisor hypergraph of a commutative ring $\mathcal{R}$, denoted by $\mathcal{H}_k^I(\mathcal{R})$.

Theorem 4. Let $\mathcal{R}$ be a commutative ring with nonzero identity, I be a nonzero proper ideal of $\mathcal{R}$, and let $x, y \in \mathcal{R} \setminus I$. The following statements hold:

1. If there exists a hyperedge E containing $x + I$ and $y + I$ in $\mathcal{H}_k(\mathcal{R}/I)$, then there exists a hyperedge E' containing x and y in $\mathcal{H}_k^I(\mathcal{R})$.
2. If there exists a hyperedge E containing x and y in $\mathcal{H}_k^I(\mathcal{R})$ and $x + I \neq y + I$, then there exists a hyperedge E' containing $x + I$ and $y + I$ in $\mathcal{H}_k(\mathcal{R}/I)$.
3. If there exists a hyperedge E containing x and y in $\mathcal{H}_k^I(\mathcal{R})$ and $x + I = y + I$, then $x^2, y^2 \in I$.

Proof.

1. Assume that there exists a hyperedge E containing $x + I$ and $y + I$ in $\mathcal{H}_k(\mathcal{R}/I)$. Then, $(x + I)(y + I) \prod_{l=3}^k (z_l + I) = 0 + I$ and the products of all elements of any $(k-1)$ subset of $\{x + I, y + I, z_3 + I, z_4 + I, z_5 + I, \dots, z_k + I\}$ are not $0 + I$. Thus, $xy \prod_{l=3}^k z_l \in I$ and the products of all elements of any $(k-1)$ subset of $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ are not in I . Therefore, there exists a hyperedge E' containing x and y in $\mathcal{H}_k^I(\mathcal{R})$.
2. Assume that there exists a hyperedge E containing x and y in $\mathcal{H}_k^I(\mathcal{R})$ and $x + I \neq y + I$. Then, $xy \prod_{l=3}^k z_l \in I$ and the products of all elements of any $(k-1)$ -subsets of $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ are not in I . Thus, $(xy \prod_{l=3}^k z_l) + I = 0 + I$. We know that $(xy \prod_{l=3}^k z_l) + I = (x + I)(y + I) \prod_{l=3}^k (z_l + I)$. Since $x + I \neq y + I$, we have $(x + I)(y + I) \prod_{l=3}^k (z_l + I) = 0 + I$ and then the products of all elements of any $(k-1)$ -subsets of $\{x + I, y + I, z_3 + I, z_4 + I, z_5 + I, \dots, z_k + I\}$ are not $0 + I$. Therefore, there exists a hyperedge E' containing $x + I$ and $y + I$ in $\mathcal{H}_k(\mathcal{R}/I)$.
3. Assume that there exists a hyperedge E containing x and y in $\mathcal{H}_k^I(\mathcal{R})$ and $x + I = y + I$. Then, $xy \prod_{l=3}^k z_l \in I$ and the products of all elements of any $(k-1)$ subset of $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ are not in I . Thus, $(xy \prod_{l=3}^k z_l) + I = 0 + I$. Since $(xy \prod_{l=3}^k z_l) + I = (x + I)(y + I) \prod_{l=3}^k (z_l + I)$, we have $(x + I)(y + I) \prod_{l=3}^k (z_l + I) = 0 + I$. Since $x + I = y + I$, we have $(x + I)^2 \prod_{l=3}^k (z_l + I) = 0 + I$. Then, $(x + I)^2 = 0 + I$. Otherwise, $\prod_{l=3}^k (z_l + I) = 0 + I$, which implies $\prod_{l=3}^k z_l \in I$, and then $y \prod_{l=3}^k z_l \in I$, which is a contradiction. Since $(x + I)^2 = 0 + I$, we have $x^2 + I = 0 + I$, which implies that $x^2 \in I$. $y^2 \in I$ can be proved using the same idea.

□

Then, we study a complete k -partite k -uniform hypergraph $\mathcal{H}_k^I(\mathcal{R})$ according to the definition modified from Kuhl and Schroeder [10].

Lemma 1. Let $\mathcal{H}_k^I(\mathcal{R})$ be a complete k -partite k -uniform hypergraph. If $xy \in I$, then x and y must be in the same partite set.

Proof. Let $\mathcal{H}_k^I(\mathcal{R})$ be a complete k -partite k -uniform hypergraph. Assume that $xy \in I$. Then, $xy \prod_{i=3}^k z_i \in I$ for some $z_i \in \mathcal{R}$. Without loss of generality, assume that $x \in V_1$ and $z_i \in V_i$ for each $3 \leq i \leq k$. Since $xy \in I$, we have $xy \prod_{j=3}^{k-1} z_j \in I$. Thus, $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ is not a hyperedge.

Case 1. Suppose that $y \in V_j$ for some $3 \leq j \leq k-1$. It contradicts the fact that $xy \in I$, and $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph.

Case 2. Suppose that $y \in V_2$. It contradicts the fact that $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ is not a hyperedge.

Hence, x and y must be in the same partite set. $\square$

By manipulating Theorem 2 of [15], we have Theorem 5.

Theorem 5. Let $\mathcal{R}$ be a commutative ring with nonzero identity and I be a nonzero proper ideal of $\mathcal{R}$. The following statements hold:

1. If $P_1, P_2, P_3, \dots, P_k$ are distinct k prime ideals of $\mathcal{R}$ and $I = \cap_{i=1}^k P_i \neq 0$, then $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph.
2. Let $r^2 \in I$ for every ideal-based k -zero-divisor $r \in \mathcal{R}$. If $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph, then there exist k prime ideals $P_1, P_2, P_3, \dots, P_{k-1}$ and P_k such that $I = \cap_{i=1}^k P_i$.

Proof.

1. Let $x_1, x_2, x_3, \dots, x_k \in \mathcal{R} \setminus I$ such that $\prod_{i=1}^k x_i \in I$ and the products of all elements of any $(k-1)$ subset of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in I . Then, $\prod_{i=1}^k x_i \in P_1, P_2, P_3, \dots, P_{k-1}$ and P_k . Since for each $1 \leq j \leq k$, P_j is a prime ideal, we have $x_1 \in P_j$ or $x_2 \in P_j$ or $x_3 \in P_j$ or ... or $x_k \in P_j$.

Since the products of all elements of any $(k-1)$ subset of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in $\cap_{i=1}^k P_i$. Then, the product of $(k-1)$ subsets of $\{x_1, x_2, x_3, \dots, x_k\}$ is not in P_i for some $1 \leq i \leq k$. Without loss of generality, assume that the product of all elements of $\{x_1, x_2, x_3, \dots, x_k\} \setminus \{x_j\}$ is not in P_j for each $1 \leq j \leq k$. Since P_j is a prime ideal, we see that $x_j \in P_j$. Then, $x_j \in P_j \setminus \cup_{l \neq j} P_l$, which is a partite set. Hence, $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph.

2. Let $r^2 \in I$ for every ideal-based k -zero-divisor $r \in \mathcal{R}$. Assume that $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph with k partite sets $V_1, V_2, V_3, \dots, V_{k-1}$ and V_k .

Let $P_i = V_i \cup I$ for all $1 \leq i \leq k$. Next, we show that P_i are prime ideals.

It suffices to show that P_1 is a prime ideal. For P_j , where $j \neq 1$, it can be proved using the same idea. First, we claim that P_1 is an ideal. Let $x, y \in P_1$.

Case 1. $x, y \in I$. Then, $x - y \in I$, which implies that $x - y \in P_1$.

Case 2. $x, y \in V_1$. Then, there exist $z_2 \in V_2, z_3 \in V_3, z_4 \in V_4, \dots, z_k \in V_k$ such that $x \prod_{i=2}^k z_i \in I$ and the products of all elements of any $(k-1)$ subset of $\{x, z_2, z_3, \dots, z_k\}$ are not in I , $y \prod_{i=2}^k z_i \in I$, and the products of all elements of any $(k-1)$ subset of $\{y, z_2, z_3, \dots, z_k\}$ are not in I . Therefore, $(x - y) \prod_{i=2}^k z_i \in I$.

Assume that $\{x - y, z_2, z_3, z_4, \dots, z_k\}$ is not a hyperedge. Otherwise, $x - y \in V_1$, which implies that $x - y \in P_1$.

Case 2.1. The products of all elements of any $(k-1)$ subset of $\{x - y, z_2, z_3, z_4, \dots, z_k\}$ are in I . Since $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph, we have that the products of all elements of any s subset of $\{z_2, z_3, \dots, z_k\}$ are not in I for all $1 \leq s \leq k-1$. Then, there exist two elements z_{i_1} and z_{i_2} such that $(x - y)z_{i_1} \in I$ and $(x - y)z_{i_2} \in I$, where $i_1 \neq i_2$. By Lemma 1, $x - y \in V_{i_1} \cap V_{i_2}$, which is a contradiction.

Case 2.2. There are some products of any $k-1$ element of $\{x - y, z_2, z_3, z_4, \dots, z_k\}$ in I . Then, there exists one element such that its product with $x - y$ in I , say z_2 ; that is, $(x - y)z_2 \in I$. By Lemma 1, $x - y \in V_2$. Thus, $x(x - y) \prod_{l=3}^k z_l \in I$, which implies that $xy \prod_{l=3}^k z_l \in I$. Since $\{x, y, z_3, z_4, z_5, \dots, z_k\}$ is not a hyperedge, we have $xy \in I$. Since $x^2 \in I$ and $xy \in I$, $x^2 - xy \in I$. It contradicts the fact that $\{x, x - y, z_3, z_4, z_5, \dots, z_k\}$ is a hyperedge.

Case 3. Without loss of generality, let $x \in V_1$ and $y \in I$. Then, there exist $z_2 \in V_2, z_3 \in V_3, z_4 \in V_4, \dots, z_k \in V_k$ such that $x \prod_{i=2}^k z_i \in I$ and $y \prod_{i=2}^k z_i \in I$. Thus, $(x - y) \prod_{i=2}^k z_i \in I$.

Since $x \in V_1$ and $z_i \in V_i$ for all $2 \leq i \leq k$, $\{x, z_2, z_3, z_4, \dots, z_k\}$ forms a hyperedge. Then, the products of all elements of any $(k-1)$ subset of $\{x, z_2, z_3, \dots, z_k\}$ are not in I .

Suppose that $(x-y) \prod_{l=3}^k z_l \in I$. Since $y \prod_{l=3}^k z_l \in I$, we have $x \prod_{l=3}^k z_l \in I$, which is a contradiction. For the other products of any $k-1$ element of $\{x-y, z_2, z_3, z_4, \dots, z_k\}$, they can all be proved using the same idea as above. Thus, $\{x-y, z_2, z_3, z_4, \dots, z_k\}$ forms a hyperedge. Therefore, $x-y \in V_1$, which implies that $x-y \in P_1$.

Now, we obtain that $x-y \in P_1$. It remains to show that $rx \in P_1$ for all $r \in \mathcal{R}$ and $x \in P_1$. Let $r \in \mathcal{R}$ and $x \in P_1$.

Case 1. $x \in I$. Then, $rx \in I$, which implies that $rx \in P_1$.

Case 2. $x \notin I$ and $x \in V_1$. Then, there exist $z_2 \in V_2, z_3 \in V_3, z_4 \in V_4, \dots, z_k \in V_k$ such that $x \prod_{i=2}^k z_i \in I$ and the products of all elements of any $(k-1)$ -subsets of $\{x, z_2, z_3, \dots, z_k\}$ are not in I . Thus, $rx \prod_{i=2}^k z_i \in I$.

Suppose that $\{rx, z_2, z_3, \dots, z_k\}$ is not a hyperedge. Otherwise, $rx \in V_1$, which implies that $rx \in P_1$.

Case 2.1. The products of all elements of any $(k-1)$ subset of $\{rx, z_2, z_3, z_4, \dots, z_k\}$ are in I . Since $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph, the products of all elements of any s subset of $\{z_2, z_3, \dots, z_k\}$ are not in I for all $1 \leq s \leq k-1$. Then, there exist two elements z_{i_1} and z_{i_2} such that $rxz_{i_1} \in I$ and $rxz_{i_2} \in I$, where $i_1 \neq i_2$. By Lemma 1, $rx \in V_{i_1} \cap V_{i_2}$, which is a contradiction.

Case 2.2. There are some products of any $k-1$ element of $\{rx, z_2, z_3, z_4, \dots, z_k\}$ in I . Then, there exists one element such that its product with rx in I , say z_2 ; that is, $rxz_2 \in I$. Thus, $rx \in V_2$. Since $x \in V_1$, by Lemma 1, $r \prod_{j=2}^{k-1} z_j \in V_1$. Then, $\{x, rx, z_3, z_4, z_5, \dots, z_k\}$ and $\{r \prod_{j=2}^{k-1} z_j, z_2, z_3, z_4, \dots, z_k\}$ are hyperedges. Since $x^2 \in I$, we have $x^2r \in I$. It contradicts the fact that $\{x, rx, z_3, z_4, z_5, \dots, z_k\}$ is a hyperedge.

Next, we show that P_1 is a prime ideal. Let $x, y \in P_1$ such that $xy \in P_1$.

Suppose that $x, y \notin P_1$.

Case 1. $xy \in V_1$ and $xy \in I$. It contradicts the fact that $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph.

Case 2. $xy \in V_1$ and $xy \notin I$. We see that $x \notin V_1$. Otherwise, $x \in V_1$, which implies $x \in P_1$, which is a contradiction. Then, there exists $i \neq 1$ such that $x \in V_i$. Without loss of generality, assume that $x \in V_2$. Thus, there exist $z_3 \in V_3, z_4 \in V_4, z_5 \in V_5, \dots, z_k \in V_k$ such that $\{xy, x, z_3, z_4, z_5, \dots, z_k\}$ is a hyperedge. Since $x^2 \in I$, we have $x^2y \prod_{l=3}^{k-1} z_l \in I$, which is a contradiction.

Case 3. $xy \notin V_1$ and $xy \in I$. By Lemma 1, x and y are in the same partite set. We see that $x, y \notin V_1$. Otherwise, $x, y \in V_1$, which implies $x, y \in P_1$, which is a contradiction. Without loss of generality, assume that $x, y \in V_2$. Then, there exist $z_1 \in V_1, z_3 \in V_3, z_4 \in V_4, \dots, z_k \in V_k$ such that $z_1x \prod_{i=3}^k z_i \in I$, and the products of all elements of any $(k-1)$ subset of $\{z_1, x, z_3, z_4, z_5, \dots, z_k\}$ are not in I , $z_1y \prod_{i=3}^k z_i \in I$, and the products of all elements of any $(k-1)$ subset of $\{z_1, y, z_3, z_4, z_5, \dots, z_k\}$ are not in I . By Lemma 1, $x \prod_{i=3}^k z_i \in V_1$ and $z_1x \prod_{j=4}^k z_j \in V_3$. Since $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph, $\{x \prod_{i=3}^k z_i, y, z_1x \prod_{j=4}^k z_j, z_4, z_5, z_6, \dots, z_k\}$ is a hyperedge. Since $x^2 \in I$, we have $z_1x^2 \prod_{i=3}^k z_i \in I$, which is a contradiction. $\square$

From Theorem 5, we can also conclude that

Theorem 6. If $r^2 \in I$ for every ideal-based k -zero-divisor $r \in \mathcal{R}$, then $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -partite k -uniform hypergraph if and only if $I = \cap_{i=1}^k P_i$, where $P_1, P_2, P_3, \dots, P_k$ are distinct k prime ideals of $\mathcal{R}$.

Example 3. For $\mathcal{R} \cong \mathbb{Z}_{30} \times \mathbb{Z}_3$ and $I \cong \bar{0} \times \mathbb{Z}_3$, we can separate $Z_I(\mathcal{R}, 3)$ into 3 partite sets $V_1 = \{(2, \bar{0}), (2, \bar{1}), (2, \bar{2}), (4, \bar{0}), (4, \bar{1}), (4, \bar{2}), (8, \bar{0}), (8, \bar{1}), (8, \bar{2}), (14, \bar{0}), (14, \bar{1}), (14, \bar{2}), (16, \bar{0}), (16, \bar{1}), (16, \bar{2}), (22, \bar{0}), (22, \bar{1}), (22, \bar{2}), (26, \bar{0}), (26, \bar{1}), (26, \bar{2}), (28, \bar{0}), (28, \bar{1}), (28, \bar{2})\}$, $V_2 = \{(\bar{3}, \bar{0}), (\bar{3}, \bar{1}), (\bar{3}, \bar{2}), (\bar{9}, \bar{0}), (\bar{9}, \bar{1}), (\bar{9}, \bar{2}), (\bar{21}, \bar{0}), (\bar{21}, \bar{1}), (\bar{21}, \bar{2}), (\bar{27}, \bar{0}), (\bar{27}, \bar{1}), (\bar{27}, \bar{2})\}$

and $V_3 = \{(\bar{5}, \bar{0}), (\bar{5}, \bar{1}), (\bar{5}, \bar{2}), (\bar{25}, \bar{0}), (\bar{25}, \bar{1}), (\bar{25}, \bar{2})\}$. We see that $\mathcal{H}_3^I(\mathcal{R})$ is a complete 3-partite 3-uniform hypergraph, but I cannot be written as the intersection of three prime ideals.

In addition to a complete k -partite k -uniform hypergraph, we investigate the completeness of $\mathcal{H}_k^I(\mathcal{R})$.

Theorem 7. Let $\mathcal{R}$ be a commutative ring with nonzero identity and I be a nonzero proper ideal of $\mathcal{R}$. The following statements hold:

1. Assume that for any k elements in $\mathcal{R} \setminus I$, their product is in I . If I is not a $(k-1)$ -absorbing ideal, then $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -uniform hypergraph.
2. If $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -uniform hypergraph, then I is not a $(k-1)$ -absorbing ideal.

Proof.

1. Assume that for any k elements in $\mathcal{R} \setminus I$, their product is in I and I is not a $(k-1)$ -absorbing ideal. Since I is not a $(k-1)$ -absorbing ideal, all products of any $(k-1)$ element in $\mathcal{R} \setminus I$ are not in I . Therefore, these k elements form a hyperedge of $\mathcal{H}_k^I(\mathcal{R})$. Therefore, $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -uniform hypergraph.
2. Assume that $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -uniform hypergraph. Then, every k element in $Z_I(\mathcal{R}, k)$ forms a hyperedge; that is, $\prod_{i=1}^k x_i \in I$ and the products of all elements of any $(k-1)$ subset of $\{x_1, x_2, x_3, \dots, x_k\}$ are not in I , for all $x_1, x_2, x_3, \dots, x_k \in Z_I(\mathcal{R}, k)$.

Suppose that I is a $(k-1)$ -absorbing ideal. Since $\prod_{i=1}^k x_i \in I$, there exists a product of some $(k-1)$ elements of $\{x_1, x_2, x_3, \dots, x_k\}$ is in I , which is a contradiction. $\square$

From Theorem 7, we can also conclude that:

Theorem 8. If for any k element in $\mathcal{R} \setminus I$, its product is in I , then $\mathcal{H}_k^I(\mathcal{R})$ is a complete k -uniform hypergraph if and only if I is not a $(k-1)$ -absorbing ideal.

Example 4. For $\mathcal{R} \cong \mathbb{Z}_8 \times \mathbb{Z}_3$ and $I \cong \bar{0} \times \mathbb{Z}_3$, we have $Z_I(\mathcal{R}, 3) = \{(\bar{2}, \bar{0}), (\bar{2}, \bar{1}), (\bar{2}, \bar{2}), (\bar{6}, \bar{0}), (\bar{6}, \bar{1}), (\bar{6}, \bar{2})\}$. We see that $\mathcal{H}_3^I(\mathcal{R})$ is a complete 3-uniform hypergraph and I is not a 2-absorbing ideal.

By extending the idea of Theorem 2.5 of [13], we obtain the following theorem. For the definition of a clique, we refer to [7].

Let $\mathcal{H}$ be a k -uniform hypergraph. A subset K of the vertex set $V(\mathcal{H})$ is called a clique if every k subset of K is a hyperedge of $\mathcal{H}$.

Theorem 9. Let $\mathcal{R}$ be a commutative ring and I be a nonzero proper ideal of $\mathcal{R}$, and let K be a clique in $\mathcal{H}_k^I(\mathcal{R})$ such that $x^2 \in I$ for all $x \in K$. If I is not a $(k-1)$ -absorbing ideal, then $K \cup I$ is an ideal of $\mathcal{R}$.

Proof. Assume that I is not a $(k-1)$ -absorbing ideal. First, we claim that $x - y \in K \cup I$ for all $x, y \in K \cup I$. Let $x, y \in K \cup I$.

Case 1. $x, y \in I$. Then, $x - y \in I$, which implies that $x - y \in K \cup I$.

Case 2. $x, y \in K$ with $x - y \notin I$. Then, there exist $z_1, z_2, z_3, \dots, z_{k-1} \in K$ such that $x \prod_{i=1}^{k-1} z_i \in I$ and the products of all elements of any $(k-1)$ subset of $\{x, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I , $y \prod_{i=1}^{k-1} z_i \in I$, and the products of all elements of any $(k-1)$ subset of $\{y, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Thus, $(x - y) \prod_{i=1}^{k-1} z_i \in I$. Since I is not a $(k-1)$ -absorbing ideal, the products of all elements of any $(k-1)$ subset of $\{x - y, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Since K is a clique, $x - y \in K$.

Case 3. Without loss of generality, $x \in K$ and $y \in I$. Then, there exist $z_1, z_2, z_3, \dots, z_{k-1} \in K$ such that $x \prod_{i=1}^{k-1} z_i \in I$ and the products of all elements of any $(k-1)$ subset of $\{x, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Since $y \in I$, we have $y \prod_{i=1}^{k-1} z_i \in I$. Thus, $(x - y) \prod_{i=1}^{k-1} z_i \in I$.

I and the products of all elements of any $(k-1)$ subset of $\{x-y, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Since K is a clique, $x-y \in K$.

Next, we show that $rx \in K \cup I$ for all $x \in K \cup I$ and $r \in R$. Let $x \in K \cup I$ and $r \in R$.

Case 1. $x \in I$. Then, $rx \in I$, which implies that $rx \in K \cup I$.

Case 2. $x \in K$ and $x \notin I$. Then, there exist $z_1, z_2, z_3, \dots, z_{k-1} \in K$ such that $x \prod_{i=1}^{k-1} z_i \in I$, and the products of all elements of any $(k-1)$ subset of $\{x, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Thus, $rx \prod_{i=1}^{k-1} z_i \in I$. Since I is not a $(k-1)$ -absorbing ideal, the products of all elements of any $(k-1)$ subset of $\{rx, z_1, z_2, z_3, \dots, z_{k-1}\}$ are not in I . Since K is a clique, $rx \in K$. $\square$

Example 5. For $\mathcal{R} \cong \mathbb{Z}_{12} \times \mathbb{Z}_3$ and $I \cong \bar{0} \times \mathbb{Z}_3$, we have $Z_I(\mathcal{R}, 3) = \{(\bar{2}, \bar{0}), (\bar{2}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{0}), (\bar{3}, \bar{1}), (\bar{3}, \bar{2}), (\bar{9}, \bar{0}), (\bar{9}, \bar{1}), (\bar{9}, \bar{2}), (\bar{10}, \bar{0}), (\bar{10}, \bar{1}), (\bar{10}, \bar{2})\}$. We see that $K = \{(\bar{2}, \bar{0}), (\bar{3}, \bar{1}), (\bar{10}, \bar{2})\}$ is a clique in $\mathcal{H}_3^I(\mathcal{R})$, and I is not a 2-absorbing ideal, but $K \cup I$ is not an ideal.

5. Discussion and Conclusions

In the study of a k -zero-divisor hypergraph, we considered a commutative ring D/I , where D is a PID and I is the appropriate ideal of D instead of considering directly a commutative ring $\mathcal{R}$.

The existence of a prime element p of D and the finiteness of D/Dp^k together with $|(Dp/Dp^k) \setminus (Dp^2/Dp^k)| \geq k$ enabled us to construct a complete hypergraph $\mathcal{H}_k(D/Dp^k)$.

Next, by assuming the existence of nonassociative distinct prime elements $p_1, p_2, p_3, \dots, p_k$ of D , the finiteness of $D/D\pi$, where $\pi = \prod_{i=1}^k p_i^{\alpha_i}$ and $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$, together with $|(Dp_i/D\pi) \setminus ((Dp_i^2/D\pi) \cup \bigcup_{j=1, j \neq i}^k (Dp_j/D\pi))| \geq \alpha_i$ for all $1 \leq i \leq k$, the vertex set $Z(D/R\pi, \sigma)$, where $\sigma = \sum_{i=1}^k \alpha_i$, can be partitioned into k partite sets. This enabled us to construct a k -partite σ -zero-divisor hypergraph of $Z(D/D\pi, \sigma)$. However, our constructed k -partite σ -zero-divisor hypergraph is not complete.

Furthermore, the diameter of $\mathcal{H}_\sigma(D/D\pi)$ is two and the girth of $\mathcal{H}_\sigma(D/D\pi)$ can be either four, two or infinity depending on σ , the cardinality of $Z(D/D\pi, \sigma)$, and the cardinality of each partition set. We notice here again that two can not be the girth of a k -partite graph.

According to an ideal-based k -zero-divisor hypergraph, we obtained the relationship between $\mathcal{H}_k(\mathcal{R}/I)$ and $\mathcal{H}_k^I(\mathcal{R})$. Under some conditions of an ideal I , we provided characterizations of a complete k -partite k -uniform hypergraph and a complete k -uniform hypergraph. We provided a result regarding a clique in $\mathcal{H}_k^I(\mathcal{R})$.

As for future research, we suggest investigating how to construct k -partite σ -zero-divisor hypergraphs to be a complete k -partite σ -uniform hypergraph according to the definition modified from Jirimutu and Wang [18]. There are also some open questions on the ideal-based k -zero-divisor hypergraph such as:

1. Is it true that $\mathcal{H}_k^I(\mathcal{R})$ is connected and $\text{diam}(\mathcal{H}_k^I(\mathcal{R})) \leq k+1$ when $\mathcal{R}$ is a commutative ring with nonzero identity and I is a nonzero proper ideal of $\mathcal{R}$?
2. Which conditions of an ideal I are required to construct a k -partite σ -uniform hypergraph $\mathcal{H}_k^I(\mathcal{R})$ according to the definition modified from Jirimutu and Wang [18]?

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Abbreviations

The following abbreviations are used in this manuscript:

PID Principal Ideal Domain

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