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Aggregation of Weak Fuzzy Norms

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Abstract: Aggregation is a mathematical process consisting in the fusion of a set of values into a unique one and representing them in some sense. Aggregation functions have demonstrated to be very important in many problems related to the fusion of information. This has resulted in the extended use of these functions not only to combine a family of numbers but also a family of certain mathematical structures such as metrics or norms, in the classical context, or indistinguishability operators or fuzzy metrics in the fuzzy context. In this paper, we study and characterize the functions through which we can obtain a single weak fuzzy (quasi-)norm from an arbitrary family of weak fuzzy (quasi-)norms in two different senses: when each weak fuzzy (quasi-)norm is defined on a possibly different vector space or when all of them are defined on the same vector space. We will show that, contrary to the crisp case, weak fuzzy (quasi-)norm aggregation functions are equivalent to fuzzy (quasi-)metric aggregation functions.

Keywords: weak fuzzy quasi-norms; aggregation function; asymmetric $*$-triangular triplet

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1. Introduction

In mathematics, an aggregation procedure amounts to a method for merging a family of structures of the same type into the only structure of this type. For example, the union or the intersection of subsets of a nonempty set X gave rise to another subset of X by aggregating the family of sets. On the other hand, given a finite family $\{d_i : i = 1, \ldots, n\}$ of metrics on X, then $\max\{d_1, \ldots, d_n\}$ is also a metric on X that is obtained by merging the original family of metrics. This metric can be constructed by means of the composition of the following functions:

- $\boldsymbol{d} : X \times X \to [0, +\infty)^n$ given by $\boldsymbol{d}(x, y) = (d_1(x, y), \ldots, d_n(x, y))$;
- $f : [0, +\infty)^n \to [0, +\infty)$ given by $f(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$.

In the literature, we can find other schemes of merging mathematical structures. If $\{(X_n, d_n) : n \in \mathbb{N}\}$ is a countable collection of metric spaces then $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} \frac{\min\{d_n(x_n, y_n), 1\}}{2^n}$ is a metric on $\prod_{n\in\mathbb{N}} X_n$, which is compatible with product topology. In this case, the new metric on $\prod_{n\in\mathbb{N}} X_n$ can be obtained with the composition of the following functions:

- $\widetilde{\boldsymbol{d}} : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \to [0, +\infty)^{\mathbb{N}}$ given by $\widetilde{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}) = (d_n(x_n, y_n))_{n\in\mathbb{N}}$;
- $f : [0, +\infty)^{\mathbb{N}} \to [0, +\infty)$ defined as $f(\boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{\min\{x_n, 1\}}{2^n}$.

In a similar manner, the sup norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ can be viewed as the aggregation of the absolute value norm on $\mathbb{R}$. This means that this norm is the composition of the following functions:

- $\widetilde{\boldsymbol{abs}} : \mathbb{R}^n \to [0, +\infty)^n$ given by $\widetilde{\boldsymbol{abs}}(x_1, \ldots, x_n) = (|x_1|, \ldots, |x_n|)$;
- $f : [0, +\infty)^n \to [0, +\infty)$ given by given by $f(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$.

In all the above cases, a function f is involved in the aggregation process, so it is natural to study which functions allow making these kinds of aggregations. This research has already been carried out for some mathematical structures. Concretely, Borsík and Doboš [1,2] have analyzed when, given a function $f : [0,+\infty)^I \to [0,+\infty)$ and an arbitrary family $\{(X_i, d_i) : i \in I\}$ of metric spaces, the function $f \circ \widetilde{\boldsymbol{d}} : (\prod_{i\in I} X_i) \times (\prod_{i\in I} X_i) \to [0,+\infty)$ given by $f \circ \widetilde{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y}) = f((d_i(\boldsymbol{x}_i,\boldsymbol{y}_i))_{i\in I})$ is a metric on the cartesian product $\prod_{i\in I} X_i$. The corresponding study for quasi-metric spaces was made by Mayor and Valero [3]. Related results appear in the papers [4–6], where the authors characterize functions $f : [0,+\infty)^k \to [0,+\infty)$ that allow combining a finite collection of pseudometrics (with respect to metrics and quasi-metrics) $\{d_i : i = 1,\ldots,k\}$ defined over the same set X into a single one pseudometric (with respect to metric and quasi-metric) on X given by $f \circ \boldsymbol{d}(x,y) = f(d_1(x,y),\ldots,d_k(x,y))$ for all $x,y \in X$.

In addition to functions that merge metrics, some researchers have characterized functions that aggregate other mathematical structures, such as norms. Thus, Herburt and Moszyńska [7] studied the functions $f : [0,+\infty)^2 \to [0,+\infty)$ which produces the function $\|\cdot\|_f : V_1 \times V_2 \to [0,+\infty)$ given by $\|(v_1,v_2)\|_f = f(\|v_1\|_1,\|v_2\|_2)$ be a norm on $V_1 \times V_2$, where $(V_1,\|\cdot\|_1),(V_2,\|\cdot\|_2)$ are two normed vector spaces. A similar study for asymmetric norms was developed by Martín, Mayor and Valero [8]. Recently, Pedraza and Rodríguez-López [9] have addressed the problem of the aggregation of norms on the same set.

Until now, we have only mentioned crisp mathematical structures. Nevertheless, several authors have considered the aggregation of fuzzy structures. Saminger, Mesiar and Bodenhofer [10] characterized the aggregation functions that preserve $*$-transitive fuzzy binary relations, where $*$ is a t-norm. Later on, a related problem about the preservation of $*$-transitivity of fuzzy binary relations was studied by Drewniak and Dudziak [11] (see also [12,13]). Moreover, Mayor and Recasens [14] obtained a characterization of the functions through which we can fuse indistinguishability operators, which are a special kind of $*$-transitive fuzzy binary relations (see also [15,16]).

Recently, Valero, Pedraza and Rodríguez-López [17] have studied the functions that permit the generation of a unique fuzzy (quasi-)metric from a collection of fuzzy (quasi-)metrics. They proved that, compared to the classical case, the functions aggregating fuzzy metrics are exactly the same than compared to the functions that aggregate fuzzy quasi-metrics. They also proved some results about the aggregation of other fuzzy structures such as fuzzy preorders and indistinguishability operators.

In this article, we continue the study of functions that aggregate a particular fuzzy structure: weak fuzzy (quasi-)norms [18]. This fuzzy structure is a generalization of the concept of fuzzy norm considered by Goleţ [19], and it is useful when studying duality in the fuzzy context [18]. Here, we characterize the functions that can afford to obtain a weak fuzzy (quasi-)norm starting from an arbitrary family of fuzzy quasi-norms. We consider two types of aggregation: on sets and on products (see Definition 4). We are able to characterize these functions with two methods: on the one hand, using the properties of $*$-supmultiplicativity (see Definition 7) and isotonicity; on the other hand, using the property of conservation of asymmetric $*$-triangular triplets (see Definition 5). These results can be considered, in some sense, similar to those obtained in the crisp case. Surprisingly, and in contrast with the crisp case, functions that aggregate weak fuzzy (quasi-)norm are the same as the functions that aggregate fuzzy (quasi-)metrics (see Corollaries 1 and 2).

2. Aggregation of Metrics and Norms

In this section, we compile some results about the aggregation of metrics and norms that constitute a necessary antecedent of our study. We first establish some notations.

We will denote by I an arbitrary index set. The elements of the Cartesian product $[0,+\infty)^I$ will be written down in bold letters $\boldsymbol{a}$. Moreover, and for the sake of simplicity, given $\boldsymbol{a} \in [0,+\infty)^I$, its ith coordinate $\boldsymbol{a}(i)$ will be denoted by $\boldsymbol{a}_i$ for any $i \in I$.

We notice that we can endow the set $[0,+\infty)^I$ with a partial order $\preceq$ defined as $\boldsymbol{a} \preceq \boldsymbol{b}$ if $\boldsymbol{a}_i \leq \boldsymbol{b}_i$ for all $i \in I$. Furthermore, $\boldsymbol{0}$ represents the element of $[0,+\infty)^I$ such that $\boldsymbol{0}_i = 0$ for all $i \in I$.

As we have sketched, in the Introduction, that classical constructions of metrics in a Cartesian product are obtained by composing an appropriate function with a Cartesian product of metrics. The study of these functions, called *metric preserving functions*, has been mainly developed by Borsík and Doboš [1,2]. The corresponding study for quasi-metrics was made by Mayor and Valero [3], who characterized the so-called *quasi-metric aggregation functions*. In both cases, the underlying idea is to construct a (quasi-)metric in the Cartesian product of a family of (quasi-)metric spaces.

Another related problem was addressed by Pradera and Trillas [6], who studied how to merge a family of pseudometrics defined over the same set into a single one. This question for metrics has been also considered recently by Mayor and Valero [4]. Therefore, we have two different but related problems, which gave rise to two different families of functions that we next define using the terminology of [9,17,20].

Definition 1 ([1,3,4])**.** *A function $f : [0,+\infty)^I \to [0,+\infty)$ is called the following:*

- *A* (quasi-)metric aggregation function on products *if given an arbitrary family of (quasi-)metric spaces $\{(X_i, d_i) : i \in I\}$ then $f \circ \widetilde{\boldsymbol{d}}$ is a (quasi-)metric on $\prod_{i\in I} X_i$ where $\widetilde{\boldsymbol{d}} : (\prod_{i\in I} X_i) \times (\prod_{i\in I} X_i) \to [0,+\infty)^I$ is defined as*
$$(\widetilde{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}))_i = d_i(x_i, y_i)$$
for all $i \in I$, $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$;
- *A* (quasi-)metric aggregation function on sets *if given a family of (quasi-)metrics $\{d_i : i \in I\}$ on an arbitrary nonempty set X, then $f \circ \boldsymbol{d}$ is a (quasi-)metric on X where $\boldsymbol{d} : X \times X \to [0,+\infty)^I$ is defined as follows:*
$$(\boldsymbol{d}(x, y))_i = d_i(x, y)$$
for all $i \in I$, $x, y \in X$.

Recall that a triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,+\infty)^I)^3$ is a *triangular triplet* if $\boldsymbol{a}_i \le \boldsymbol{b}_i + \boldsymbol{c}_i$, $\boldsymbol{b}_i \le \boldsymbol{a}_i + \boldsymbol{c}_i$ and $\boldsymbol{c}_i \le \boldsymbol{a}_i + \boldsymbol{b}_i$ for all $i \in I$ (see [2]). As we next observe, this concept was introduced in [1] for characterizing metric aggregation functions on products. Moreover, if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ solely verifies that $\boldsymbol{a}_i \le \boldsymbol{b}_i + \boldsymbol{c}_i$ for all $i \in I$, then it is called an *asymmetric triangular triplet* ([3]). A function $f : [0,+\infty)^I \to [0,+\infty)$ is said to preserve (asymmetric) triangular triplets if given an asymmetric triangular triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,+\infty)^I)^3$, then $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}))$ is an asymmetric triangular triplet.

Borsík and Doboš [1] characterized metric aggregation functions on products in the following manner.

Theorem 1 ([1])**.** *A function $f : [0,+\infty)^I \to [0,+\infty)$ is a metric aggregation function on products if and only if $f^{-1}(0) = \{\boldsymbol{0}\}$ and f preserves triangular triplets.*

On its part, Mayor and Valero [3] proved the next result which characterizes the functions merging quasi-metrics on products.

Theorem 2 ([3])**.** *Consider a function $f : [0,+\infty)^I \to [0,+\infty)$. Then, the following statements are equivalent:*

(1) *f is a quasi-metric aggregation function on products;*
(2) *$f^{-1}(0) = \{\boldsymbol{0}\}$ and f preserves asymmetric triangular triplets;*
(3) *$f^{-1}(0) = \{\boldsymbol{0}\}$, f is subadditive and isotone.*

The two previous theorems bring to light that every quasi-metric aggregation function on products is a metric aggregation function on products. However, the reciprocal implication does not hold in general [3] (Example 8).

On the other hand, if you consider (asymmetric) normed vector spaces rather than (quasi-)metric spaces, the concepts of (asymmetric) norm aggregation function on products and (asymmetric) norm aggregation function on sets can be considered in a natural manner. The former has been characterized in [7,8], while the latter has been studied in [9]. As the main objective of this paper is to study this problem in the fuzzy context, we recall the known results for crisp (asymmetric) norms. In the following, a function $f : [0, +\infty)^I \to [0, +\infty)$ is said to be positive homogeneous if $f(\lambda \cdot \boldsymbol{x}) = \lambda f(\boldsymbol{x})$ for all $\lambda \geq 0, \boldsymbol{x} \in [0, +\infty)^I$.

We first recall the following result showing that the family of asymmetric norm aggregation functions on products is equal to the family of norm aggregation functions on products.

Theorem 3 ([7–9])**.** *Given a function $f : [0, +\infty)^I \to [0, +\infty)$, the following statements are equivalent:*

(1) f is an asymmetric norm aggregation function on products;
(2) f is a norm aggregation function on products;
(3) $((\mathbb{R}^2)^I, \|\cdot\|_f)$ is a normed space where $\|\boldsymbol{x}\|_f = f((\|\boldsymbol{x}_i\|)_{i\in I})$ and $\|\cdot\|$ is the Euclidean norm for all $\boldsymbol{x} \in (\mathbb{R}^2)^I$;
(4) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous, and it preserves asymmetric triangular triplets;
(5) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous, and it preserves triangular triplets;
(6) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous, subadditive and isotone.

From Theorems 1, 2 and 3, we have it that every (asymmetric) norm aggregation function on products is also a (quasi-)metric aggregation function on products. However, the reciprocal implication does not hold in general. We can provide an easy example.

Example 1. *Let $f : [0, +\infty) \to [0, +\infty)$ be given by $f(x) = \min\{x, 1\}$. It is straightforward to check that $f^{-1}(0) = 0$, and f is subadditive and isotone. Therefore, f is a (quasi-)metric aggregation function on sets. However, f is not positive homogeneous since, for example, $f\left(4 \cdot \frac{1}{2}\right) = 1 \neq 2 = 4f\left(\frac{1}{2}\right)$. Hence, f is not an asymmetric norm aggregation function on products.*

In the case of the aggregation on sets, norm aggregation functions and asymmetric norm aggregation functions are different classes of functions.

Theorem 4 ([9])**.** *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function and let g be the restriction of f to $(0, +\infty)^I \cup \{\mathbf{0}\}$. The following conditions are equivalent:*

(1) f is a norm aggregation function on sets;
(2) For every family of norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, f \circ \boldsymbol{n})$ is a normed space;
(3) $g^{-1}(0) = \mathbf{0}$, g is positive homogeneous, and it preserves asymmetric triangular triplets;
(4) $g^{-1}(0) = \mathbf{0}$, g is positive homogeneous, and it preserves triangular triplets;
(5) $g^{-1}(0) = \mathbf{0}$, g is positive homogeneous, subadditive and isotone.

Theorem 5 ([9])**.** *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function. The following statements are equivalent:*

(1) f is an asymmetric norm aggregation function on sets;
(2) For every family of asymmetric norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, f \circ \boldsymbol{n})$ is an asymmetric normed space;
(3) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous, and it preserves asymmetric triangular triplets;
(4) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous, and it preserves triangular triplets;
(5) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous, subadditive and isotone.

3. Weak Fuzzy (Quasi-)Norms

As the goal of the paper is to study in the fuzzy context those functions that aggregate fuzzy norms in the spirit of the results of the previous section, in the following, we present the basic definitions about fuzzy norms and some examples. First, we remind the reader of the well-known notion of a triangular norm.

Definition 2 ([21]). *We say that a binary operation* $*\colon [0,1] \times [0,1] \to [0,1]$ *is a* triangular norm *or a* t-norm *if, for every* $a, b, c, d \in [0,1]$, *it satisfies the following properties:*

- $a * (b * c) = (a * b) * c$;
- $a * b = b * a$;
- $a * 1 = a$;
- $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$.

A t-norm $*$ *is said to be* continuous *if* $*$ *is a continuous function.*

Example 2 ([21]). *Some of the most renowned examples of triangular norms are the following:*

- $x \wedge y := \min\{x, y\}$; *(minimum t-norm)*
- $x *_P y := x \cdot y$; *(product t-norm)*
- $x *_D y := \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$ *(drastic t-norm)*

The origins of fuzzy normed spaces can be found in the concept of probabilistic normed space and Šerstnev space [22,23]. This notion was first adapted to the fuzzy context by Katsaras [24]. Later on, Cheng and Mordeson [25] introduced a new definition of a fuzzy norm, which induces a fuzzy metric in the sense of Kramosil and Michalek [26]. Bag and Samanta considered a more general concept of fuzzy norm [27] by removing the left-continuity condition (see next definition). In this paper, we use the concept of fuzzy norm as considered by Goleţ [19] as well as the terminology of [18,28] relative to (weak) fuzzy (quasi-)norms.

Definition 3 ([18,19,28]). *A* weak fuzzy quasi-norm *on a real vector space* V *is a pair* $(N, *)$ *such that* $*$ *is a continuous t-norm and* N *is a fuzzy set on* $V \times [0, +\infty)$ *such that, for any vectors* $x, y \in V$ *and for any parameters* $t, s > 0$, *it satisfies the following conditions:*

(FQN1) $N(x, 0) = 0$;
(FQN2) $N(x, t) = N(-x, t) = 1$ *for all* $t > 0$ *if and only if* $x = 0_V$;
(FQN3) $N(\lambda x, t) = N(x, \frac{t}{\lambda})$ *for all* $\lambda > 0$;
(FQN4) $N(x, t) * N(y, s) \leq N(x + y, t + s)$;
(FQN5) $N(x, \cdot) : [0, \infty) \to [0, 1]$ *is left-continuous.*

If N also satisfies the following:

(FQN6) $\lim_{t \to +\infty} N(x, t) = 1$

then $(N, *)$ *is called a* fuzzy quasi-norm.

A (weak) fuzzy norm on a real vector space V *is a (weak) fuzzy quasi-norm* $(N, *)$ *on* V *such that the following is the case.*

(FQN3') $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right)$ *for all* $\lambda \in \mathbb{R} \backslash \{0\}$.

A (weak) fuzzy (quasi-)normed *space is a triple* $(V, N, *)$ *such that* V *is a real vector space and* $(N, *)$ *is a (weak) fuzzy (quasi-)norm on* V.

Remark 1. *Notice that the definition of fuzzy norm that we have considered is that of Goleţ. It differs slightly from that defined in [25] by Cheng and Mordeson since they considered a real or complex vector space* V. *They allow that the parameter* t *also takes negative values by considering* *that $N(x,t) = 0$ for every $t < 0$, and they only create the definition for the minimum t-norm. The above definition is equal to that given in [18,28].*

There are also other notions of a fuzzy norm that modify the previous conditions as the elimination of (FQN5) [27].

Remark 2. *The definition of a (weak) fuzzy (quasi-)normed space $(V, N, *)$ given above requires the continuity of the triangular norm $*$. This property is used for ensuring that $(N, *)$ endows the vector space V with a classical topology τ_N (see [18,28]) having as its base, $\{B_N(x,r,t) : x \in V, r \in]0,1[, t > 0\}$, where*

$$B_N(x,r,t) = \{y \in V : N(y - x, t) > 1 - r\}.$$

Despite this, since we do not need $(N, *)$ generating a topology, as we can suppose that $*$ is an arbitrary t-norm rather than a continuous one.

Remark 3. *If $(V, N, *)$ is a (weak) fuzzy (quasi-)normed space then $N(x, \cdot) : [0, +\infty) \to [0, +\infty)$ is an isotone function for every $x \in V$. In fact, given $x \in V$, if $t < s$ then, by (FQN2) and (FQN4), we have that $N(v,t) = N(v,t) * 1 = N(v,t) * N(0_V, s-t) \leq N(v + 0_V, t + s - t) = N(v,s)$. We will use this fact throughout the paper.*

Example 3 (cf. [28,29] [Example 1])**.** *Let (V, q) be a quasi-normed space. Let $k, m, n \in \mathbb{R}^+$ be fixed. Define $N : V \times [0, +\infty) \to [0,1]$ as the following.*

$$N(x,t) = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{kt^n}{kt^n + mq(x)} & \text{if } t > 0 \end{cases}$$

Then, $(V, N, *)$ is a fuzzy quasi-normed space for every continuous t-norm $*$. If $k = n = m = 1$ then $(N, *)$ is called the standard fuzzy quasi-norm, and it will be denoted by $(N_q, *)$.

Example 4. *Let $a \in [0,1[$ and consider $N_a : \mathbb{R} \times [0, +\infty) \to [0,1]$ defined as the following.*

$$N_a(x,t) = \begin{cases} 1 & \text{if } t > |x| \\ a & \text{if } 0 < t \leq |x| \\ 0 & \text{if } t = 0 \end{cases}.$$

We can easily prove that $(\mathbb{R}, N_a, *)$ is a fuzzy normed space for every continuous t-norm $*$.

The reason for having chosen the definition of weak fuzzy (quasi-)norm as considered in 3 instead of other definition of fuzzy norm is that every weak fuzzy (quasi-)norm $(N, *)$ on a vector space V induces a fuzzy (quasi-)metric $(M_N, *)$ on V (in the sense of the paper [26]) given as $M_N(x,y,t) = N(y - x, t)$ for all $x, y \in V$ and all $t \geq 0$ (see [18]). Since (quasi-)metric aggregation functions have been already characterized in [17] and we plan to characterize here the functions that aggregate fuzzy (quasi-)norms, it is natural to select a concept of fuzzy (quasi-)norm that allows constructing a fuzzy (quasi-)metric.

4. Aggregation of Weak Fuzzy (Quasi-)Norms

In Section 2 we have summarized the known results about the aggregation of (quasi-)metrics and asymmetric norms on products and on sets. On the other hand, in [17], the functions that aggregate fuzzy (quasi-)metrics on products and on sets were completely characterized. Nevertheless, to the best of our knowledge, the problem for fuzzy (quasi-)norms has not been already solved. The goal of this section is fill in this gap. We first set the definitions of the functions that we intend to characterize.

Definition 4. *A function $F : [0,1]^I \to [0,1]$ is said to be the following:*

- *A* weak fuzzy (quasi-)norm aggregation function on products *if given a t-norm $*$ and a collection of weak fuzzy (quasi-)normed spaces* $\{(V_i, N_i, *) : i \in I\}$ *then* $(F \circ \widetilde{\mathbf{N}}, *)$ *is a weak fuzzy (quasi-)norm on* $\prod_{i\in I} V_i$ *where* $\widetilde{\mathbf{N}} : \prod_{i\in I} V_i \times [0, +\infty) \to [0,1]^I$ *is given by the following:*

$$(\widetilde{\mathbf{N}}(\mathbf{x}, t))_i = N_i(x_i, t)$$

  for every $\mathbf{x} \in \prod_{i\in I} V_i$ *and* $t \geq 0$.
  If the previous condition is only verified for a t-norm $*$, then F is called an $*$-weak fuzzy (quasi-)norm aggregation function on products.

- *A* weak fuzzy (quasi-)norm aggregation function on sets *if given a t-norm $*$ and a collection of weak fuzzy (quasi-)norms* $\{(N_i, *) : i \in I\}$ *defined on a real vector space V then* $(F \circ \mathbf{N}, *)$ *is a weak fuzzy (quasi-)norm on V where* $\mathbf{N} : V \times [0, +\infty) \to [0,1]^I$ *is given by the following:*

$$(\mathbf{N}(x, t))_i = N_i(x, t)$$

  for every $x \in V$ and $t \geq 0$.
  If the previous condition is only verified for a t-norm $*$, then F is called a $*$-weak fuzzy (quasi-)norm aggregation function on sets.

Remark 4. *It can be easily proved that a weak fuzzy (quasi-)norm aggregation function on products* $F : [0,1]^I \to [0,1]$ *is also a weak fuzzy (quasi-)norm aggregation function on sets. Trivially, if the cardinality of I is one, then the two notions are equivalent. However, if the cardinality of I is greater than one, the two concepts are different in general as we next illustrate with an example.*

Example 5. *Consider I an index set and $j \in I$ is fixed. Denote by P_j the jth projection. It is obvious that P_j is a weak fuzzy (quasi-)norm aggregation function on sets since if* $\{(N_i, *) : i \in I\}$ *is a collection of weak fuzzy (quasi-)norms on a vector space V, then* $(P_j \circ \mathbf{N}, *) = (N_j, *)$.

Nevertheless, if $\{(V_i, N_i, *) : i \in I\}$ *is a collection of nontrivial weak fuzzy (quasi-)normed spaces, consider* $\mathbf{x} \in \prod_{i\in I} V_i$ *such that* $\mathbf{x}_j = 0_{V_j}$ *and* $\mathbf{x}_i \neq 0_{V_i}$ *whenever $i \neq j$. Then* $F \circ \widetilde{\mathbf{N}}(\mathbf{x}) = 0$, *but* $\mathbf{x} \neq 0_{\prod_{i\in I} V_i}$, *so* $(F \circ \widetilde{\mathbf{N}}, *)$ *is not a weak fuzzy quasi-norm on* $\prod_{i\in I} V_i$.

In the following, we will introduce several valuable concepts used in [17] for proving a characterization of fuzzy (quasi-)metric aggregation functions which will be also useful in our work. Actually, it will be shown that weak fuzzy (quasi-)norm aggregation functions are exactly the fuzzy (quasi-)metric aggregation functions. Notice that this is not true in the crisp context where norm aggregation functions are metric aggregation functions, but the reciprocal implication does not hold in general [2,3,8,9].

We begin by recalling the notion of (asymmetric) $*$-triangular triplet [14,17] in which we will use the operation $*^I$ on $[0,1]^I$ defined as $(\mathbf{a} *^I \mathbf{b})_i = \mathbf{a}_i * \mathbf{b}_i$ for every $i \in I$ and every $\mathbf{a}, \mathbf{b} \in [0,1]^I$.

Definition 5 ([14,17])**.** *Consider an index set I and a t-norm $*$. A triplet* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in ([0,1]^I)^3$ *is said to be the following:*

- $*$-triangular *if*

$$\mathbf{a} *^I \mathbf{b} \preceq \mathbf{c}, \quad \mathbf{a} *^I \mathbf{c} \preceq \mathbf{b} \quad \text{and} \quad \mathbf{b} *^I \mathbf{c} \preceq \mathbf{a},$$

  in other words,

$$a_i * b_i \leq c_i, \quad a_i * c_i \leq b_i \quad \text{and} \quad b_i * c_i \leq a_i, \quad \text{for every } i \in I.$$

- Asymmetric $*$-triangular *if* $\mathbf{a} *^I \mathbf{b} \preceq \mathbf{c}$.

If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is an asymmetric $*$-triangular triplet for every t-norm $*$, then* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is called a(n) (asymmetric)* triangular triplet.

Example 6. *Consider a weak fuzzy (quasi-)normed space* $(V, N, *)$. *Then* $(N(x,t), N(y,s), N(x+y, t+s))$ *is an asymmetric* $*$*-triangular triplet for every vector* $x, y, z \in V$ *and every* $t, s > 0$.

Definition 6 ([17])**.** *Consider an index set* I *and a t-norm* $*$. *A function* $F : [0,1]^I \to [0,1]$ *is said to* preserve $*$-triangular (asymmetric $*$-triangular) triplets *if for every* $*$*-triangular (asymmetric* $*$*-triangular) triplet* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,1]^I)^3$ *then* $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ *is a* $*$*-triangular (an asymmetric* $*$*-triangular) triplet.*

If F *preserves* $*$*-triangular (asymmetric* $*$*-triangular) triplets for every t-norm* $*$, *then* F *is said to* preserve triangular (asymmetric triangular) triplets.

We next provide a concept that is a particular case of the concept of domination as considered in [10]. Domination has been demonstrated to be useful in the study of the preservation by means of aggregation functions of some properties of certain fuzzy structures [10,17,30]. It has also been used for constructing other fuzzy structures such as m-polar $*$-orderings [31].

Definition 7 ([10,17])**.** *Given a triangular norm* $*$, *a function* $F : [0,1]^I \to [0,1]$ *is* $*$-supmultiplicative *if the following is the case.*

$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b}), \quad \text{for every } \boldsymbol{a}, \boldsymbol{b} \in [0,1]^I.$$

We will say that F *is* supmultiplicative *if* F *is* $*$*-supmultiplicative for every t-norm* $*$.

$*$-supmultiplicative functions and functions preserving (asymmetric) $*$-triangular triplets can appear to be unrelated concepts. However, both have been used for characterizing, in different senses, the functions preserving the property of $*$-transitivity of fuzzy binary relations [10,14]. Its relationship has been disclosed in [17] in the following way.

Proposition 1 ([17] (Proposition 3.30))**.** *Let* $F : [0,1]^I \to [0,1]$ *be a function and* $*$ *be a t-norm. Each of the following statements implies its successor:*

(1) F preserves asymmetric $*$*-triangular triplets;*
(2) F preserves $*$*-triangular triplets;*
(3) F is $*$*-supmultiplicative.*

If F *is isotone then all the above statements are equivalent.*

We still need to recall two concepts that will be needed in our characterization.

Definition 8 ([32,33])**.** *A function* $F : [0,1]^I \to [0,1]$ *is called* sequentially left-continuous *if* F *is sequentially continuous when* $[0,1]^I$ *is endowed with the product topology of the upper limit topology, and* $[0,1]$ *carries the usual topology. Recall that the upper limit topology on* $[0,1]$ *has as base* $\{(a,b] : a, b \in [0,1], a \leq b\}$.

Remark 5. *Notice that if* F *is isotone then, for proving that* F *is sequentially left-continuous, it is enough to consider nondecreasing sequences on* $[0,1]^I$. *In fact, let* $\{\boldsymbol{t}_n\}_{n\in\mathbb{N}}$ *be a sequence in* $[0,1]^I$ *converging to* $\boldsymbol{t}$ *with respect to the product topology of the upper limit topology. For all* $n \in \mathbb{N}$, *define* $\boldsymbol{s}_n \in [0,1]^I$ *as* $(\boldsymbol{s}_n)_i = \inf_{k \geq n} (\boldsymbol{t}_k)_i$ *for all* $i \in I$. *It is obvious that* $\{\boldsymbol{s}_n\}_{n\in\mathbb{N}}$ *is a nondecreasing sequence and* $\boldsymbol{s}_n \preceq \boldsymbol{t}_n$ *for all* $n \in \mathbb{N}$. *Moreover,* $\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n = \bigvee_{n\in\mathbb{N}} \boldsymbol{s}_n = \boldsymbol{t}$. *Suppose that* $\{F(\boldsymbol{s}_n)\}_{n\in\mathbb{N}}$ *converges to* $F(\boldsymbol{t}) = F(\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n)$, *that is,* $\bigvee_{n\in\mathbb{N}} F(\boldsymbol{s}_n) = F(\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n)$. *Then, the following is the case:*

$$F(\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n) = \bigvee_{n\in\mathbb{N}} F(\boldsymbol{s}_n) \leq \bigvee_{n\in\mathbb{N}} F(\boldsymbol{t}_n) \leq F(\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n)$$

since F *is isotone. Therefore,* $\bigvee_{n\in\mathbb{N}} F(\boldsymbol{t}_n) = F(\bigvee_{n\in\mathbb{N}} \boldsymbol{t}_n) = F(\boldsymbol{t})$.

Recall (see Section 2) that the characterizations of an asymmetric norm aggregation function or a (quasi-)metric aggregation function f require the imposition of some conditions on the set $f^{-1}(0)$. These conditions will be substituted by certain properties of the core in the fuzzy framework.

Definition 9 ([17])**.** *Given a function $F : [0,1]^I \to [0,1]$, its* core *is the set $F^{-1}(1)$. We say the following:*

- *F has a* trivial core *if $F^{-1}(1) = \{\mathbf{1}\}$;*
- *The* core of F is countably included in a unitary face *if given $\{\boldsymbol{a}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ there exists $i \in I$ such that $(\boldsymbol{a}_n)_i = 1$ for all $n \in \mathbb{N}$.*

The following result characterizes the weak fuzzy (quasi-)norm aggregation functions on products (compare with [17] (Theorem 4.15)).

Theorem 6. *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

(1) F is a $(*$-$)$weak fuzzy quasi-norm aggregation function on products;
(2) F is a $(*$-$)$weak fuzzy norm aggregation function on products;
(3) $F(\mathbf{0}) = 0$, F is isotone, $(*$-$)$supmultiplicative, sequentially left-continuous and F has trivial core;
(4) $F(\mathbf{0}) = 0$, F is sequentially left-continuous with trivial core, and F preserves asymmetric $(*$-$)$triangular triplets.

Proof. (1) $\Rightarrow$ (2) This implication can be easily observed. (2) $\Rightarrow$ (3) We begin showing that $F(\mathbf{0}) = 0$. Let $(V, N, *)$ be a weak fuzzy normed space and $x \in V$. Let $\{(V_i, N_i, *) : i \in I\}$ be the family of weak fuzzy normed spaces such that $(V_i, N_i, *) = (V, N, *)$ for all $i \in I$. By assumption, $(F \circ \widetilde{\mathbf{N}}, *)$ is a weak fuzzy norm on V^I so $0 = F \circ \widetilde{\mathbf{N}}(\boldsymbol{x}, 0) = F(\mathbf{0})$ where $\boldsymbol{x}_i = x$ for all $i \in I$.

Let us check that F has a trivial core. We first notice that if $t > 0$ then $F(\mathbf{1}) = F((N(0_V, t))_{i\in I}) = F \circ \widetilde{\mathbf{N}}(\mathbf{0}_{V^I}, t) = 1$ by (FQN2).

In order to obtain a contradiction, suppose that there exists $\boldsymbol{a} \in [0,1]^I$ verifying that $F(\boldsymbol{a}) = 1$ but $\boldsymbol{a} \neq \mathbf{1}$. Let $J = \{i \in I : \boldsymbol{a}_i \neq 1\}$, which is nonempty. Consider the family of weak fuzzy normed spaces $\{(\mathbb{R}, N_i, *) : i \in I\}$ where the following is the case.

- If $i \in J$, then $(N_i, *) = (N_{\boldsymbol{a}_i}, *)$ is the fuzzy norm of Example 4;
- If $i \notin J$, then $(N_i, *) = (N, *)$ is an arbitrary fixed weak fuzzy norm on $\mathbb{R}$.

By assumption, $(F \circ \widetilde{\mathbf{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}^I$. Let $\boldsymbol{x} \in \mathbb{R}^I$ such that $\boldsymbol{x}_i = 1$ if $i \in J$ and $\boldsymbol{x}_i = 0$ otherwise. Then, given $t > 0$, we have the following:

$$F \circ \widetilde{\mathbf{N}}(\boldsymbol{x}, t) = F((N_i(\boldsymbol{x}_i, t))_{i\in I}) = \begin{cases} F(\mathbf{1}) = 1 & \text{if } t > 1 \\ F(\boldsymbol{a}) = 1 & \text{if } 0 < t \leq 1. \end{cases}.$$

which contradicts (FQN2). Hence, F has trivial core.

We next show the isotonicity of F. Consider two elements $\boldsymbol{a}, \boldsymbol{b}$ belonging to $[0,1]^I$ verifying $\boldsymbol{a} \preceq \boldsymbol{b}$. Let us consider the real vector space $\mathbb{R}$ and, for every index $i \in I$, we define $N_i : \mathbb{R} \times [0, +\infty) \to [0,1]$ as follows.

$$N_i(x, t) = \begin{cases} 0 & \text{if } 0 \leq t \leq |x|, \\ \boldsymbol{a}_i & \text{if } |x| < t \leq 2|x|, \\ \boldsymbol{b}_i & \text{if } 2|x| < t \leq 3|x|, \\ 1 & \text{if } t > 3|x|. \end{cases}$$

It is simple to show that $(\mathbb{R}, N_i, *)$ is a weak fuzzy normed space for every $i \in I$. Furthermore, $N_i(1,2) = \boldsymbol{a}_i$ and $N_i(1,3) = \boldsymbol{b}_i$ for every $i \in I$. Since $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}$, then $F \circ \widetilde{\boldsymbol{N}}(\mathbf{1}, \cdot)$ is increasing. Consequently, we have the following.

$$\begin{aligned} F \circ \widetilde{\boldsymbol{N}}(\mathbf{1}, 2) &\leq F \circ \widetilde{\boldsymbol{N}}(\mathbf{1}, 3) \\ F((N_i(1,2))_{i\in I}) &\leq F((N_i(1,3))_{i\in I}) \\ F((\boldsymbol{a}_i)_{i\in I}) &\leq F((\boldsymbol{b}_i)_{i\in I}) \\ F(\boldsymbol{a}) &\leq F(\boldsymbol{b}) \end{aligned}$$

Thus, F is isotone.

Now, we show the $*$-supmultiplicativity of F. Let $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$. Define $L_1 = \{(x,y) \in \mathbb{R}^2 : x \neq 0, y = 0\}$, $L_2 = \{(x,y) \in \mathbb{R}^2 : x = 0, y \neq 0\}$ and $L_3 = \mathbb{R}^2 \backslash (L_1 \cup L_2 \cup \{(0,0)\})$. For each $i \in I$, define a function $N_i : \mathbb{R}^2 \times [0, +\infty) \to [0,1]$ as follows:

$$N_i(\boldsymbol{x}, t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \|\boldsymbol{x}\|, \\ \boldsymbol{a}_i & \text{if } \boldsymbol{x} \in L_1 \text{ and } t > \|\boldsymbol{x}\|, \\ \boldsymbol{b}_i & \text{if } \boldsymbol{x} \in L_2 \text{ and } t > \|\boldsymbol{x}\|, \\ \boldsymbol{a}_i * \boldsymbol{b}_i & \text{if } \boldsymbol{x} \in L_3 \text{ and } t > \|\boldsymbol{x}\|, \\ 1 & \text{if } \boldsymbol{x} = \mathbf{0} \text{ and } t > 0, \end{cases}$$

where $\|\cdot\|$ is the Euclidean norm. Then $(N_i, *)$ is a weak fuzzy norm on $\mathbb{R}^2$ for all $i \in I$. We only verify that $(N_i, *)$ satisfies (FQN4). Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and $t, s > 0$. If $\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$, the inequality is trivially true. If $\boldsymbol{x} = \mathbf{0}$ or $\boldsymbol{y} = \mathbf{0}$, we also obtain the inequality since $N_i(\boldsymbol{z}, \cdot)$ is isotone for every $\boldsymbol{z} \in \mathbb{R}^2$. Thus, let us suppose that $\boldsymbol{x} + \boldsymbol{y} \neq \mathbf{0}$, $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}$. Let $j \in \{1,2,3\}$ such that $\boldsymbol{x} + \boldsymbol{y} \in L_j$. If $\boldsymbol{x} \in L_j$ or $\boldsymbol{y} \in L_j$, then it is clear that $N_i(\boldsymbol{x}, t) * N_i(\boldsymbol{y}, s) \leq N_i(\boldsymbol{x} + \boldsymbol{y}, t + s)$. If $\boldsymbol{x} \notin L_j$ and $\boldsymbol{y} \notin L_j$, we distinguish the following three cases:

- If $j = 3$, then $\boldsymbol{x} \in L_1$ and $\boldsymbol{y} \in L_2$ or viceversa. If $t \leq \|\boldsymbol{x}\|$ or $s \leq \|\boldsymbol{y}\|$, then $N_i(\boldsymbol{x}, t) = 0$ or $N_i(\boldsymbol{y}, s) = 0$. so the inequality holds. Otherwise $t > \|\boldsymbol{x}\|$ and $s > \|\boldsymbol{y}\|$, which implies that $t + s > \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \geq \|\boldsymbol{x} + \boldsymbol{y}\|$. Hence, $N_i(\boldsymbol{x}, t) * N_i(\boldsymbol{y}, s) = \boldsymbol{a}_i * \boldsymbol{b}_i \leq N_i(\boldsymbol{x} + \boldsymbol{y}, t + s) = \boldsymbol{a}_i * \boldsymbol{b}_i$;
- If $j = 2$, then at least one of $\boldsymbol{x}, \boldsymbol{y}$ belongs to L_3. Without loss of generality, we can suppose that $\boldsymbol{x} \in L_3$. As above, if $t \leq \|\boldsymbol{x}\|$ or $s \leq \|\boldsymbol{y}\|$, then $N_i(\boldsymbol{x}, t) = 0$ or $N_i(\boldsymbol{y}, s) = 0$, so the inequality holds. Otherwise $t > \|\boldsymbol{x}\|$ and $s > \|\boldsymbol{y}\|$, which implies that $t + s > \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \geq \|\boldsymbol{x} + \boldsymbol{y}\|$. Hence, $N_i(\boldsymbol{x}, t) * N_i(\boldsymbol{y}, s) = \boldsymbol{a}_i * \boldsymbol{b}_i * N_i(\boldsymbol{y}, s) \leq \boldsymbol{b}_i = N_i(\boldsymbol{x} + \boldsymbol{y}, t + s)$.
- If $j = 1$, we can reason as in the previous case.

Since $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a weak fuzzy norm on $(\mathbb{R}^2)^I$, it verifies (FQN4). By defining $(\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{1}) \in (\mathbb{R}^2)^I$ such that $(\mathbf{1}, \mathbf{0})_i = (1,0)$, $(\mathbf{0}, \mathbf{1})_i = (0,1)$ and $(\mathbf{1}, \mathbf{1})_i = (1,1)$ for all $i \in I$, we have the following.

$$\begin{aligned} F \circ \widetilde{\boldsymbol{N}}((\mathbf{1}, \mathbf{0}), 2) * F \circ \widetilde{\boldsymbol{N}}((\mathbf{0}, \mathbf{1}), 2) &\leq F \circ \widetilde{\boldsymbol{N}}((\mathbf{1}, \mathbf{1}), 4) \\ F((\boldsymbol{a}_i)_{i\in I}) * F((\boldsymbol{b}_i)_{i\in I}) &\leq F((\boldsymbol{a}_i * \boldsymbol{b}_i)_{i\in I}) \end{aligned}$$

Thus, F is $*$-supmultiplicative.

Finally, we demonstrate that the function F is sequentially left-continuous. By Remark 5, let $\{\boldsymbol{s}_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence in $[0,1]^I$ having as limit $\boldsymbol{s} \in [0,1]^I$ in the product topology of the lower limit topology.

For each $i \in I$, define $N_i : \mathbb{R} \times [0, +\infty) \to [0, 1]$ as the following.

$$N_i(x,t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{|x|}{2}, \\ (\boldsymbol{s}_n)_i & \text{if } |x|\left(1 - \frac{1}{n+1}\right) < t \le |x|\left(1 - \frac{1}{n+2}\right),\ n \in \mathbb{N} \\ \boldsymbol{s}_i & \text{if } 0 < |x| \le t, \\ 1 & \text{if } x = 0,\ t > 0. \end{cases}$$

Then, $(\mathbb{R}, N_i, *)$ is a weak fuzzy normed space for all $i \in I$. We only check (FQN4). Let $x, y \in \mathbb{R}$ and $t, s > 0$. If $N_i(x,t) = 0$ or $N_i(y,s) = 0$, then the conclusion is obvious, so we suppose that $N_i(x,t) \neq 0$ and $N_i(y,s) \neq 0$. We may also assume that $x \neq 0$, $y \neq 0$ and $x + y \neq 0$ (otherwise, the conclusion follows trivially). If $|x| \le t$ and $|y| \le s$ then $|x+y| \le |x| + |y| \le t + s$, so $N_i(x+y, t+s) = \boldsymbol{s}_i \ge N_i(x,t) * N_i(y,s) = \boldsymbol{s}_i * \boldsymbol{s}_i$.

Now suppose that $|x| > t$ and $|y| \le s$. Since $N_i(x,t) \neq 0$ and $x \neq 0$, we also have it that $0 < \frac{|x|}{2} < t$. Then, there exists $n_x \in \mathbb{N}$ such that the following is the case.

$$|x|\left(1 - \frac{1}{n_x+1}\right) < t \le |x|\left(1 - \frac{1}{n_x+2}\right).$$

Therefore, the following obtains.

$$\begin{aligned} t + s &> |x|\left(1 - \frac{1}{n_x+1}\right) + |y| > |x|\left(1 - \frac{1}{n_x+1}\right) + |y|\left(1 - \frac{1}{n_x+1}\right) \\ &= (|x| + |y|)\left(1 - \frac{1}{n_x+1}\right) \ge |x+y|\left(1 - \frac{1}{n_x+1}\right). \end{aligned}$$

From this and since $\{(\boldsymbol{s}_n)_i\}_{n\in\mathbb{N}}$ is nondecreasing, $N_i(x+y, t+s) \ge (\boldsymbol{s}_{n_x})_i = N_i(x,t) \ge N_i(x,t) * N_i(y,s)$.

If $|x| \le t$ and $|y| > s$, we can reason as above.

Finally, let us suppose that $|x| > t$ and $|y| > s$. Then, we can find $n_x, n_y \in \mathbb{N}$ such that the following is the case.

$$|x|\left(1 - \frac{1}{n_x+1}\right) < t \le |x|\left(1 - \frac{1}{n_x+2}\right) \text{ and } |y|\left(1 - \frac{1}{n_y+1}\right) < s \le |y|\left(1 - \frac{1}{n_y+2}\right).$$

Then, we have the following.

$$\begin{aligned} |x+y|\left(1 - \frac{1}{(n_x \wedge n_y)+1}\right) &\le |x| + |y| - \frac{|x|}{(n_x \wedge n_y)+1} - \frac{|y|}{(n_x \wedge n_y)+1} \\ &\le |x| + |y| - \frac{|x|}{n_x+1} - \frac{|y|}{n_y+1} < t + s. \end{aligned}$$

Hence, the following is the case.

$$N_i(x+y, t+s) \ge (\boldsymbol{s}_{n_x \wedge n_y})_i \ge (\boldsymbol{s}_{n_x})_i * (\boldsymbol{s}_{n_y})_i = N_i(x,t) * N_i(y,s).$$

Consequently, $(N_i, *)$ satisfies (FQN4).

Consider the collection of fuzzy normed spaces $\{(\mathbb{R}, N_i, *) : i \in I\}$. By assumption, $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a weak fuzzy norm on $\mathbb{R}^I$. Then $F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{1}, \cdot)$ is left-continuous so $\{F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{1}, 1 - \frac{1}{n+2})\}_{n\in\mathbb{N}}$ converges to $F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{1}, 1)$. We observe that the following is the case:

$$F \circ \widetilde{\boldsymbol{N}}\left(\boldsymbol{1}, 1 - \frac{1}{n+2}\right) = F\left(\left(N_i\left(1, 1 - \frac{1}{n+2}\right)\right)_{i\in I}\right) = F(((\boldsymbol{s}_n)_i)_{i\in I}) = F(\boldsymbol{s}_n)$$

for every $n \in \mathbb{N}$ and

$$F \circ \widetilde{\boldsymbol{N}}(\mathbf{1}, 1) = F((N_i(1,1))_{i \in I}) = F((\boldsymbol{s}_i)_{i \in I}) = F(\boldsymbol{s}).$$

Thus, F is sequentially left-continuous.

(3) $\Rightarrow$ (4) We are required to demonstrate that F preserves asymmetric ($*$-)triangular triplets. This was proved in [17] (Proposition 3.30), but we will reproduce it here. Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,1]^I)^3$ such that $\boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}$. Since F is $*$-supmultiplicative and isotone, then the following is the case.

$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b}) \leq F(\boldsymbol{c}).$$

Hence, $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an asymmetric $*$-triangular triplet.

(4) $\Rightarrow$ (1) Let $\{(V_i, N_i, *) : i \in I\}$ be a collection of weak fuzzy quasi-normed spaces. We need to prove that $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a weak fuzzy quasi-norm on $\prod_{i \in I} V_i$.

Given $\boldsymbol{x} \in \prod_{i \in I} V_i$, then $F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{x}, 0) = F((N_i(\boldsymbol{x}_i, 0))_{i \in I}) = F(\mathbf{0}) = 0$; thus, (FQN1) holds.

Now, suppose that $F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{x}, t) = F \circ \widetilde{\boldsymbol{N}}(-\boldsymbol{x}, t) = 1$ for all $t > 0$. F has trivial core this is equivalent to $\widetilde{\boldsymbol{N}}(\boldsymbol{x}, t) = \widetilde{\boldsymbol{N}}(-\boldsymbol{x}, t) = \mathbf{1}$ for all $t > 0$, that is, $N_i(\boldsymbol{x}_i, t) = N_i(-\boldsymbol{x}_i, t) = 1$ for all $i \in I$ and all $t > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-norm for all $i \in I$, then $\boldsymbol{x} = (0_{V_i})_{i \in I} = 0_{\prod_{i \in I} V_i}$; thus, $(FQN2)$ is true.

Obviously, $F \circ \widetilde{\boldsymbol{N}}$ verifies (FQN3) since given $\lambda, t > 0$ and $\boldsymbol{x} \in \prod_{i \in I} V_i$, we the following.

$$F \circ \widetilde{\boldsymbol{N}}(\lambda \boldsymbol{x}, t) = F((N_i(\lambda \boldsymbol{x}_i, t))_{i \in I}) = F\left(\left(N_i\left(\boldsymbol{x}_i, \frac{t}{\lambda}\right)\right)_{i \in I}\right) = F \circ \widetilde{\boldsymbol{N}}\left(\boldsymbol{x}, \frac{t}{\lambda}\right).$$

Now, we verify (FQN4). Let $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i \in I} V_i$ and $t, s > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-norm for all $i \in I$, it is clear that the triplet $((N_i(\boldsymbol{x}_i, t))_{i \in I}, (N_i(\boldsymbol{y}_i, s))_{i \in I}, (N_i(\boldsymbol{x}_i + \boldsymbol{y}_i, t + s))_{i \in I})$ is asymmetric $*$-triangular. By hypothesis, $(F((N_i(\boldsymbol{x}_i, t))_{i \in I}), F((N_i(\boldsymbol{y}_i, s))_{i \in I}),$ $F((N_i(\boldsymbol{x}_i + \boldsymbol{y}_i, t + s))_{i \in I}))$ is an asymmetric $*$-triangular triplet, so the following is the case:

$$F((N_i(\boldsymbol{x}_i, t))_{i \in I}) * F((N_i(\boldsymbol{y}_i, s))_{i \in I}) \leq F((N_i(\boldsymbol{x}_i + \boldsymbol{y}_i, t + s))_{i \in I})$$

which means that $F \circ \widetilde{\boldsymbol{N}}$ verifies (FQN4).

At last, we must prove (FQN5), that is, $F \circ \widetilde{N}(\boldsymbol{x}, \cdot)$ is sequentially left-continuous for every $\boldsymbol{x} \in \prod_{i \in I} V_i$. Let $\{t_n\}_{n \in \mathbb{N}}$ be a nondecreasing sequence on $[0,1]$ converging to t. It is clear that $\{\widetilde{\boldsymbol{N}}(\boldsymbol{x}, t_n)\}_{n \in \mathbb{N}}$ is a nondecreasing sequence on $[0,1]^I$ due to the fact that $N_i(\boldsymbol{x}_i, \cdot)$ is isotone for every $i \in I$. Furthermore, the sequence $\{N_i(\boldsymbol{x}_i, t_n)\}_{n \in \mathbb{N}}$ is convergent to $N_i(\boldsymbol{x}_i, t)$ for all $i \in I$. Thus, $\{\widetilde{\boldsymbol{N}}(\boldsymbol{x}, t_n)\}_{n \in \mathbb{N}}$ converges to $\widetilde{\boldsymbol{N}}(\boldsymbol{x}, t)$. Since F is sequentially left-continuous, the conclusion follows. □

Observe that, in the crisp context, the norm aggregation functions on products also coincide with the asymmetric norm aggregation functions on products [8,9]. Nevertheless, these functions are different from the (quasi-)metric aggregation functions on products [2,3]. Surprisingly, this does not occur in the fuzzy framework.

Corollary 1. *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. Then F is a ($*$-)weak fuzzy (quasi-)norm aggregation function on products if and only if F is a ($*$-)fuzzy (quasi-)metric aggregation function on products.*

Proof. This is a direct consequence of the previous theorem and [17] (Theorem 4.15) (notice that, in that paper, sequentially left-continuity is called simply left-continuity). □

Example 7.

- *Let $F : [0,1] \to [0,1]$ be an isotone and left-continuous function such that $F(0) = 0$ and $F^{-1}(1) = \{1\}$. Then, F is a $\wedge$-weak fuzzy (quasi-)norm aggregation function on products since it is a $\wedge$-supmultiplicative function;*
- *Let $F : [0,1]^n \to [0,1]$ be an isotone and left-continuous function such that $F(0) = 0$ and $F^{-1}(1) = \{\mathbf{1}\}$. Then, F is a $*_D$-weak fuzzy (quasi-)norm aggregation function on products where $*_D$ is the drastic t-norm since it is a $*_D$-supmultiplicative function;*
- *Given a continuous t-norm $*$ and $n \in \mathbb{N}$, the function $F_* : [0,1]^n \to [0,1]$ given by $F_*(a_1, \ldots, a_n) = a_1 * \ldots * a_n$ satisfies the conditions of Theorem 6. Therefore, F is a $*$-weak fuzzy (quasi-)norm aggregation function on products.*
- *Consider an index set I. Then, the function $\mathsf{Inf} : [0,1]^I \to [0,1]$ given by $\mathsf{Inf}(\boldsymbol{x}) = \inf_{i \in I} \boldsymbol{x}_i$ satisfies the conditions of Theorem 6. Therefore, it is a weak fuzzy (quasi-)norm aggregation function on products.*

The following theorem, which must be compared with [17] (Theorem 4.19), provides a characterization of the functions that aggregate weak fuzzy (quasi-)norm aggregation on sets.

Theorem 7. *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

(1) F is a ($*$-)weak fuzzy quasi-norm aggregation function on sets;
(2) F is a ($*$-)weak fuzzy norm aggregation function on sets;
(3) $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$. The core of F is countably included in a unitary face, and F is isotone, ($*$-)supmultiplicative and sequentially left-continuous ;
(4) $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$. The core of F is countably included in a unitary face, and F is sequentially left-continuous, and F preserves asymmetric ($*$-)triangular triplets.

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (3) We first prove that $F(\mathbf{0}) = 0$. Let $(V, N, *)$ be an arbitrary weak fuzzy normed space, $v \in V$ and $t > 0$. Considering the collection of weak fuzzy normed spaces $\{(V, N_i, *) : i \in I\}$ where $N_i = N$ for all $i \in I$, we have that $(F \circ \mathbf{N}, *)$ is a weak fuzzy norm on V so $0 = F \circ \mathbf{N}(v, 0) = F((N(v,0))_{i \in I}) = F(\mathbf{0})$.

On the other hand, $F(\mathbf{1}) = F((N(0_V, t))_{i \in I}) = F \circ \mathbf{N}(0_V, t) = 1$ by (FQN2).

For proving that F is $*$-supmultiplicative, we can proceed as in the proof of this fact in the implication (2) $\Rightarrow$ (3) of Theorem 6.

Now, we check that the core of F is countably included in a unitary face.

Suppose, contrary to our claim, that we can find a sequence $\{\boldsymbol{a}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ such that for any $i \in I$ there exists $n_i \in \mathbb{N}$ verifying $(\boldsymbol{a}_{n_i})_i \neq 1$. Let us consider the vector space $\mathbb{R}$ and, for each $i \in I$, we define $N_i : \mathbb{R} \times [0, +\infty) \to [0,1]$ as the following.

$$N_i(x,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } x = 0,\ t > 0, \\ (\boldsymbol{a}_1)_i & \text{if } x \neq 0,\ t > |x|, \\ (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i & \text{if } x \neq 0,\ \frac{|x|}{n+1} < t \leq \frac{|x|}{n},\ n \in \mathbb{N}. \end{cases}$$

Notice that $(\mathbb{R}, N_i, *)$ is a weak fuzzy normed space for all $i \in I$. Let us verify this. It is obvious that (FQN1) is satisfied. On the other hand, let $x \in \mathbb{R}$ such that $N_i(x,t) = N_i(-x,t) = 1$ for all $t > 0$. By assumption, we can find $n_i \in \mathbb{N}$ such that $(\boldsymbol{a}_{n_i}) \neq 1$. Hence, if $x \neq 0$ then $N_i(x, \frac{|x|}{n_i}) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n_i+1})_i \leq (\boldsymbol{a}_1)_i \wedge \ldots \wedge (\boldsymbol{a}_{n_i+1})_i < 1$, which is a contradiction. Therefore $x = 0$.

Furthermore, let $x \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. If $x = 0$, it is clear that $N_i(\lambda 0, t) = 1 = N_i\left(0, \frac{t}{|\lambda|}\right)$. If $x \neq 0$, the equality $N_i(\lambda x, t) = N_i\left(x, \frac{t}{|\lambda|}\right)$ follows from the equivalences of the inequalities $t > |\lambda x|$ and $\frac{|\lambda x|}{n+1} < t \leq \frac{|\lambda x|}{n}$ with $\frac{t}{|\lambda|} > x$ and $\frac{|x|}{n+1} < \frac{t}{|\lambda|} \leq \frac{|x|}{n}$, respectively, so (FQN3′) is proved.

We next check (FQN4). Let $x, y \in \mathbb{R}$ and $t, s > 0$. If $x + y = 0$, it is obvious that $N_i(x,t) * N_i(y,s) \leq N_i(x+y, t+s) = 1$. Moreover, if $x + y \neq 0$ and $x = 0$ or $y = 0$, the inequality is also clear since if, for example, $y = 0$ then $N_i(x,t) * N_i(y,s) = N_i(x,t) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i$ for some $n \in \mathbb{N}$. Since $t < s + t$, the factors that appear in multiplication by the t-norm $*$ in the value of $N_i(x, t+s)$ are less or equal than the factors in $N_i(x,t)$, so $N_i(x,t) \leq N_i(x, t+s)$. Finally, suppose that $x + y \neq 0$ and $x \neq 0$, $y \neq 0$. If $t + s > |x+y|$, the inequality is clear since $N_i(x+y, t+s) = (\boldsymbol{a}_1)_i$. Otherwise, $t + s \leq |x+y| \leq |x| + |y|$. Then, $t \leq |x|$ or $s \leq |y|$. We distinguish some of the following cases:

- $t \leq |x|$ and $s > |y|$. Then, there exists $n_x \in \mathbb{N}$ such that $\frac{|x|}{n_x+1} < t \leq \frac{|x|}{n_x}$. Then, the following is the case.

$$t + s > \frac{|x|}{n_x + 1} + |y| > \frac{|x| + |y|}{n_x + 1} \geq \frac{|x + y|}{n_x + 1}$$

  This means that the number of factors that appear in $N_i(x+y, t+s)$ is less than or equal to $n_x + 1$, which is the number of factors that appear in $N_i(x,t)$. Consequently, $N_i(x,t) * N_i(y,s) \leq N_i(x,t) \leq N_i(x+y, t+s)$.
- $t > |x|$ and $s \leq |y|$. In this case, we can reason as above.
- $t \leq |x|$ and $s \leq |y|$. Let $n_x, n_y \in \mathbb{N}$ such that the following is the case.

$$\frac{|x|}{n_x + 1} < t \leq \frac{|x|}{n_x} \text{ and } \frac{|y|}{n_y + 1} < s \leq \frac{|y|}{n_y}.$$

  Then, we have the following.

$$t + s > \frac{|x|}{n_x + 1} + \frac{|y|}{n_y + 1} \geq \frac{|x| + |y|}{\max\{n_x, n_y\} + 1} \geq \frac{|x + y|}{\max\{n_x, n_y\} + 1}.$$

  This means that the number of factors that appear in $N_i(x+y, t+s)$ is less than or equal to $\max\{n_x, n_y\} + 1$. By reasoning as above, we obtain the desired inequality.

What remains is proving that $N_i(x, \cdot)$ is left-continuous. If $x = 0$, it is obvious. Suppose now that $x \neq 0$ and let $t > 0$. By construction, if $\{t_n\}_{n \in \mathbb{N}}$ is a sequence in $(0, +\infty)$ for which its upper limit is t, we can find $n_0 \in \mathbb{N}$ such that $N(x, t_n)$ is constant for every $n \geq n_0$; thus, the conclusion follows. We conclude that $(N_i, *)$ is a weak fuzzy norm on $\mathbb{R}$ for all $i \in I$.

Notice that if $t > 1$, then $N_i(1,t) = (\boldsymbol{a}_1)_i$. Thus, we have the following.

$$F \circ \boldsymbol{N}(1,t) = F((N_i(1,t))_{i \in I}) = F(((\boldsymbol{a}_1)_i)_{i \in I}) = F(\boldsymbol{a}_1) = 1.$$

If $0 < t \leq 1$, then we can find $n \in \mathbb{N}$ such that $N_i(1,t) = (\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i$ for all $i \in I$. Since F is $*$-supmultiplicative, then we have the following.

$$\begin{aligned} F \circ \boldsymbol{N}(1,t) &= F((N_i(1,t))_{i \in I}) = F(((\boldsymbol{a}_1)_i * \ldots * (\boldsymbol{a}_{n+1})_i)_{i \in I}) = F(\boldsymbol{a}_1 * \ldots * \boldsymbol{a}_{n+1}) \\ &\geq F(\boldsymbol{a}_1) * \ldots * F(\boldsymbol{a}_n) = 1. \end{aligned}$$

Therefore, $F \circ \boldsymbol{N}(1,t) = 1$ for all $t > 0$, which contradicts the fact that $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy norm. Consequently, the core of F is countably included in a unitary face.

The proofs that F is isotone and the proof that F is sequentially left-continuous are similar to the same proofs in the implication (2)$\Rightarrow$ (3) of Theorem 6.

(3) $\Rightarrow$ (4) This is similar to the implication (3) $\Rightarrow$ (4) of Theorem 6.

(4) $\Rightarrow$ (1) Let $\{(V, N_i, *) : i \in I\}$ be a collection of weak fuzzy quasi-normed spaces. Let us check that $(F \circ \boldsymbol{N}, *)$ is a weak fuzzy quasi-norm on V.

Given $x \in V$, then $F \circ \boldsymbol{N}(x, 0) = F((N_i(x,0))_{i \in I}) = F(\boldsymbol{0}) = 0$; thus, (FQN1) holds.

Now, suppose that there exists $x \in V$ such that $F \circ \boldsymbol{N}(x,t) = F \circ \boldsymbol{N}(-x,t) = 1$ for all $t > 0$. Since $(N_i(x,t), N_i(-x,t), N_i(x_i,t) * N_i(-x,t))$ is an asymmetric $*$-triangular triplet for all $i \in I$, by assumption $(F((N_i(x,t))_{i\in I}), F((N_i(-x,t))_{i\in I}), F((N_i(x,t) * N_i(-x,t))_{i\in I}))$ is also an asymmetric $*$-triangular triplet. Thus, the following is the case.

$$1 = 1 * 1 = F((N_i(x,t))_{i\in I}) * F((N_i(-x,t))_{i\in I}) \leq F((N_i(x,t) * N_i(-x,t))_{i\in I}).$$

Hence, $F((N_i(x,t) * N_i(-x,t))_{i\in I}) = 1$ for all $t > 0$. Let us define $\boldsymbol{a}_n = (N_i(x, \frac{1}{n}) * N_i(-x, \frac{1}{n}))_{i\in I}$. Then, $\{\boldsymbol{a}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$. Since the core of F is countably included in a unitary face, then $(\boldsymbol{a}_n)_j = 1$ for some $j \in I$ and for all $n \in \mathbb{N}$, that is, $N_j(x, \frac{1}{n}) * N_j(-x, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Consequently, $N_j(x, \frac{1}{n}) = N_j(-x, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Moreover, since $N_i(x, \cdot)$ and $N_i(-x, \cdot)$ are isotone, we immediately obtain that $N_j(x,t) = N_j(-x,t) = 1$ for all $t > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-metric on V, then $x = 0_V$. Therefore, $F \circ \boldsymbol{N}$ satisfies (FQN2).

It is clear that $F \circ \boldsymbol{N}$ verifies (FQN3) since, given $\lambda, t > 0$ and $x \in V$, we have the following.

$$F \circ \boldsymbol{N}(\lambda x, t) = F((N_i(\lambda x, t))_{i\in I}) = F\left(\left(N_i\left(x, \frac{t}{\lambda}\right)\right)_{i\in I}\right) = F \circ \boldsymbol{N}\left(x, \frac{t}{\lambda}\right).$$

In order to prove (FQN4), let $x, y \in V$ and $t, s > 0$. Since $(N_i, *)$ is a weak fuzzy quasi-norm for all $i \in I$, it is obvious that $((N_i(x,t))_{i\in I}, (N_i(y,s))_{i\in I}, (N_i(x+y, t+s))_{i\in I})$ is an asymmetric $*$-triangular triplet. By hypothesis, $(F((N_i(x,t))_{i\in I}), F((N_i(y,s))_{i\in I}), F((N_i(x+y, t+s))_{i\in I}))$ is an asymmetric $*$-triangular triplet. Thus, the following is the case.

$$F((N_i(x,t))_{i\in I}) * F((N_i(y,s))_{i\in I}) \leq F((N_i(x+y, t+s))_{i\in I})$$

Thus, $F \circ \boldsymbol{N}$ verifies (FQN4).

To this end, (FQN5) follows in a similar manner as in the implication (4)$\Rightarrow$ (1) of Theorem 6. $\square$

Corollary 2. *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. Then, F is a ($*$-)weak fuzzy (quasi-)norm aggregation function on sets if and only if F is a ($*$-)fuzzy (quasi-)metric aggregation function on sets.*

Proof. This is a direct consequence of the previous theorem and [17] (Theorem 4.19) (notice that in that, in the paper, sequentially left-continuity is called simply left-continuity). $\square$

We provide an example of a $\wedge$-weak fuzzy (quasi-)norm aggregation function on sets that are not a $\wedge$-weak fuzzy (quasi-)norm aggregation function on products.

Example 8. *Let I be a subset of $[0,1]$ having minimum greater than 0 with cardinality greater than 1. Let $F : [0,1]^I \to [0,1]$, given by the following:*

$$F(\boldsymbol{x}) = \alpha \cdot \inf\{i \cdot \boldsymbol{x}_i : i \in I\}$$

for all $\boldsymbol{x} \in [0,1]^I$, where $\alpha = \frac{1}{\min I}$.

It is obvious that $F(\boldsymbol{1}) = 1$ and $F(\boldsymbol{0}) = 0$. Moreover, F is clearly isotone. We prove that F is $\wedge$-supmultiplicative. Let $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^I$. Then, we have the following:

$$F(\boldsymbol{x}) \wedge F(\boldsymbol{y}) \leq (\alpha \cdot i \cdot \boldsymbol{x}_i) \wedge (\alpha \cdot i \cdot \boldsymbol{y}_i) = \alpha \cdot i \cdot (\boldsymbol{x}_i \wedge \boldsymbol{y}_i)$$

for all $i \in I$. Hence, $F(\boldsymbol{x}) \wedge F(\boldsymbol{y}) \leq \alpha \cdot \inf\{i \cdot (\boldsymbol{x}_i \wedge \boldsymbol{y}_i) : i \in I\} = F(\boldsymbol{x} \wedge^I \boldsymbol{y})$.

Furthermore, if $F(\boldsymbol{a}) = 1$, then $\boldsymbol{a}_{\min I} = 1$. Otherwise, $\min I \cdot \boldsymbol{a}_{\min I} < \min I$, which implies that $\inf\{i \cdot \boldsymbol{x}_i : i \in I\} \leq \min I \cdot \boldsymbol{a}_{\min I} < \min I$. Thus, $F(\boldsymbol{a}) < 1$ follows, which is a contradiction. Consequently, the core of F is countably included in a unitary face. Thus, F is a $\wedge$-weak fuzzy (quasi-)norm aggregation functions on sets.

However, the core of F is not trivial. In fact, given $j \in I\backslash\{\min I\}$, then $F(\boldsymbol{b}) = 1$ where $\boldsymbol{b} \in [0,1]^I$ is given by $\boldsymbol{b}_i = 1$ whenever $i \neq j$ and $\boldsymbol{b}_j = \frac{\min I}{j}$.

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