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Continuous Dependence on the Heat Source of 2D Large-Scale Primitive Equations in Oceanic Dynamics

Yuanfei Li *,† and Peng Zeng †

School of Data Science, Guangzhou Huashang College, Guangzhou 511300, China; zyx1120@gdhsc.edu.cn
* Correspondence: 201610104816@mail.scut.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, we consider the initial-boundary value problem for the two-dimensional primitive equations of the large-scale oceanic dynamics. These models are often used to predict weather and climate change. Using the differential inequality technique, rigorous a priori bounds of solutions and the continuous dependence on the heat source are established. We show the application of symmetry in mathematical inequalities in practice.

Keywords: a priori bounds; 2D primitive equations; continuous dependence; heat source

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1. Introduction

Primitive equations are very useful models which are often used to study the climate and weather prediction. It was Lions, Teman and Wang (see [1–4]) who first started the mathematical study of the primitive equations of the atmosphere, the ocean and the coupled atmosphere–ocean. Assuming that all unknown functions are independent of the latitude y, Petcu et al. [5] obtained the two-dimensional primitive equations of the ocean from the three-dimensional primitive equations. The existence and uniqueness of strong solutions of the primitive equations were derived. In a following paper, Huang and Guo [6] considered the two-dimensional primitive equations of large-scale oceanic motion. They obtained the the existence and uniqueness of global strong solutions. Huang et al. [7] studied the two-dimensional primitive equations of large-scale ocean in geophysics driven by degenerate noise. They proved the asymptotically strong Feller property of the probability transition semigroups. Due to the importance of primitive equations, there are many papers to study the problems (see, e.g., [8–14]).

Recently, many authors began to study the structural stability of large-scale primitive equations. Li [15] obtained the continuous dependence on the viscosity coefficient of primitive equations of the atmosphere with vapor saturation. By using the energy analysis methods, Li [16] proved that the primitive equations of the coupled atmosphere-ocean depended continuously on the boundary parameters. The inspiration of the study came from the fluid equations. There have been a lot of articles in the literature to study the stability of fluid equations (for interest, see [17–29]).

In this paper, we also assume that all the unknown functions are independent of the latitude y as in [5,6]. We consider the following two-dimensional large-scale primitive equations with heat source:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \mu_1 \Delta u + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - fv + \frac{\partial p}{\partial x} = 0,\\
&\frac{\partial v}{\partial t} - \mu_2 \Delta v + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + fu = 0,\\
&\frac{\partial p}{\partial z} + \rho g = 0,\\
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,\\
&\frac{\partial T}{\partial t} - \mu_3 \Delta T + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = Q(x,t),\\
&\rho = \rho_0(1 - \beta_T(T - T_{ref})).
\end{aligned} \tag{1}$$

The domain is defined as

$$\Omega = (0,1) \times (-h,0),$$

where h is the depth of the oceanic which is always assumed to be a positive constant in this paper. In (1) the unknown functions (u,v), w, ρ, p, T are the horizontal velocity field, the vertical velocity, the density, the pressure, the temperature, respectively. Q is the heat source function which is given. f is a function of the Earth's rotation which is taken to be constant here, and $\mu_i > 0 (i = 1,2,3)$ are the viscosity coefficients. ρ_0, T_{ref} are the reference values of the density and the temperature. β_T is the expansion coefficient (constants), $\Delta = \partial_x^2 + \partial_z^2$. We observe that, in the case of ocean dynamics, one has to add the diffusion-transport equation of the salinity to the system (1). The salinity equation is not present in (1), but this would raise little additional difficulty to take into account the salinity.

The boundary of Ω is denoted by $\partial\Omega$ which can be partitioned into

$$\begin{aligned}
\Gamma_0 &= \{(x,z) \in \overline{\Omega} : 0 < x < 1,\ z = 0\},\\
\Gamma_{-h} &= \{(x,z) \in \overline{\Omega} : 0 < x < 1,\ z = -h\},\\
\Gamma_s &= \{(x,z) \in \overline{\Omega} : x = 0, or\ x = 1, -h \le z \le 0\}.
\end{aligned}$$

The system (1) also has the following boundary conditions:

$$\begin{aligned}
&\frac{\partial u}{\partial z} = 0,\ \frac{\partial v}{\partial z} = 0,\ w = 0,\ \frac{\partial T}{\partial z} = -\beta T, \quad on\ \Gamma_0,\\
&u = v = w = 0,\ \frac{\partial T}{\partial z} = 0, \quad on\ \Gamma_{-h},\\
&u = v = w = 0,\ \frac{\partial T}{\partial z} = 0, \quad on\ \Gamma_s,
\end{aligned} \tag{2}$$

where β is a positive constant. In addition, the initial conditions can be written as

$$u(x,z,0) = u_0(x,z), \quad v(x,z,0) = v_0(x,z), \quad T(x,z,0) = T_0(x,z), \quad in\ \Omega. \tag{3}$$

The aim of this paper is to prove the continuous dependence on the heat source of problem (1)–(3) by using the energy methods. This type of study is devoted to know whether a small change in the equation can cause a large change in the solutions. While we take advantage of the mathematical analysis and the symmetry in mathematical inequalities to study these equations, it is helpful for us to know their applicability in physics. Since there will appear some inevitable errors in reality, the study of continuous dependence or convergence results becomes more and more significant. At present, most articles in the literature mainly focused on the existence and long-time behavior of the solutions of the primitive equations. Obviously, the structural stability of the primitive equations has not been paid enough attentions. The research of this paper will bring reference to the study of structural stability of other types of primitive equations.

The present paper is organized as follows. In next section we give some preliminaries of the problem and some well-known inequalities which will be used in the whole paper. We establish rigorous a priori bounds of the solutions in Section 2. In Section 3 we want to prove that the energy is exponential decay with time. Finally, we show how to derive a continuous dependence on the the heat source of our problem in Section 4.

2. Preliminaries of the Problem

We formulate the Equations (1)–(3). Since $w|_{z=-h} = 0$, we integrate the Equation $(1)_4$ from $-h$ to z to obtain

$$w(x,z,t) = w(x,-h,t) - \int_{-h}^{z} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta = -\frac{\partial}{\partial x}\int_{-h}^{z} u(x,\zeta,t)d\zeta. \tag{4}$$

In view of $w|_{z=0} = 0$

$$\int_{-h}^{0} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta = \frac{\partial}{\partial x}\int_{-h}^{0} u(x,\zeta,t)d\zeta = 0. \tag{5}$$

This means that $\int_{-h}^{0} u(x,\zeta,t)d\zeta$ is a constant for arbitrary $0 \le x \le 1$. Realizing the boundary conditions $(2)_3$ we deduce that

$$\int_{-h}^{0} u(x,\zeta,t)d\zeta = 0.$$

By integrating $(1)_3$ and using $(1)_6$ we have

$$\frac{\partial}{\partial x} p(x,z,t) = \frac{\partial}{\partial x} p_s - \mu \int_{z}^{0} \frac{\partial}{\partial x} T(x,\zeta,t)d\zeta, \tag{6}$$

where $p_s = p(x,0,t)$ is the pressure on the surface of the ocean which is unknown and a function of the horizontal variable only, and $\mu = \rho_0 \beta_T$. Inserting (4)–(6) into (1)–(3), our problem can be rewritten as

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \mu_1 \Delta u + u\frac{\partial u}{\partial x} - \Big(\int_{-h}^{z} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z} - fv + \frac{\partial p_s}{\partial x} - \mu\Big(\int_{z}^{0} \frac{\partial}{\partial x} T(x,\zeta,t)d\zeta\Big) = 0,\\
&\frac{\partial v}{\partial t} - \mu_2 \Delta v + u\frac{\partial v}{\partial x} - \Big(\int_{-h}^{z} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta\Big)\frac{\partial v}{\partial z} + fu = 0,\\
&\frac{\partial T}{\partial t} - \mu_3 \Delta T + u\frac{\partial T}{\partial x} - \Big(\int_{-h}^{z} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z} = Q,\\
&\int_{-h}^{0} \frac{\partial}{\partial x} u(x,\zeta,t)d\zeta = 0,
\end{aligned} \tag{7}$$

with the following boundary conditions

$$\begin{aligned}
&\frac{\partial u}{\partial z}\Big|_{z=0} = \frac{\partial v}{\partial z}\Big|_{z=0} = 0,\ u\Big|_{z=-h} = v\Big|_{z=-h} = 0,\ (u,v)\Big|_{\Gamma_s} = 0,\\
&\frac{\partial T}{\partial z}\Big|_{z=0} = -\beta T,\ \frac{\partial T}{\partial z}\Big|_{z=-h} = \frac{\partial T}{\partial z}\Big|_{\Gamma_s} = 0,
\end{aligned} \tag{8}$$

and the initial conditions

$$(u,v,T)\Big|_{t=0} = (u_0, v_0, T_0). \tag{9}$$

In this paper, we also use some well-known inequalities. We list them here.

Lemma 1. *If $\omega(x) \in C^1(0,h)$ and $\omega(0) = \omega(h) = 0$, then*

$$\int_0^h \omega^2 dx \leq \frac{h^2}{\pi^2} \int_0^h \Big(\frac{\partial \omega}{\partial x}\Big)^2 dx. \tag{10}$$

For proof of Lemma 2.1 one can see Refs. [30,31].

Lemma 2. *If $\omega(x,z,t)$ is a sufficiently smooth function in $\Omega = (0,1) \times (-h,0)$ and $\omega(0,z,t) = \omega(1,z,t) = 0$, then*

$$\Big(\int_\Omega \omega^4 dA\Big)^{\frac{1}{2}} \leq C\Big[\Big(\int_\Omega \omega^2 dA\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla\omega|^2 dA\Big)^{\frac{1}{2}} + \Big(\int_\Omega \omega^2 dA\Big)^{\frac{1}{4}}\Big(\int_\Omega |\nabla\omega|^2 dA\Big)^{\frac{3}{4}}\Big],$$

or

$$\Big(\int_\Omega \omega^4 dA\Big)^{\frac{1}{2}} \leq C\Big[\int_\Omega \omega^2 dA + \delta\int_\Omega |\nabla\omega|^2 dA\Big], \tag{11}$$

where $\nabla = (\partial_x, \partial_z)$, C is a positive computable constant and δ is a positive arbitrary constant.

Proof. By the Hölder inequality, we then write

$$\int_\Omega \omega^4 dA \leq \int_{-h}^0 \Big(\int_0^1 \omega^6 dx\Big)^{\frac{1}{2}}\Big(\int_0^1 \omega^2 dx\Big)^{\frac{1}{2}} dz. \tag{12}$$

Since $\omega(0,z,t) = \omega(1,z,t) = 0$, we have

$$\omega^3 = 3\int_0^x \omega^2(\xi,z,t)\frac{\partial\omega(\xi,z,t)}{\partial\xi}d\xi = -3\int_x^1 \omega^2(\xi,z,t)\frac{\partial\omega(\xi,z,t)}{\partial\xi}d\xi. \tag{13}$$

Therefore

$$|\omega|^3 \leq \frac{3}{2}\int_0^1 \omega^2(x,z,t)\Big|\frac{\partial\omega(x,z,t)}{\partial x}\Big|dx. \tag{14}$$

Then we have

$$\Big(\int_0^1 \omega^6 dx\Big)^{\frac{1}{2}} \leq \frac{3}{2}\Big(\int_0^1 \omega^2\Big|\frac{\partial\omega}{\partial x}\Big|dx\Big). \tag{15}$$

Inserting (15) into (12) we get

$$\begin{aligned}\int_\Omega \omega^4 dA &\leq \frac{3}{2}\int_{-h}^0 \Big(\int_0^1 \omega^2\Big|\frac{\partial\omega}{\partial x}\Big|dx\Big)\Big(\int_0^1 \omega^2 dx\Big)^{\frac{1}{2}} dz \\ &\leq \frac{3}{2}\max_{-h\leq z\leq 0}\Big\{\Big(\int_0^1 \omega^2 dx\Big)^{\frac{1}{2}}\Big\}\int_\Omega \omega^2\Big|\frac{\partial\omega}{\partial x}\Big|dA.\end{aligned} \tag{16}$$

Obviously, we have

$$\begin{aligned}\omega^2 &= 2\int_{-h}^z \omega(x,\zeta,t)\frac{\partial\omega(x,\zeta,t)}{\partial\zeta}d\zeta + \omega^2(x,-h,t) \\ &= -2\int_z^0 \omega(x,\zeta,t)\frac{\partial\omega(x,\zeta,t)}{\partial\zeta}d\zeta + \omega^2(x,0,t),\end{aligned} \tag{17}$$

so,

$$\omega^2 \leq \int_{-h}^0 |\omega|\Big|\frac{\partial\omega}{\partial z}\Big|dz + \frac{1}{2}[\omega^2(x,0,t) + \omega^2(x,-h,t)]. \tag{18}$$

To bound the last term of (18), we define a new known function, $f(z)$, satisfying

$$f(0) > 0,\ f(-h) < 0,\ |f'(z)| \le m_1,\ |f(z)| \le m_2,\ for\ -h \le z \le 0, \tag{19}$$

where m_1, m_2 are positive constants. For example, $f(z) = \frac{m_1}{2}(z + \frac{h}{2}),\ m_1 h < 4m_2$ satisfies all the conditions in (19). Using the above estimates and employing the divergence theorem allow us to write

$$\begin{aligned}\min\{f(0), -f(-h)\}([\omega^2(x,0,t) + \omega^2(x,-h,t)] &\le f(0)\omega^2(x,0,t) - f(-h)\omega^2(x,-h,t) \\ &= \int_{-h}^{0} \frac{\partial}{\partial z}\Big(f\omega^2\Big)dz = \int_{-h}^{0} f'(z)\omega^2 dz + 2\int_{-h}^{0} f\omega\frac{\partial \omega}{\partial z}dz \\ &\le m_1 \int_{-h}^{0} \omega^2 dz + 2m_2 \int_{-h}^{0} |\omega|\Big|\frac{\partial \omega}{\partial z}\Big|dz.\end{aligned} \tag{20}$$

Inserting (20) into (18), we have

$$\omega^2 \le m_3 \int_{-h}^{0} \omega^2 dz + m_4 \int_{-h}^{0} |\omega|\Big|\frac{\partial \omega}{\partial z}\Big|dz, \tag{21}$$

where

$$m_3 = \frac{m_1}{2\min\{f(0), -f(-h)\}},\ m_4 = 1 + \frac{m_2}{\min\{f(0), -f(-h)\}}. \tag{22}$$

Therefore

$$\max_{-h\le z\le 0}\Big\{\Big(\int_0^1 \omega^2 dx\Big)^{\frac{1}{2}}\Big\} \le \Big(m_3\int_\Omega \omega^2 dA + m_4\int_\Omega |\omega|\Big|\frac{\partial \omega}{\partial z}\Big|dA\Big)^{\frac{1}{2}}. \tag{23}$$

Thus, from (16) and (23), by the Hölder inequality we have

$$\begin{aligned}\int_\Omega \omega^4 dA \le \frac{3}{2}\Big[m_3 \int_\Omega \omega^2 dA \\ + m_4\Big(\int_\Omega \omega^2 dA\Big)^{\frac{1}{2}}\Big(\int_\Omega \Big|\frac{\partial \omega}{\partial z}\Big|^2 dA\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big(\int_\Omega \omega^4 dA\Big)^{\frac{1}{2}}\Big(\int_\Omega \Big|\frac{\partial \omega}{\partial x}\Big|^2 dA\Big)^{\frac{1}{2}}.\end{aligned} \tag{24}$$

We have after simplification

$$\begin{aligned}\Big(\int_\Omega \omega^4 dA\Big)^{\frac{1}{2}} \le C\Big[\Big(\int_\Omega \omega^2 dA\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla\omega|^2 dA\Big)^{\frac{1}{2}} \\ + \Big(\int_\Omega \omega^2 dA\Big)^{\frac{1}{4}}\Big(\int_\Omega |\nabla\omega|^2 dA\Big)^{\frac{3}{4}}\Big].\end{aligned} \tag{25}$$

□

3. A *priori* Estimates

Now we derive some a priori estimates for the solutions of (7)–(9).

3.1. Estimates for $||u||_2^2,\ ||v||_2^2$ *and* $||T||_2^2$

Multiplying Equation $(7)_3$ with T and integrating over Ω and using $(2.5)_2$ we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega T^2 dA + \mu_3\int_\Omega |\nabla T|^2 dA = -\mu_3\int_0^1 T^2 dx|_{z=0} + \int_\Omega TQdA \\ - \int_\Omega \Big[u\frac{\partial T}{\partial x} - \Big(\int_{-h}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z}\Big]TdA.\end{aligned} \tag{26}$$

Integrating by parts we have

$$-\int_{\Omega}\Big[u\frac{\partial T}{\partial x}-\Big(\int_{-h}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z}\Big]TdA=0. \tag{27}$$

By the Cauchy–Schwarz inequality and the Hölder inequality we deduce

$$\int_{\Omega}TQdA\leq\frac{1}{2}\int_{\Omega}T^2dA+\frac{1}{2}\int_{\Omega}Q^2dA. \tag{28}$$

By (26)–(28) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}T^2dA+\mu_3\int_{\Omega}|\nabla T|^2dA\leq\frac{1}{2}\int_{\Omega}T^2dA+\frac{1}{2}\int_{\Omega}Q^2dA. \tag{29}$$

By the Gronwall inequality, we have

$$\begin{aligned}\int_{\Omega}T^2dA+2\mu_3\int_0^t\int_{\Omega}|\nabla T|^2dAd\eta&\leq\int_{\Omega}T_0^2dA\cdot e^t+\int_0^t\int_{\Omega}e^{t-\eta}Q^2dAd\eta\\&\doteq F_1(t).\end{aligned} \tag{30}$$

Taking the inner product of Equation $(7)_1$ with u, in $L^2(\Omega)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega}u^2dA+\mu_1\int_{\Omega}|\nabla u|^2dA=&-f\int_{\Omega}uvdA\\&-\int_{\Omega}\Big[u\frac{\partial u}{\partial x}-\Big(\int_{-h}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z}\Big]udA\\&-\int_{\Omega}\frac{\partial p_s}{\partial x}udA+\mu\int_{\Omega}\Big(\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)d\zeta\Big)udA.\end{aligned} \tag{31}$$

An integration leads to

$$-\int_{\Omega}\Big[u\frac{\partial u}{\partial x}-\Big(\int_{-h}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z}\Big]udA=0. \tag{32}$$

Integrating by parts and using $(7)_4$ we get

$$-\int_{\Omega}\frac{\partial p_s}{\partial x}udA=-\int_0^1\frac{\partial p_s}{\partial x}\Big(\int_{-h}^{0}udz\Big)dx=\int_0^1p_s\Big(\int_{-h}^{0}\frac{\partial u}{\partial x}dz\Big)dx=0. \tag{33}$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned}\mu\int_{\Omega}\Big(\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)d\zeta\Big)udA&=-\mu\int_{\Omega}\Big(\int_{z}^{0}T(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial x}dA\\&\leq\frac{\mu_1}{2}\int_{\Omega}\Big(\frac{\partial u}{\partial x}\Big)^2dA+\frac{h^2\mu^2}{2\mu_1}\int_{\Omega}T^2dA.\end{aligned} \tag{34}$$

By (31)–(34) we get

$$\frac{d}{dt}\int_{\Omega}u^2dA+\mu_1\int_{\Omega}|\nabla u|^2dA\leq-2f\int_{\Omega}uvdA+\frac{h^2\mu^2}{2\mu_1}\int_{\Omega}T^2dA. \tag{35}$$

Similarly, we can have from $(7)_2$

$$\frac{d}{dt}\int_{\Omega}v^2dA+2\mu_2\int_{\Omega}|\nabla v|^2dA\leq2f\int_{\Omega}uvdA. \tag{36}$$

Combining (35) and (36) and using (30) we get

$$\frac{d}{dt}\Big(\int_{\Omega}u^2dA+\int_{\Omega}v^2dA\Big)+\mu_1\int_{\Omega}|\nabla u|^2dA+2\mu_2\int_{\Omega}|\nabla v|^2dA\leq\frac{h^2\mu^2}{2\mu_1}F_1(t). \tag{37}$$

We integrate (37) from 0 to t to find

$$\begin{aligned}&\int_{\Omega}u^2dA+\int_{\Omega}v^2dA+\mu_1\int_0^t\int_{\Omega}|\nabla u|^2dAd\eta+2\mu_2\int_0^t\int_{\Omega}|\nabla v|^2dAd\eta\\&\qquad\leq\int_{\Omega}u_0^2dA+\int_{\Omega}v_0^2dA+\frac{h^2\mu^2}{2\mu_1}\int_0^tF_1(\eta)d\eta\doteq F_2(t).\end{aligned} \tag{38}$$

3.2. *Estimate for* $|T|$

We multiply $(7)_3$ by T^{p-1}, and integrate by parts to find

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\Omega}T^pdA+\frac{p-1}{p^2}\mu_3\int_{\Omega}|\nabla T^{\frac{p}{2}}|^2dA&=-\mu_3\beta\int_0^1T^pdx|_{z=0}+\int_{\Omega}QT^{p-1}dA\\&-\int_{\Omega}\Big[u\frac{\partial T}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z}\Big]T^{p-1}dA.\end{aligned} \tag{39}$$

After integrating by parts on the third term of (39) and realizing the boundary condition (8) we get

$$-\int_{\Omega}\Big[u\frac{\partial T}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z}\Big]T^{p-1}dA=0. \tag{40}$$

By the Hölder inequality and the Cauchy–Schwarz inequality we have

$$\int_{\Omega}QT^{p-1}dA\leq\frac{1}{p}\int_{\Omega}Q^pdA+\frac{p-1}{p}\int_{\Omega}T^pdA. \tag{41}$$

Therefore,

$$\frac{d}{dt}\int_{\Omega}T^pdA\leq\int_{\Omega}Q^pdA+(p-1)\int_{\Omega}T^pdA. \tag{42}$$

By the Gronwall inequality we have

$$\int_{\Omega}T^pdA\leq\int_{\Omega}T_0^pdA\cdot e^{(p-1)t}+\int_0^t\int_{\Omega}e^{(p-1)(t-\eta)}Q^pdAd\eta.$$

Therefore

$$\Big(\int_{\Omega}T^pdA\Big)^{\frac{1}{p}}\leq\Big\{\int_{\Omega}T_0^pdA\cdot e^{(p-1)t}+\int_0^t\int_{\Omega}e^{(p-1)(t-\eta)}Q^pdAd\eta\Big\}^{\frac{1}{p}}. \tag{43}$$

Letting now $p\to\infty$ in (43) we can obtain

$$\sup_{\Omega}|T|\leq T_m, \tag{44}$$

where $T_m=\ \sup_{\Omega}\{||Q||_{\infty},\ ||T_0||_{\infty}\}$.

3.3. Estimate for $||\frac{\partial u}{\partial z}||_{L^4(\Omega)}$

Using $(7)_1$ we start with

$$\begin{aligned}\int_0^t\int_\Omega \frac{\partial}{\partial z}\Big\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z}-fv+\frac{\partial p_s}{\partial x}\\ -\mu\Big(\int_z^0\frac{\partial}{\partial x}T(x,\zeta,t)d\zeta\Big)-\mu_1\Delta u\Big\}\frac{\partial u}{\partial z}dAd\eta=0.\end{aligned} \tag{45}$$

Integrating by parts we have

$$\begin{aligned}\frac{1}{2}\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dA+\mu_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta&=\frac{1}{2}\int_\Omega\Big(\frac{\partial u_0}{\partial z}\Big)^2dA\\ &-\int_0^t\int_\Omega\Big[u\frac{\partial^2u}{\partial x\partial z}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,\eta)d\zeta\Big)\frac{\partial^2u}{\partial z^2}\Big]\frac{\partial u}{\partial z}dAd\eta\\ &+f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}dAd\eta-\mu\int_0^t\int_\Omega\frac{\partial T}{\partial x}\frac{\partial u}{\partial z}dAd\eta.\end{aligned} \tag{46}$$

Upon integrating by parts we get

$$-\int_0^t\int_\Omega\Big[u\frac{\partial^2u}{\partial x\partial z}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,\eta)d\zeta\Big)\frac{\partial^2u}{\partial z^2}\Big]\frac{\partial u}{\partial z}dAd\eta=0. \tag{47}$$

By (30), (38) and the Hölder inequality we have

$$\begin{aligned}-\mu\int_0^t\int_\Omega\frac{\partial T}{\partial x}\frac{\partial u}{\partial z}dAd\eta&\le\mu\Big(\int_0^t\int_\Omega\Big(\frac{\partial T}{\partial x}\Big)^2dAd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dAd\eta\Big)^{\frac{1}{2}}\\ &\le\mu\sqrt{\frac{F_1(t)F_2(t)}{2\mu_1\mu_3}}.\end{aligned} \tag{48}$$

Inserting the above bounds into (46) we write

$$\begin{aligned}\frac{1}{2}\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dA&+\mu_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta\\ &\le\frac{1}{2}\int_\Omega\Big(\frac{\partial u_0}{\partial z}\Big)^2dA+f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}dAd\eta\\ &+\mu\sqrt{\frac{F_1(t)F_2(t)}{2\mu_1\mu_3}}.\end{aligned} \tag{49}$$

We now carry out a similar procedure starting from $(7)_2$ to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dA&+\mu_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\\ &=\frac{1}{2}\int_\Omega\Big(\frac{\partial v_0}{\partial z}\Big)^2dA-f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}dAd\eta\\ &-\int_0^t\int_\Omega\Big[u\frac{\partial^2v}{\partial x\partial z}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial^2v}{\partial z^2}\Big]\frac{\partial v}{\partial z}dAd\eta\\ &-\int_0^t\int_\Omega\Big[\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\Big]\frac{\partial v}{\partial z}dAd\eta.\end{aligned} \tag{50}$$

Upon integrating by parts we get

$$-\int_0^t\int_\Omega\Big[u\frac{\partial^2v}{\partial x\partial z}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u(x,\zeta,t)d\zeta\Big)\frac{\partial^2v}{\partial z^2}\Big]\frac{\partial v}{\partial z}dAd\eta=0. \tag{51}$$

Upon using the Cauchy–Schwarz inequality, (11), (38)

$$
\begin{aligned}
&-\int_0^t\int_\Omega\Big[\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\Big]\frac{\partial v}{\partial z}dAd\eta\\
&\leq\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^4dAd\eta\Big]^{\frac14}\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^4dAd\eta\Big]^{\frac14}\\
&+\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^4dAd\eta\Big]^{\frac12}\\
&\leq\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dAd\eta+\delta_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta\Big]^{\frac12}\\
&\cdot\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dAd\eta+\delta_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\Big]^{\frac12}\\
&+\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dAd\eta+\delta_3\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\Big]\\
&\leq\frac12\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dAd\eta+\delta_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta\Big]\\
&+\frac12\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dAd\eta+\delta_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\Big]\\
&+\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x}\Big)^2dAd\eta\Big]^{\frac12}\Big[\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dAd\eta+\delta_3\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\Big]\\
&\leq\Big[\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\frac{F_2(t)}{\mu_1}+\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\frac{F_2(t)}{2\mu_2}+\sqrt{\frac{F_2(t)}{\mu_1}}\frac{F_2(t)}{2\mu_2}\Big]\\
&+\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\delta_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta\\
&+\Big[\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\delta_2+\sqrt{\frac{F_2(t)}{\mu_1}}\delta_3\Big]\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta,
\end{aligned}
\tag{52}
$$

where $\delta_1,\ \delta_2,\ \delta_3$ are positive constants which will be given later.

The idea is to insert (51) and (52) into (50) and then choose $\delta_2=\frac12\sqrt{\frac{2\mu_2}{F_2(t)}}\mu_2,\ \delta_3=\frac14\sqrt{\frac{\mu_1}{F_2(t)}}\mu_2$. We may have

$$
\begin{aligned}
&\frac12\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dA+\frac12\mu_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\\
&\qquad\leq\frac12\int_\Omega\Big(\frac{\partial v_0}{\partial z}\Big)^2dA-f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}dAd\eta\\
&\qquad+\Big[\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\frac{F_2(t)}{\mu_1}+\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\frac{F_2(t)}{2\mu_2}+\sqrt{\frac{F_2(t)}{\mu_1}}\frac{F_2(t)}{2\mu_2}\Big]\\
&\qquad+\frac12\sqrt{\frac{F_2(t)}{2\mu_2}}\delta_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta.
\end{aligned}
\tag{53}
$$

We add (49) and (53) and choose that $\delta_1=\sqrt{\frac{2\mu_2}{F_2(t)}}\mu_1$ to find

$$
\begin{aligned}
&\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^2dA+\int_\Omega\Big(\frac{\partial v}{\partial z}\Big)^2dA\\
&\qquad+\mu_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial z}\Big|^2dAd\eta+\mu_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial z}\Big|^2dAd\eta\\
&\qquad\leq F_3(t),
\end{aligned}
\tag{54}
$$

where

$$
\begin{aligned}
F_3(t) = {} & \int_\Omega \Big(\frac{\partial u_0}{\partial z}\Big)^2 dA + \int_\Omega \Big(\frac{\partial v_0}{\partial z}\Big)^2 dA + 2\mu\sqrt{\frac{F_1(t)F_2(t)}{2\mu_1\mu_3}} \\
& + \sqrt{\frac{F_2(t)}{2\mu_2}\frac{F_2(t)}{\mu_1}} + \sqrt{\frac{F_2(t)}{2\mu_2}\frac{F_2(t)}{2\mu_2}} + \sqrt{\frac{F_2(t)}{\mu_1}\frac{F_2(t)}{\mu_2}}.
\end{aligned} \tag{55}
$$

Using (11) with $\delta = 1$ then we have

$$
\begin{aligned}
\int_0^t \int_\Omega \Big(\frac{\partial u}{\partial z}\Big)^4 dA d\eta & \leq C\Big(\int_0^t \int_\Omega \Big(\frac{\partial u}{\partial z}\Big)^2 dA d\eta + \int_0^t \int_\Omega \Big|\nabla \frac{\partial u}{\partial z}\Big|^2 dA d\eta\Big)^2 \\
& \leq C\Big(\int_0^t F_3(\eta) d\eta + \frac{1}{\mu_1} F_3(t)\Big)^2 \doteq F_4(t).
\end{aligned} \tag{56}
$$

4. Exponential Decay Estimates with Time When $Q = 0$

In this section we want to prove the following theorem basis on Section 3.

Theorem 1. *If $(u_0, v_0, T_0) \in H(\Omega)$, $Q = 0$, then the global weakly strong solution (u, v, T) for the system (7)–(9) satisfies*

$$
||u||^2,\ ||v||^2,\ ||T||^2,\ \int_0^t ||\partial_{x_2} T(\eta)||^2 d\eta,\ \int_0^t ||\partial_{x_2} u(\eta)||^2 d\eta,\ \int_0^t ||\partial_{x_2} v(\eta)||^2 d\eta
$$

decay exponentially with time.

Proof. Since T, u and v satisfy the conditions of Lemma 2.1, we have

$$
\begin{aligned}
\int_\Omega \Big|\frac{\partial T}{\partial x}\Big|^2 dA & \geq \pi^2 \int_\Omega T^2 dA, \\
\int_\Omega \Big|\frac{\partial u}{\partial x}\Big|^2 dA & \geq \pi^2 \int_\Omega u^2 dA, \\
\int_\Omega \Big|\frac{\partial v}{\partial x}\Big|^2 dA & \geq \pi^2 \int_\Omega v^2 dA.
\end{aligned} \tag{57}
$$

It follows from (29) with $Q = 0$ that

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega T^2 dA + \pi^2\mu_3 \int_\Omega T^2 dA + \mu_3 \int_\Omega \Big|\frac{\partial T}{\partial z}\Big|^2 dA \leq 0. \tag{58}
$$

So, by the Gronwall inequality, we get

$$
\int_\Omega T^2 dA + \mu_3 \int_0^t \int_\Omega \Big|\frac{\partial T}{\partial z}\Big|^2 dA d\eta \leq \int_\Omega T_0^2 dA \cdot e^{-\tau_1 t}. \tag{59}
$$

where $\tau_1 = \pi^2 \mu_3$. In view of (35), (36) and (57), we have

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_\Omega u^2 dA + \int_\Omega v^2 dA\Big) + \pi^2\mu_1 \int_\Omega u^2 dA + 2\pi^2\mu_2 \int_\Omega v^2 dA & \\
+ \mu_1 \int_\Omega \Big|\frac{\partial u}{\partial z}\Big|^2 dA + \mu_2 \int_\Omega \Big|\frac{\partial v}{\partial z}\Big|^2 dA & \\
\leq \frac{h^2\mu^2}{2\mu_1}\int_\Omega T^2 dA. &
\end{aligned} \tag{60}
$$

Letting

$$
\tau_2 = \pi^2 \min\{\mu_1,\ 2\mu_2\},
$$

and using (59) we may have from (60)

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^2 dA + \int_\Omega v^2 dA\Big) + \tau_2\Big(\int_\Omega u^2 dA + \int_\Omega v^2 dA\Big) \\
&\qquad\qquad + \mu_1 \int_\Omega \Big|\frac{\partial u}{\partial z}\Big|^2 dA + \mu_2 \int_\Omega \Big|\frac{\partial v}{\partial z}\Big|^2 dA \\
&\qquad\qquad \le \frac{h^2\mu^2}{2\mu_1}\int_\Omega T_0^2 dA \cdot e^{-\tau_1 t}.
\end{aligned}
\tag{61}
$$

By the Gronwall inequality again, we get

$$
\begin{aligned}
&\int_\Omega u^2 dA + \int_\Omega v^2 dA + \mu_1\int_0^t\int_\Omega \Big|\frac{\partial u}{\partial z}\Big|^2 dA d\eta + \mu_2\int_0^t\int_\Omega \Big|\frac{\partial v}{\partial z}\Big|^2 dA d\eta \\
&\qquad \le \Big(\int_\Omega u_0^2 dA + \int_\Omega v_0^2 dA\Big)\cdot e^{-\tau_2 t} + \frac{h^2\mu^2}{2\mu_1(\tau_2-\tau_1)}\int_\Omega T_0^2 dA \cdot e^{-\tau_1 t}, if\ \tau_2 > \tau_1, \\
&\int_\Omega u^2 dA + \int_\Omega v^2 dA + \mu_1\int_0^t\int_\Omega \Big|\frac{\partial u}{\partial z}\Big|^2 dA d\eta + \mu_2\int_0^t\int_\Omega \Big|\frac{\partial v}{\partial z}\Big|^2 dA d\eta \\
&\qquad \le \Big(\int_\Omega u_0^2 dA + \int_\Omega v_0^2 dA\Big)\cdot e^{-\tau_2 t} + \frac{h^2\mu^2}{2\mu_1(\tau_2-\tau_1)}\int_\Omega T_0^2 dA \cdot e^{-\tau_2 t}, if\ \tau_2 < \tau_1, \\
&\int_\Omega u^2 dA + \int_\Omega v^2 dA + \mu_1\int_0^t\int_\Omega \Big|\frac{\partial u}{\partial z}\Big|^2 dA d\eta + \mu_2\int_0^t\int_\Omega \Big|\frac{\partial v}{\partial z}\Big|^2 dA d\eta \\
&\qquad \le \Big(\int_\Omega u_0^2 dA + \int_\Omega v_0^2 dA\Big)\cdot e^{-\tau_2 t} + \frac{h^2\mu^2 t}{2\mu_1}\int_\Omega T_0^2 dA \cdot e^{-\tau_2 t}, if\ \tau_2 = \tau_1.
\end{aligned}
$$

□

5. Continuous Dependence on the Heat Source

Supposing (u^*, v^*, T^*, p_s^*) also be the solutions of (7)–(9) with the same initial-boundary conditions as (u, v, T, p_s), but with different heat source Q^*. Let

$$
\widetilde{u} = u - u^*,\ \widetilde{v} = v - v^*,\ \widetilde{T} = T - T^*,\ \pi_s = p_s - p_s^*,\ \widetilde{Q} = Q - Q^*,
\tag{62}
$$

then $(\widetilde{u}, \widetilde{v}, \pi_s)$ satisfies the following initial-boundary problem

$$
\begin{aligned}
&\frac{\partial \widetilde{u}}{\partial t} - \mu_1\Delta\widetilde{u} + \widetilde{u}\frac{\partial u}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x}\widetilde{u}(x,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z} + u^*\frac{\partial \widetilde{u}}{\partial x} \\
&- \Big(\int_{-h}^z \frac{\partial}{\partial x}u^*(x,\zeta,t)d\zeta\Big)\frac{\partial \widetilde{u}}{\partial z} - f\widetilde{v} + \frac{\partial \pi_s}{\partial x} - \Big(\int_z^0 \frac{\partial}{\partial x}\widetilde{T}(x,\zeta,t)d\zeta\Big) = 0, \\
&\frac{\partial \widetilde{v}}{\partial t} - \mu_2\Delta\widetilde{v} + \widetilde{u}\frac{\partial v}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x}\widetilde{u}(x,\zeta,t)d\zeta\Big)\frac{\partial v}{\partial z} + u^*\frac{\partial \widetilde{v}}{\partial x} \\
&- \Big(\int_{-h}^z \frac{\partial}{\partial x}u^*(x,\zeta,t)d\zeta\Big)\frac{\partial \widetilde{v}}{\partial z} - f\widetilde{u} = 0, \\
&\frac{\partial \widetilde{T}}{\partial t} - \mu_3\Delta\widetilde{T} + \widetilde{u}\frac{\partial T}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x}\widetilde{u}(x,\zeta,t)d\zeta\Big)\frac{\partial T}{\partial z} + u^*\frac{\partial \widetilde{T}}{\partial x} \\
&- \Big(\int_{-h}^z \frac{\partial}{\partial x}u^*(x,\zeta,t)d\zeta\Big)\frac{\partial \widetilde{T}}{\partial z} = \widetilde{Q}, \\
&\int_{-h}^0 \frac{\partial}{\partial x}\widetilde{u}(x,\zeta,t)d\zeta = 0,
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
&\frac{\partial \widetilde{u}}{\partial z}\Big|_{z=0} = 0,\ \frac{\partial \widetilde{v}}{\partial z}\Big|_{z=0} = 0,\ \widetilde{u}\Big|_{z=-h} = \widetilde{v}\Big|_{z=-h} = 0,\ (\widetilde{u}, \widetilde{v})\Big|_{\Gamma_s} = 0, \\
&\frac{\partial \widetilde{T}}{\partial z}\Big|_{z=0} = -\beta\widetilde{T},\ \frac{\partial \widetilde{T}}{\partial z}\Big|_{z=-h} = 0,\ \frac{\partial \widetilde{T}}{\partial x}\Big|_{\Gamma_s} = 0,
\end{aligned}
\tag{64}
$$

$$(\widetilde{u},\widetilde{v},\widetilde{T})\Big|_{t=0}=(0,0,0). \tag{65}$$

We have the following theorem:

Theorem 2. *Let $(\widetilde{u},\widetilde{v},\widetilde{T})$ be the solutions of (63)–(65) with $Q,T_0\in L^\infty(\Omega)$ and $T_0,u_0,v_0\in L^2(\Omega)$. Then $(\widetilde{u},\widetilde{v},\widetilde{T})$ satisfy the inequality for $\theta>0,\gamma_1(t)>0$*

$$\begin{aligned}\int_\Omega(\widetilde{u}^2+\widetilde{v}^2+\theta\widetilde{T}^2)dA+\int_0^t\int_\Omega\Big(\frac{1}{2}\mu_1|\nabla\widetilde{u}|^2+\mu_2|\nabla\widetilde{v}|^2+\theta\mu_3|\nabla\widetilde{T}|^2\Big)dAd\eta\\ \leq\gamma_1(t)\theta\int_0^t\int_0^\tau\int_\Omega e^{\int_\tau^t\gamma_1(s)ds}\widetilde{Q}^2dAd\eta d\tau+\theta\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta,\end{aligned} \tag{66}$$

which is the continuous dependence result on the heat source Q.

Proof. Now taking the inner product of the first equation of (63) with $\widetilde{u}$, in $L^2(\Omega)$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega\widetilde{u}^2dA+\mu_1\int_0^t\int_\Omega|\nabla\widetilde{u}|^2dAd\eta&=f\int_0^t\int_\Omega\widetilde{u}\widetilde{v}dAd\eta-\int_0^t\int_\Omega\frac{\partial\pi_s}{\partial x}\widetilde{u}dAd\eta\\&+\mu\int_0^t\int_\Omega\Big(\int_z^0\frac{\partial}{\partial x}\widetilde{T}(x,\zeta,\eta)d\zeta\Big)\widetilde{u}dAd\eta\\&-\int_0^t\int_\Omega\Big[\widetilde{u}\frac{\partial u}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)\frac{\partial u}{\partial z}\Big]\widetilde{u}dAd\eta\\&-\int_0^t\int_\Omega\Big[u^*\frac{\partial\widetilde{u}}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u^*(x,\zeta,\eta)d\zeta\Big)\frac{\partial\widetilde{u}}{\partial z}\Big]\widetilde{u}dAd\eta.\end{aligned} \tag{67}$$

An integration by parts leads to

$$-\int_0^t\int_\Omega\Big[u^*\frac{\partial\widetilde{u}}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}u^*(x,\zeta,\eta)d\zeta\Big)\frac{\partial\widetilde{u}}{\partial z}\Big]\widetilde{u}dAd\eta=0, \tag{68}$$

$$-\int_0^t\int_\Omega\frac{\partial\pi_s}{\partial x}\widetilde{u}dAd\eta=-\int_0^t\int_0^1\frac{\partial\pi_s}{\partial x}\Big(\int_{-h}^0\widetilde{u}(x,z,\eta)dz\Big)dxd\eta=0. \tag{69}$$

By the Hölder inequality, Lemma 2.2, (38), Lemma 2.1, (56) and the AG mean inequality, we have

$$\begin{aligned}&-\int_0^t\int_\Omega\Big[\widetilde{u}\frac{\partial u}{\partial x}-\Big(\int_{-h}^z\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)\frac{\partial u}{\partial z}\Big]\widetilde{u}dAd\eta\\&\leq\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x}\Big)^2dAd\eta\Big]^{\frac{1}{2}}\Big[\int_0^t\int_\Omega\widetilde{u}^4dAd\eta\Big]^{\frac{1}{2}}\\&+\Big[\int_0^t\int_\Omega\Big(\int_{-h}^z\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)^2dAd\eta\Big]^{\frac{1}{2}}\Big[\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial z}\Big)^4dAd\eta\Big]^{\frac{1}{4}}\\&\cdot\Big[\int_0^t\int_\Omega\widetilde{u}^4dAd\eta\Big]^{\frac{1}{4}}\\&\leq\sqrt{\frac{F_2(t)}{\mu_1}}C\Big[\int_0^t\int_\Omega\widetilde{u}^2dAd\eta+\delta_1\int_0^t\int_\Omega|\nabla\widetilde{u}|^2dAd\eta\Big]\\&+\frac{\sqrt{C}h}{\pi}\sqrt[4]{F_4(t)}\Big[\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{u}}{\partial x}\Big)^2dAd\eta\Big]^{\frac{1}{2}}\Big[\int_0^t\int_\Omega\widetilde{u}^2dAd\eta\\&+\delta_1\int_0^t\int_\Omega|\nabla\widetilde{u}|^2dAd\eta\Big]^{\frac{1}{2}}\\&\leq b_1(t)\int_0^t\int_\Omega\widetilde{u}^2dAd\eta+b_2(t)\delta_1\int_0^t\int_\Omega|\nabla\widetilde{u}|^2dAd\eta\end{aligned} \tag{70}$$

for computable $b_1(t),b_2(t)$ and positive arbitrary constant δ_1.

Applying the Cauchy–Schwarz inequality again we have

$$\begin{aligned}
&\mu \int_0^t \int_\Omega \Big(\int_z^0 \frac{\partial}{\partial x} \widetilde{T}(x,\zeta,\eta) d\zeta \Big) \widetilde{u} dA d\eta \\
&\qquad = -\mu \int_0^t \int_\Omega \Big(\int_z^0 \widetilde{T}(x,\zeta,\eta) d\zeta \Big) \frac{\partial \widetilde{u}}{\partial x} dA d\eta \\
&\qquad \leq \frac{h^2\mu^2}{\mu_1} \int_0^t \int_\Omega \widetilde{T}^2 dA d\eta + \frac{\mu_1}{4} \int_0^t \int_\Omega \Big(\frac{\partial \widetilde{u}}{\partial x} \Big)^2 dA d\eta.
\end{aligned} \tag{71}$$

Inserting (68)—(71) into (67) and choosing $\delta_1 = \frac{\mu_1}{4b_2(t)}$ we have

$$\begin{aligned}
\frac{1}{2} \int_\Omega \widetilde{u}^2 dA + \frac{1}{2}\mu_1 \int_0^t \int_\Omega |\nabla \widetilde{u}|^2 dA d\eta &\leq f \int_0^t \int_\Omega \widetilde{u}\widetilde{v} dA d\eta \\
&\quad + \frac{h^2\mu^2}{\mu_1} \int_0^t \int_\Omega \widetilde{T}^2 dA d\eta + b_1(t) \int_0^t \int_\Omega \widetilde{u}^2 dA d\eta.
\end{aligned} \tag{72}$$

Now, taking the inner product of Equation $(63)_2$ with $\widetilde{v}$, we have

$$\begin{aligned}
\frac{1}{2} \int_\Omega \widetilde{v}^2 dA + \mu_2 \int_0^t \int_\Omega |\nabla \widetilde{v}|^2 dA d\eta &= -f \int_0^t \int_\Omega \widetilde{u}\widetilde{v} dA d\eta \\
&\quad - \int_0^t \int_\Omega \Big[\widetilde{u} \frac{\partial v}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x} \widetilde{u}(x,\zeta,\eta) d\zeta \Big) \frac{\partial v}{\partial z} \Big] \widetilde{u} dA d\eta \\
&\quad - \int_0^t \int_\Omega \Big[u^* \frac{\partial \widetilde{v}}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x} u^*(x,\zeta,\eta) d\zeta \Big) \frac{\partial \widetilde{v}}{\partial z} \Big] \widetilde{v} dA d\eta.
\end{aligned} \tag{73}$$

Computing as previous we arrive at

$$\frac{1}{2} \int_\Omega \widetilde{v}^2 dA + \frac{1}{2}\mu_2 \int_0^t \int_\Omega |\nabla \widetilde{v}|^2 dA d\eta \leq -f \int_0^t \int_\Omega \widetilde{u}\widetilde{v} dA d\eta + b_3(t) \int_0^t \int_\Omega \widetilde{v}^2 dA d\eta \tag{74}$$

for computable positive function $b_3(t)$. A combination of (74) and (78) leads to

$$\begin{aligned}
&\int_\Omega \widetilde{u}^2 dA + \int_\Omega \widetilde{v}^2 dA + \mu_1 \int_0^t \int_\Omega |\nabla \widetilde{u}|^2 dA d\eta + \mu_2 \int_0^t \int_\Omega |\nabla \widetilde{v}|^2 dA d\eta \\
&\quad \leq \frac{2h^2\mu^2}{\mu_1} \int_0^t \int_\Omega \widetilde{T}^2 dA d\eta + 2b_1(t) \int_0^t \int_\Omega \widetilde{u}^2 dA d\eta + 2b_3(t) \int_0^t \int_\Omega \widetilde{v}^2 dA d\eta.
\end{aligned} \tag{75}$$

We take the inner product of Equation $(63)_3$ with $\widetilde{T}$, we have

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \widetilde{T}^2 dA + \mu_3 \int_0^t \int_\Omega |\nabla \widetilde{T}|^2 dA d\eta \\
&\quad = - \int_0^t \int_\Omega \Big[\widetilde{u} \frac{\partial T}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x} \widetilde{u}(x,\zeta,\eta) d\zeta \Big) \frac{\partial T}{\partial z} \Big] \widetilde{T} dA d\eta \\
&\quad - \int_0^t \int_\Omega \Big[u^* \frac{\partial \widetilde{T}}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x} u^*(x,\zeta,\eta) d\zeta \Big) \frac{\partial \widetilde{T}}{\partial z} \Big] \widetilde{T} dA d\eta \\
&\quad + \int_0^t \int_\Omega \widetilde{Q}\widetilde{T} dA d\eta.
\end{aligned} \tag{76}$$

On integrating by parts we have

$$- \int_0^t \int_\Omega \Big[u^* \frac{\partial \widetilde{T}}{\partial x} - \Big(\int_{-h}^z \frac{\partial}{\partial x} u^*(x,\zeta,\eta) d\zeta \Big) \frac{\partial \widetilde{T}}{\partial z} \Big] \widetilde{T} dA d\eta = 0. \tag{77}$$

Integrating by parts and using the Hölder inequality, (44), Lemma 3.2, the AG mean inequality we get

$$
\begin{aligned}
&-\int_0^t\int_\Omega\Big[\widetilde{u}\frac{\partial T}{\partial x}-\Big(\int_{-h}^{z}\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)\frac{\partial T}{\partial z}\Big]\widetilde{T}dAd\eta\\
&=\int_0^t\int_\Omega\widetilde{u}T\frac{\partial\widetilde{T}}{\partial x}dAd\eta-\int_0^t\int_\Omega\Big(\int_{-h}^{z}\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)T\frac{\partial\widetilde{T}}{\partial z}dAd\eta\\
&\le T_m\Big(\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{T}}{\partial x}\Big)^2dAd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_\Omega\widetilde{u}^2dAd\eta\Big)^{\frac{1}{2}}\\
&+T_m\Big(\int_0^t\int_\Omega\Big(\int_{-h}^{z}\frac{\partial}{\partial x}\widetilde{u}(x,\zeta,\eta)d\zeta\Big)^2dAd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{T}}{\partial z}\Big)^2dAd\eta\Big)^{\frac{1}{2}}\\
&\le\frac{1}{2}T_m\delta_2\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{T}}{\partial x}\Big)^2dAd\eta+\frac{1}{2\delta_2}T_m\int_0^t\int_\Omega\widetilde{u}^2dAd\eta\\
&+\frac{h^2}{2\pi^2\delta_2}T_m\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{u}}{\partial x}\Big)^2dAd\eta+\frac{1}{2}\delta_2T_m\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{T}}{\partial z}\Big)^2dAd\eta,
\end{aligned}
\tag{78}
$$

where δ_2 is a positive constant.

By the Hölder inequality and the AG mean inequality it follows that

$$
\int_0^t\int_\Omega\widetilde{Q}\widetilde{T}dAd\eta\le\frac{1}{2}\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta+\frac{1}{2}\int_0^t\int_\Omega\widetilde{T}^2dAd\eta. \tag{79}
$$

Inserting (77)–(79) into (76) and choosing $\delta_2=\frac{\mu_3}{2T_m}$ we get

$$
\begin{aligned}
\int_\Omega\widetilde{T}^2dA+\mu_3\int_0^t\int_\Omega|\nabla\widetilde{T}|^2dAd\eta\le{}&\frac{1}{\delta_2}T_m\int_0^t\int_\Omega\widetilde{u}^2dAd\eta+\frac{h^2}{\pi^2\delta_2}T_m\int_0^t\int_\Omega\Big(\frac{\partial\widetilde{u}}{\partial x}\Big)^2dAd\eta\\
&+\int_0^t\int_\Omega\widetilde{T}^2dAd\eta+\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta.
\end{aligned}
\tag{80}
$$

Then, using (75) and (80), we find that for a positive constant $\theta=\frac{\pi^2\delta_2\mu_1}{2h^2T_m}$

$$
\begin{aligned}
\int_\Omega\Big(\widetilde{u}^2+\widetilde{v}^2dA+\theta\widetilde{T}^2\Big)dA+\int_0^t\int_\Omega\Big(\frac{1}{2}\mu_1|\nabla\widetilde{u}|^2+\mu_2|\nabla\widetilde{v}|^2+\theta\mu_3|\nabla\widetilde{T}|^2\Big)dAd\eta&\\
\le\gamma_1(t)\int_0^t\int_\Omega\Big(\widetilde{u}^2+\widetilde{v}^2+\theta\widetilde{T}^2\Big)dAd\eta&\\
+\theta\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta,&
\end{aligned}
\tag{81}
$$

where

$$
\gamma_1(t)=\max\{1+\frac{2h^2}{\mu_1\theta},\ 2b_1(t)+\frac{T_m\theta}{\delta_2},\ 2b_3(t)\}. \tag{82}
$$

Therefore

$$
\frac{d}{dt}\Big\{\int_0^t\int_\Omega(\widetilde{u}^2+\widetilde{v}^2+\theta\widetilde{T}^2)dAd\eta\cdot e^{-\int_0^t\gamma_1(s)ds}\Big\}\le\theta\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta\cdot e^{-\int_0^t\gamma_1(s)ds}. \tag{83}
$$

An integration of (83) yields that

$$
\int_0^t\int_\Omega(\widetilde{u}^2+\widetilde{v}^2+\theta\widetilde{T}^2)dAd\eta\le\theta\int_0^t\int_0^\tau\int_\Omega e^{\int_\tau^t\gamma_1(s)ds}\widetilde{Q}^2dAd\eta d\tau. \tag{84}
$$

Then returning to (81), we obtain

$$\begin{aligned}\int_\Omega(\widetilde{u}^2+\widetilde{v}^2+\theta\widetilde{T}^2)dA+\int_0^t\int_\Omega\Big(\frac{1}{2}\mu_1|\nabla\widetilde{u}|^2+\mu_2|\nabla\widetilde{v}|^2+\theta\mu_3|\nabla\widetilde{T}|^2\Big)dAd\eta\\ \leq\gamma_1(t)\theta\int_0^t\int_0^\tau\int_\Omega e^{\int_\tau^t\gamma_1(s)ds}\widetilde{Q}^2dAd\eta d\tau+\theta\int_0^t\int_\Omega\widetilde{Q}^2dAd\eta.\end{aligned} \tag{85}$$

6. Conclusions

In this paper, we obtain the continuous dependence of the two-dimensional large-scale primitive equations in oceanic dynamics, where the depth of the ocean is assumed to be a positive constant. When the depth of the ocean is positive but not always a constant, Huang and Guo [32] have obtained the existence and uniqueness of a global strong solution for the problem. The study of the continuous dependence of the primitive equations in this case may be more interesting.

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