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Generalized Higher Order Preinvex Functions and Equilibrium-like Problems

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Abstract: Equilibrium problems and variational inequalities are connected to the symmetry concepts, which play important roles in many fields of sciences. Some new preinvex functions, which are called generalized preinvex functions, with the bifunction $\zeta(.,.)$ and an arbitrary function k , are introduced and studied. Under the normed spaces, new parallelograms laws are taken as an application of the generalized preinvex functions. The equilibrium-like problems are represented as the minimum values of generalized preinvex functions under the $k\zeta$ -invex sets. Some new inertial methods are proposed and researched to solve the higher order directional equilibrium-like problem, Convergence criteria of the our methods is discussed, along with some unresolved issues.

Keywords: directional derivatives; k -convex functions; k -convex sets; variational inequalities



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1. Introduction

Symmetric concepts play an important and significant role in all the branches of pure and applied sciences. Variational like inequalities represent the optimality criteria of the differentiable preinvex functions and represent novel applications extensions of the variational principles. Many interesting properties, applications, and generalizations of the variational inequalities, which are proposed by Stampacchia [1], have been discussed in signal processing, linear inverse problems, and machine learning; see References [2–15] and the references therein. The concepts and applications of convex functions and convex sets are investigated; see Hanson [16], Ben-Israel et al. [17], and Noor [18–21], as well as References [22–26] for more details.

The so-called (h, k) convex set and convex functions are defined by Micherda and Rajba [26], Hazy [27], and Crestescu et al. [28]. The definition and characterizations of k -convex functions are introduced and studied by Noor [29]. The modified $k\eta$ -invex sets and $k\eta$ -preinvex is studied. Generalized preinvex functions can be regarded as an important improvement of the (h, k) convex functions, which are investigated by Micherda et al. [26] and Hazy [27]. Their results help us to further consider the problems of directional equilibrium-like instances.

Some computing methods to solve variational inequalities, optimization problems, and equilibrium problems are discussed. Glowinski et al. [5] use the auxiliary principle approach of involving Bregman distance function to consider some iterative schemes that solve the higher order directional equilibrium-like problems. Some convergence properties of these methods are discussed by applying either pseudomonotonicity or partially relaxed strongly monotonicity, which is a weaker condition than monotonicity.

In Section 2, some new concepts and properties of generalized preinvex functions are set. Preinvex functions are viewed as novel extensions of the convex functions that are associated with variational-like inequalities. Naturally, all the results are closely related with symmetry concepts. In Section 3, main characterizations of the higher-order strongly generalized preinvex are investigated. In Section 4, we derive various parallelograms.

In Section 5, several approximate methods for solving equilibrium-like problems are proposed. We get the convergence of the methods by the pseudomonotone operators. The applications of our results are discussed in some special cases. Further research in the field of applied science is promoted by our ideas and techniques.

2. Preliminaries

Assuming that Ω_k is a nonempty and closed under the normed space H . The norm and inner product are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively.

In this paper, only if explained specially, we assume that $\lambda \in [0, 1]$ and $p \geq 1$.

Definition 1. $\Omega_{k\zeta}$ is a k -invex set about the bifunction $\zeta(\cdot, \cdot)$ and arbitrary function k , if

$$\mu + k(\lambda)\zeta(v, \mu) \in \Omega_{k\zeta}, \quad \forall \mu, v \in \Omega_{k\zeta}.$$

For $k(\lambda) = \lambda$, the set $\Omega_{k\zeta}$ which is an invex set Ω_ζ , is researched by Ben-Israel and Mond [17].

Definition 2. Ω is a k -invex set about the bifunction $\zeta(\cdot, \cdot)$ and arbitrary function k , if

$$\mu + \lambda\zeta(v, \mu) \in \Omega_\zeta \quad \forall \mu, v \in \Omega_\zeta.$$

When $\zeta(v, \mu) = v - \mu$, the Definition 1 is reduced to:

Definition 3. Ω_k is a k -convex set about arbitrary function k , if

$$\mu + k(\lambda)(v - \mu) \in \Omega_k \quad \forall \mu, v \in \Omega_k,$$

which was introduced by Hazy [27]. Relevant research is found in Reference [9,13,19,30,31].

Next, the definitions of higher-order generalized preinvex function with bifunction $\zeta(\cdot, \cdot)$ and an arbitrary function k are introduced.

Definition 4. The function G on $\Omega_{k\zeta}$ is defined as higher-order generalized preinvex function, if a constant β , bifunction $\zeta(\cdot, \cdot)$ and an arbitrary function k are present, that is:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq (1 - k(\lambda))G(\mu) + k(\lambda)G(v) \\ &\quad - \beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

When $k(\lambda) = \lambda$, the Definition 4 is reduced to:

Definition 5. The function G on Ω_ζ is defined as higher-order preinvex function, if a constant β , bifunction $\zeta(\cdot, \cdot)$ and an arbitrary function k are present, such that

$$\begin{aligned} G(\mu + \lambda\zeta(v, \mu)) &\leq (1 - \lambda)G(\mu) + \lambda G(v) \\ &\quad - \beta\{\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_\zeta. \end{aligned}$$

Noor and Noor [21] introduce and study the higher-order preinvex functions. Obviously, when $k(\lambda) = \lambda$, a higher-order $k(\lambda)$ preinvex function appears, which is a higher-order preinvex function.

When $\zeta(v, \mu) = v - \mu$, the higher-order k -preinvex function is collapsed to:

Definition 6. The function G on $\Omega_{k\zeta}$ is defined as higher-order k -convex function, if a constant β and an arbitrary function k are present, that is:

$$\begin{aligned} G(\mu + k(\lambda)(v - \mu)) &\leq (1 - k(\lambda))G(\mu) + k(\lambda)G(v) \\ &\quad - \beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|v - \mu\|^p, \\ &\quad \forall \mu, v \in \Omega_k. \end{aligned}$$

If $k(\lambda) = \lambda$, $\zeta(v, \mu) = v - \mu$, thus, Definition 6 is reduced to:

Definition 7. The function G on Ω_ζ is defined as higher-order convex function, if a constant β and an arbitrary function k are present, such that

$$\begin{aligned} G(\mu + \lambda(v - \mu)) &\leq (1 - \lambda)G(\mu) + \lambda G(v) \\ &\quad - \beta\{\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p\}\|v - \mu\|^p, \quad \forall \mu, v \in \Omega. \end{aligned}$$

Mohsin et al. [32] study higher-order convex functions which represent a significant improvement to the concepts for higher-order convex functions that are studied by Alabdali et al. [33], Lin et al. [34], Mako et al. [35], and Olbrys [36].

When $\lambda = 1$, the k -convex function is reduced to:

Condition A

$$G(\mu + k(1)\zeta(v, \mu)) \leq G(v), \quad \forall \mu, v \in \Omega_{k\zeta},$$

which is very important to the derivation of these main results.

Definition 8. The function G on $\Omega_{k\zeta}$ is defined as higher-order quasi k -preinvex function if a constant β and a function k are present, that is:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq \max\{G(\mu), G(v)\} \\ &\quad - \beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

Definition 9. A function G on $\Omega_{k\zeta}$ is defined as higher-order logarithmic generalized preinvex function, if the bifunction $\eta(., .)$ and a function k are present, that is:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq (G(\mu))^{1-k(\lambda)}(G(v))^{k(\lambda)} \\ &\quad - \beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

where $G(\cdot) > 0$.

When $\beta = 0$, Definition 9 is reduced to:

Definition 10. The function G on $\Omega_{k\zeta}$ is defined as higher-order logarithmic generalized preinvex function, if a function k is present, that is:

$$G(\mu + k(\lambda)\zeta(v, \mu)) \leq (G(\mu))^{1-k(\lambda)}(G(v))^{k(\lambda)}, \quad \forall \mu, v \in \Omega_{k\zeta},$$

or, equivalently:

Definition 11. A function G on $\Omega_{k\zeta}$ is defined as higher-order logarithmic generalized preinvex function, if a function k is present, that is:

$$\log G(\mu + k(\lambda)\zeta(v, \mu)) \leq (1 - k(\lambda)) \log G(\mu) + k(\lambda) \log G(v), \quad \forall \mu, v \in \Omega_{k\zeta}.$$

From this idea, the higher-order logarithmic generalized preinvex function is defined as follows:

Definition 12. A function G on $\Omega_{k\zeta}$ is defined as higher-order logarithmic generalized preinvex function, if a function k is present, that is:

$$\begin{aligned} \log G(\mu + k(\lambda)\zeta(v, \mu)) &\leq (1 - k(\lambda)) \log G(\mu) + k(\lambda) \log G(v), \\ &\quad -\beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\} \|\zeta(v, \mu)\|^p \\ &\quad \forall \mu, v \in \Omega_{k\zeta}, \end{aligned}$$

where it seems to be a new one.

Through the above definitions, we obtain:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq (G(\mu))^{1-k(\lambda)}(G(v))^{k(\lambda)} \\ &\quad -\beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\} \|\zeta(v, \mu)\|^p \\ &\leq (1 - k(\lambda))G(\mu) + k(\lambda)G(v) \\ &\quad -\beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\} \|\zeta(v, \mu)\|^p \\ &\leq \max\{G(\mu), G(v)\} \\ &\quad -\beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\} \|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

From the above equation, we know that higher-order logarithmic generalized preinvex functions \implies higher-order generalized preinvex functions and higher-order generalized preinvex functions \implies higher-order generalized quasi preinvex functions. Otherwise, it does not hold.

If the function G satisfies these two conditions of higher-order generalized preinvex function and higher-order generalized preincave function, then, a new concept is defined as follows.

Definition 13. The function G on $\Omega_{k\zeta}$ is defined as higher-order generalized preinvex affine function, if an arbitrary function k and a constant β are present, that is:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &= (1 - k(\lambda))G(\mu) + k(\lambda)G(v) \\ &\quad -\beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\} \|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

If the functions $k(\lambda)$ and the bifunction $\zeta(\cdot, \cdot)$ are selected as the suitable form, several categories of higher-order generalized preinvex functions and their variant forms are obtained.

The further assumption of the bifunction $\zeta(\cdot, \cdot)$ and the function $k(\lambda)$ is defined as follows.

Condition M. Assuming that $\zeta(\cdot, \cdot): \Omega_{k\zeta} \times \Omega_{k\zeta} \rightarrow H$ satisfies the following equations

$$\begin{aligned} \zeta(\mu, \mu + k(\lambda)\zeta(v, \mu)) &= -k(\lambda)\zeta(v, \mu) \\ \zeta(v, \mu + k(\lambda)\zeta(v, \mu)) &= (1 - k(\lambda))\zeta(v, \mu), \quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

When $k(\lambda) = 0$, if and only if $\mu = v$, where $\forall \mu, v \in \Omega_{k\zeta}$, we obtain $\zeta(\mu, v) = 0$. It shows [9,10] that $\zeta(\mu + k(\lambda)\zeta(v, \mu), \mu) = k(\lambda)\zeta(v, \mu)$, where $\forall \mu, v \in \Omega_{k\zeta}$.

For $k(\lambda) = \lambda$, Condition M is reduced to the condition C of Mohan and Neogy [18].

Next, the definition of k -directional derivative is set.

Definition 14 (Reference [29]). The k -directional derivative of G at $\mu \in \Omega_{k\zeta}$ in the direction $v \in \Omega_{k\zeta}$ is

$$G'_{k\zeta}(\mu, \zeta(v, \mu)) = \lim_{\lambda \rightarrow 0^+} \left\{ \frac{G(\mu + k(\lambda)\zeta(v, \mu)) - G(\mu)}{k(\lambda)} \right\}.$$

For $\zeta(v, \mu) = v$ and $k(\lambda) = \lambda$, the k -directional derivative of G at $\mu \in \Omega$ in the direction $v \in \Omega$ is consistent with the usual directional derivative of G at u in a direction v which is defined by

$$G'(\mu, v) = \lim_{\lambda \rightarrow 0^+} \frac{G(\mu + \lambda v) - G(\mu)}{\lambda}.$$

The $v \rightarrow G'_{k\eta}(\mu, \zeta(v, \mu))$ is positively homogeneous, subadditive.

Definition 15. The differentiable function G on $\Omega_{k\zeta}$ is defined as higher-order strongly pseudo k -invex function, iff, if a constant $\beta > 0$ is present, that is:

$$G'_k(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \geq 0 \Rightarrow G(v) - G(\mu) \geq 0, \quad \forall \mu, v \in \Omega_{k\zeta}.$$

Definition 16. The differentiable function G on $\Omega_{k\zeta}$ is defined as higher order strongly quasi- k -invex function, iff, a constant $\beta > 0$ is present, that is:

$$\begin{aligned} G(v) &\leq G(\mu) \\ \Rightarrow \\ G'_k(\mu, \eta(v, \mu)) + \beta \|\zeta(\mu, v)\|^p &\leq 0, \quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

Definition 17. The G on $\Omega_{k\zeta}$ is defined as pseudo- k -invex, iff,

$$G'_k(\mu, \zeta(v, \mu)) \geq 0 \Rightarrow G(v) \geq G(\mu), \quad \forall \mu, v \in \Omega_{k\zeta}.$$

Definition 18. The differentiable function G on $\Omega_{k\zeta}$ is defined as quasi k -invex function, iff,

$$G(v) \leq G(\mu) \Rightarrow G'_k(\mu, \zeta(v, \mu)) \leq 0, \quad \forall \mu, v \in \Omega_{k\zeta}.$$

When $\zeta(v, \mu) = -\zeta(v, \mu), \forall \mu, v \in \Omega_{k\zeta}, \zeta(\cdot, \cdot)$ is skew-symmetric. Thus, Definitions 7–14 are changed to the known ones. The concepts of our paper have a great improvement on the previously known concepts. The new concepts are very important to the optimization and mathematical programming.

3. Characterizations

Next, under the invex set $\Omega_{k\zeta}$, some features of higher-order strongly generalized preinvex functions are researched.

Theorem 1. Under the condition M and the assumption of G which is a differentiable function on $\Omega_{k\zeta}$ in H . G is a higher-order generalized preinvex function, if and only if, G is a higher-order k -invex function.

Proof. Because G is a higher-order generalized preinvex function which is based on $\Omega_{k\zeta}$,

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq (1 - k(\lambda))G(\mu) + \lambda G(v) \\ &- \beta \{ (k(\lambda))^p (1 - (k\lambda)) + k(\lambda)(1 - k(\lambda))^p \} \|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}, \lambda \in [0, 1], \quad p \geq 1, \end{aligned}$$

which is changed as

$$G(v) - G(\mu) \geq \left\{ \frac{G(\mu + \lambda\zeta(v, \mu)) - G(\mu)}{k(\lambda)} + \beta\{(k(\lambda))^{p-1}(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^{p-1}\} \|\zeta(v, \mu)\|^p \right\}$$

When $\lambda \rightarrow 0$, the following inequality is obtained:

$$G(v) - G(\mu) \geq G'_k(u, \zeta(v, \mu)) + \beta\|\zeta(v, \mu)\|^p.$$

Thus, G is a higher-order generalized invex function.

However, under the condition M , if G is a higher-order strongly generalized invex function which is based on $\Omega_{k\zeta}$, where $\forall \mu, v \in \Omega_{k\zeta}, \lambda \in [0, 1], v_\lambda = \mu + k(\lambda)\zeta(v, \mu) \in \Omega_{k\zeta}$, we obtain:

$$\begin{aligned} &G(v) - G(\mu + k(\lambda)\zeta(v, \mu)) \\ &\geq \langle G'(\mu + k(\lambda)\zeta(v, \mu)), \zeta(v, \mu + k(\lambda)\zeta(v, \mu)) \rangle + \beta\|\zeta(v, \mu + k(\lambda)\zeta(v, \mu))\|^p \\ &= (1 - k(\lambda))\langle G'_k(\mu + k(\lambda)\zeta(v, \mu)), \zeta(v, \mu) \rangle + \beta(1 - k(\lambda))^p\|\zeta(v, \mu)\|^p. \end{aligned} \tag{1}$$

Using the similar method, the following results are obtained:

$$\begin{aligned} &G(\mu) - G(\mu + k(\lambda)\zeta(v, \mu)) \\ &\geq \langle G'_k(\mu + k(\lambda)\zeta(v, \mu)), \zeta(\mu, \mu + k(\lambda)\zeta(v, \mu)) \rangle + \beta\|\zeta(\mu, \mu + k(\lambda)\zeta(v, \mu))\|^p \\ &= -k(\lambda)\langle G'_k(\mu + k(\lambda)\zeta(v, \mu)), \zeta(v, \mu) \rangle + \beta k(\lambda)^p\|\zeta(v, \mu)\|^p. \end{aligned} \tag{2}$$

From multiplying (1) by $k(\lambda)$ plus multiplying (2) by $(1 - k(\lambda))$, we obtain:

$$G(\mu + k(\lambda)\zeta(v, \mu)) \leq (1 - k(\lambda))G(\mu) + k(\lambda)G(v) - \beta(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\|\zeta(v, \mu)\|^p.$$

From the above equation, we find that G is a higher-order generalized preinvex function. \square

Theorem 2. Under the assumption of G which is a differentiable higher-order generalized preinvex function on $\Omega_{k\zeta}$. If G is a higher-order generalized invex function, thus,

$$G'_k(\mu, \zeta(v, \mu)) + G'_k(v, \zeta(\mu, v)) \leq -\beta\{\|\zeta(v, \mu)\|^p + \|\zeta(\mu, v)\|^p\}, \forall \mu, v \in \Omega_{k\zeta}. \tag{3}$$

Proof. Because G is a higher-order generalized invex function which is based on $\Omega_{k\zeta}$,

$$G(v) - G(\mu) \geq G'_k(\mu, \zeta(v, \mu)) + \beta\|\zeta(v, \mu)\|^p. \tag{4}$$

By exchanging the positions of u and v in Equation (4), the following inequality is obtained:

$$G(\mu) - G(v) \geq G'_k(v, \zeta(\mu, v)) + \beta\|\zeta(\mu, v)\|^p. \tag{5}$$

From adding Equations (4) and (5), the following result is:

$$G'_k(\mu, \zeta(v, \mu)) + G'_k(v, \zeta(\mu, v)) \leq -\beta\{\|\zeta(v, \mu)\|^p + \|\zeta(\mu, v)\|^p\}. \tag{6}$$

We find that $G'_k(\cdot)$ is a higher-order generalized monotone operator. \square

When $p = 2$, the converse of Theorem 2 holds. Then, the following result is obtained:

Theorem 3. Assuming that the $G'_k(\cdot)$ is a higher order generalized monotone, that is:

$$G(v) - G(\mu) \geq G'_k(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \int_0^1 k(\lambda)^{p-1} d\lambda.$$

Proof. Because $G'_k(\cdot)$ is a higher-order strongly $k\zeta$ -monotone. Through Equation (6), the following inequality is obtained:

$$G'_k(v, \zeta(\mu, v)) \leq -G'_k(\mu, \zeta(v, \mu)) - \beta \{\|\zeta(v, \mu)\|^p + \|\zeta(\mu, v)\|^p\}. \tag{7}$$

Under the Condition M, because Ω is an invex set, $\forall \mu, v \in \Omega_\zeta, \lambda \in [0, 1], v_\lambda = \mu + k(\lambda)\zeta(v, \mu) \in \Omega_\zeta$. Bringing $v = v_\lambda$ into Equation (7), we obtain:

$$\begin{aligned} G'_k(v_\lambda, \zeta(\mu, \mu + k(\lambda)\zeta(v, \mu))) &\leq G'_k(\mu, \zeta(\mu + k(\lambda)\zeta(v, \mu))) \\ &\quad - \beta \{\|\zeta(\mu + k(\lambda)\zeta(v, \mu), \mu)\|^p \\ &\quad + \|\zeta(\mu, \mu + k(\lambda)\zeta(v, \mu))\|^p\} \\ &= -k(\lambda)G'_k(\mu, \zeta(v, \mu)) - 2k(\lambda)^p \beta \|\zeta(v, \mu)\|^p; \end{aligned}$$

then,

$$G'_k(v_\lambda, \zeta(v, \mu)) \geq G'_k(\mu, \zeta(v, \mu)) + 2\beta k(\lambda)^{p-1} \|\zeta(v, \mu)\|^p. \tag{8}$$

Assuming $\zeta(\lambda) = G(\mu + k(\lambda)\zeta(v, \mu))$, through Equation (8), we obtain:

$$\begin{aligned} \zeta'(\lambda) &= G'_k(\mu + k(\lambda)\zeta(v, \mu), \zeta(v, \mu)) \\ &\geq G'_k(\mu, \zeta(v, \mu)) + 2\beta k(\lambda)^{p-1} \|\zeta(v, \mu)\|^p. \end{aligned} \tag{9}$$

Equation (9) is integrated from 0 and 1, and the following inequality is obtained:

$$\zeta(1) - \zeta(0) \geq G'_k(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \int_0^1 k(\lambda)^{p-1} d\lambda.$$

That is,

$$G(\mu + k(\lambda)\zeta(v, \mu)) - G(\mu) \geq G'_k(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \int_0^1 k(\lambda)^{p-1} d\lambda.$$

From Condition A, the following inequality is obtained:

$$G(v) - G(\mu) \geq \langle G'(\mu), \zeta(v, \mu) \rangle + \beta \|\zeta(v, \mu)\|^p \int_0^1 k(\lambda)^{p-1} d\lambda.$$

□

Now, a necessary condition of higher-order strongly generalized pseudo-invex function is given.

Theorem 4. Under the Conditions A and M and assumption of $G'_k(\cdot)$ which is a higher-order relaxed $k\zeta$ -pseudomonotone operator, the inequality is obtained as follows:

$$G(v) - G(\mu) \geq \beta \|\zeta(v, \mu)\|^p \int_0^1 k(\lambda) d\lambda. \tag{10}$$

Proof. Because G' is higher order relaxed $k\zeta$ -pseudomonotone. For $\forall \mu, v \in \Omega_\zeta$,

$$G'_k(\mu, \zeta(v, \mu)) \geq 0,$$

where it implies

$$-G'_k(v, \eta(\mu, \nu)) \geq \beta \|\zeta(\mu, \nu)\|^p. \tag{11}$$

Because Ω is an invex set, $\forall \mu, \nu \in \Omega_{k\zeta}$ $\lambda \in [0, 1]$, $\nu_\lambda = \mu + k(\lambda)\zeta(\nu, \mu) \in \Omega_{k\zeta}$. Under the Condition M , $\nu = \nu_\lambda$ is taken in Equation (11), and we obtain:

$$-G'_k(\mu + \lambda\zeta(\nu, \mu), \zeta(\mu, \nu)) \geq k(\lambda)\beta \|\zeta(\nu, \mu)\|^p. \tag{12}$$

Let

$$\xi(\lambda) = G(\mu + k(\lambda)\zeta(\nu, \mu)), \quad \forall \mu, \nu \in \Omega_{k\zeta}, \lambda \in [0, 1].$$

From Equation (12), we obtain:

$$\xi'(\lambda) = G'_k(\mu + k(\lambda)\zeta(\nu, \mu), \zeta(\mu, \nu)) \geq k(\lambda)\beta \|\zeta(\nu, \mu)\|^p.$$

The above equation is integrated between 0 to 1, and the following inequality is obtained:

$$\xi(1) - \xi(0) \geq \beta \|\zeta(\nu, \mu)\|^p \int_0^1 k(\lambda) d\lambda,$$

that is:

$$G(\mu + \lambda\zeta(\nu, \mu)) - G(\mu) \geq \beta \|\zeta(\nu, \mu)\|^p \int_0^1 k(\lambda) d\lambda,$$

under the Condition A ,

$$G(\nu) - G(\mu) \geq \beta \|\zeta(\nu, \mu)\|^p \int_0^1 k(\lambda) d\lambda,$$

which is the required Equation (10). \square

Theorem 5. Assuming that the differential (μ) of a differentiable preinvex function $G(\mu)$ is Lipschitz continuous on $\Omega_{k\zeta}$ under a constant $\beta > 0$, the inequality is obtained as follows:

$$G(\mu + \zeta(\nu, \mu)) - G(\mu) \leq \langle G'(\mu), \zeta(\nu, \mu) \rangle + \beta \|\zeta(\nu, \mu)\|^p \int_0^1 k(\lambda) d\lambda, \quad \forall \mu, \nu \in \Omega_{k\zeta}.$$

Proof. Through Noor and Noor [20,21], its proof is obtained easily. \square

Definition 19. The function G is defined as a sharply higher order strongly generalized pseudo preinvex; if a constant $\beta > 0$ is present, then,

$$\begin{aligned} G'_k(\mu, \zeta(\nu, \mu)) &\geq 0 \\ \Rightarrow \\ G(\nu) &\geq G(\nu + k(\lambda)\zeta(\nu, \mu)) \\ &+ \beta \{k(\lambda)^p(1 - k(\lambda)) + \lambda(1 - k(\lambda))^p\} \|\zeta(\nu, \mu)\|^p, \forall \mu, \nu \in \Omega_{k\zeta}. \end{aligned}$$

Theorem 6. Assuming that G is a higher-order sharply generalized pseudo preinvex function on $\Omega_{k\zeta}$ under a constant $\beta > 0$, the inequality is obtained as follows:

$$-\langle G'_k(\nu, \zeta(\nu, \mu)) \rangle \geq \beta \|\zeta(\nu, \mu)\|^p, \quad \forall \mu, \nu \in \Omega_{k\zeta}.$$

Proof. Because G is a higher sharply pseudo generalized preinvex function which is based on $\Omega_{k\zeta}$,

$$G(v) \geq G(v + k(\lambda)\zeta(v, \mu)) + \beta\{(k(\lambda))^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p,$$

then,

$$\frac{G(v + k(\lambda)\zeta(v, \mu)) - G(v)}{k(\lambda)} + \beta\{k(\lambda)^{p-1}(1 - k(\lambda)) + (1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p \leq 0.$$

When $\lambda \rightarrow 0$, we obtain:

$$-G'_k(v, \zeta(v, \mu)) \geq \beta\|\zeta(v, \mu)\|^p.$$

□

Definition 20. G is defined as a higher-order pseudo generalized preinvex function about strictly positive bifunction $W(\cdot, \cdot)$, if

$$\begin{aligned} G(v) &< G(\mu) \\ \Rightarrow \\ G(\mu + k(\lambda)\zeta(v, \mu)) &< G(\mu) + k(\lambda)(k(\lambda) - 1)W(v, \mu), \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

Theorem 7. Assuming that G is a higher-order generalized preinvex function and satisfies $G(v) < G(\mu)$, G is a higher-order generalized pseudo preinvex function.

Proof. Because $G(v) < G(\mu)$ and G are higher-order generalized preinvex functions, for $\forall \mu, v \in \Omega_{k\zeta}$, $\lambda \in [0, 1]$, the inequality is obtained as follows:

$$\begin{aligned} G(\mu + k(\lambda)\zeta(v, \mu)) &\leq G(\mu) + k(\lambda)(G(v) - G(\mu)) \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p \\ &< G(\mu) + k(\lambda)(1 - k(\lambda))(G(v) - G(\mu)) \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p \\ &= G(\mu) + k(\lambda)(k(\lambda) - 1)(G(\mu) - G(v)) \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p \\ &< G(v) + k(\lambda)(k(\lambda) - 1)W(\mu, v) \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}, \end{aligned}$$

where $W(\mu, v) = G(\mu) - G(v) > 0$. Therefore, G is a higher-order generalized pseudo preinvex function. □

4. Parallelogram Laws

Next, we obtain some new parallelogram laws.

Through Definition 13, we obtain:

$$\begin{aligned} \|G(\mu + k(\lambda)\zeta(v, \mu))\|^p &= (1 - k(\lambda))\|\mu\|^p + k(\lambda)\|v\|^p \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + (\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned} \tag{13}$$

Bringing $\lambda = \frac{1}{2}$ into Equation (13), we obtain:

$$\begin{aligned} & \|G(\mu + k(\frac{1}{2})\zeta(v, \mu))\|^p + \beta\{k(\frac{1}{2})\}^p(1 - k(\frac{1}{2})) + k(\frac{1}{2})(1 - k(\frac{1}{2}))^p\} \|\zeta(v, \mu)\|^p \\ & = k(\frac{1}{2})\|\mu\|^p + k(\frac{1}{2})\|v\|^p, \forall \mu, v \in \Omega_{\zeta}. \end{aligned} \tag{14}$$

Thus, for higher order generalized preinvex functions, the above equation is called the generalized parallelogram-like laws under the Banach spaces.

Several special situations of the generalized parallelogram-like laws are discussed.

(I). If $k(\frac{1}{2}) = \frac{1}{2}$, then, Equation (14) becomes

$$\|G(\mu + \frac{1}{2}\zeta(v, \mu))\|^p + \beta\frac{1}{2^p}\|\zeta(v, \mu)\|^p = k(\frac{1}{2})\|\mu\|^p + k(\frac{1}{2})\|v\|^p, \forall \mu, v \in \Omega_{\zeta}. \tag{15}$$

(II). If $\zeta(v, \mu) = v - \mu$, $k(\frac{1}{2}) = \frac{1}{2}$, and the function G is homogeneous, then, (15) is reduced to the following parallelogram-like law:

$$\|v + \mu\|^p + \beta\|v - \mu\|^p = 2^{p-1}\{\|\mu\|^p + \|v\|^p\}, \tag{16}$$

which is called the parallelogram-like law under the Banach spaces with the preinvex functions. Under the Banach spaces, parallelogram laws are used by Xu [37], Bynum [31], and Chen et al. [38,39] in the field of prediction theory and applied sciences.

(III). If $p = 2$, $k(\frac{1}{2}) = \frac{1}{2}$, the parallelogram law (14) is reduced to

$$\|2\mu + \zeta(v, \mu)\|^2 + \beta\|\zeta(v, \mu)\|^2 = 2\{\|\mu\|^2 + \|v\|^2\}. \tag{17}$$

Using bifunction $\zeta(., .)$, the inner product spaces are described by the new parallelogram law.

(IV). When $\zeta(v, \mu) = v - \mu$, parallelogram law Equation (17) becomes

$$\|v + \mu\|^2 + \beta\|v - \mu\|^2 = 2\{\|\mu\|^2 + \|v\|^2\}. \tag{18}$$

From proper choice of k , bifunction $\zeta(., .)$ and p , various inner products are described by several types of parallelogram laws.

5. Equilibrium-like Problems

The directional equilibrium-like problems are introduced in this section.

Next, the optimality of the differentiable higher-order generalized preinvex functions are discussed.

Theorem 8. Assuming that Φ is a differentiable higher-order generalized preinvex function, when $\mu \in \Omega_{k\zeta}$ is the minimum of G , the following inequality is obtained:

$$\Phi(v) - \Phi(\mu) \geq \beta\|\zeta(v, \mu)\|^p, \quad \forall \mu, v \in \Omega_{k\zeta}. \tag{19}$$

Proof. Because $\mu \in \Omega_{k\zeta}$ is the minimum of Φ ,

$$\Phi(\mu) \leq \Phi(v). \tag{20}$$

When $\Omega_{k\zeta}$ is an invex set, $\forall \mu, v \in \Omega_{k\zeta}$, $\lambda \in [0, 1]$, $v_{\lambda} = \mu + k(\lambda)\zeta(v, \mu) \in \Omega_{k\zeta}$, bringing $v = v_{\lambda}$ in Equation (20), we obtain:

$$0 \leq \lim_{\lambda \rightarrow 0} \left\{ \frac{\Phi(\mu + k(\lambda)\zeta(v, \mu)) - \Phi(\mu)}{k(\lambda)} \right\} = \Phi'_k(\mu, \zeta(v, \mu)). \tag{21}$$

Since Φ is a differentiable higher-order generalized preinvex function,

$$\begin{aligned} \Phi(\mu + k(\lambda)\zeta(v, \mu)) &\leq \Phi(\mu) + k(\lambda)(F(v) - F(\mu)) \\ &\quad - \beta\{k(\lambda)^p(1 - k(\lambda)) + k(\lambda)(1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p, \\ &\forall \mu, v \in \Omega_{k\zeta}, \end{aligned}$$

from Equation (21), we obtain:

$$\begin{aligned} \Phi(v) - \Phi(\mu) &\geq \lim_{\lambda \rightarrow 0} \left\{ \frac{\Phi(\mu + k(\lambda)\zeta(v, \mu)) - \Phi(\mu)}{k(\lambda)} \right\} \\ &\quad + \beta\{k(\lambda)^{p-1}(1 - k(\lambda)) + (1 - k(\lambda))^p\}\|\zeta(v, \mu)\|^p \\ &= \Phi'_k(\mu, \zeta(v, \mu)) + \beta\|\zeta(v, \mu)\|^p. \end{aligned}$$

□

When $\mu \in \Omega_{k\zeta}$ satisfies the following inequality:

$$\Phi'_k(\mu, \zeta(v, \mu)) + \beta\|\zeta(v, \mu)\|^p \geq 0, \quad \forall \mu, v \in \Omega_{\zeta}, \tag{22}$$

we find that $\mu \in \Omega_{k\zeta}$ is the minimum of Φ . The Equation (22) is defined as the higher order generalized equilibrium-like problem.

However, the Equation (22) may not appear as the minimum value of the higher-order generalized preinvex functions. It leads us to research a more general equilibrium-like problem, where Equation (22) is a special case.

For given a constant $\beta > 0$, operator \mathcal{D} and bifunction $\zeta(\cdot, \cdot)$, we investigate the problem of solving $\mu \in \Omega_{k\zeta}$,

$$\mathcal{D}(\mu, \zeta(v, \mu)) + \beta\|\zeta(v, \mu)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta}, p \geq 1, \tag{23}$$

which is said to be the higher-order directional equilibrium-like problem.

1. If $\beta = 0$, then, the higher-order directional equilibrium-like problem Equation (23) is reduced to solving $\mu \in \Omega_{k\zeta}$, such that

$$\mathcal{D}(\mu, \zeta(v, \mu)) \geq 0, \quad \forall v \in \Omega_{k\zeta},$$

which is considered as the bifunction invex equilibrium problem. References [27,29] give the numerical methods and other formulas of the bifunction invex equilibrium problem.

2. For the nonlinear operator \mathcal{A} , if

$$\mathcal{D}(\mu, \zeta(v, \mu)) = \langle \mathcal{A}\mu, \zeta(v, \mu) \rangle,$$

the higher order directional equilibrium-like problem Equation (23) is equivalent to solving $\mu \in \Omega_{k\zeta}$, that is:

$$\langle \mathcal{A}\mu, \zeta(v, \mu) \rangle + \beta\|\zeta(v, \mu)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta},$$

which seems to be a new higher-order strongly *variational-like inequality*.

In order to properly select the spaces, operators, and $k\zeta$ -invex sets, a large group of optimization programming, variational-like inequalities, and equilibrium problem is obtained. We find that the higher-order strongly directional equilibrium-like problems are relatively unified and flexible.

From an auxiliary principle technique which is based on Bergman functions, some iterative methods for equilibrium-like problems (23) are investigated.

For given $\mu \in \Omega_{k\zeta}$ satisfying the equilibrium-like problem (23), the auxiliary problem of solving $w \in \Omega_{k\zeta}$ is considered, that is:

$$\rho \mathcal{D}(w, \zeta(v, w)) + \langle \varphi'_k(w) - \varphi'_k(\mu), v - w \rangle + \rho \|\zeta(v, w)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta}, \quad (24)$$

where $\varphi'_k(\mu)$ is the differential of the strongly preinvex function $\varphi(\mu)$ at $\mu \in \Omega_{k\zeta}$, and $\rho > 0$, which is a constant.

Remark 1. The function $\mathcal{B}(w, \mu) = \varphi(w) - \varphi(\mu) - \varphi'_k(\mu, \zeta(w, \mu))$ related to the preinvex function $\varphi(\mu)$ is defined as the generalized Bregman function. From the strongly preinvexity of $\varphi(\mu)$, the $\mathcal{B}(\cdot, \cdot)$ is nonnegative and $\mathcal{B}(w, \mu) = 0$, if and only if $w = \mu, \forall \mu, w \in \Omega_{k\zeta}$.

If $w = \mu$, then, w is a solution of Equation (23). From the result, the iterative method to solve Equation (23) is obtained as follows.

Algorithm a. For $\mu_0 \in H$, from the iterative scheme

$$\begin{aligned} \rho \mathcal{D}(\mu_{n+1}, \zeta(v, \mu_{n+1})) + \langle \varphi'_k(\mu_{n+1} - \varphi'_k(\mu_n, \zeta(v, \mu_{n+1}))) \\ + \rho \|\zeta(v, \mu)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta}, \end{aligned} \quad (25)$$

computing method of the approximate solution μ_{n+1} is obtained, where $\rho > 0$ is a constant. Algorithm a is defined as the proximal method which solves directional equilibrium-like problem Equation (23).

When $\beta = 0$, then, Algorithm a is changed into:

Algorithm b. For $\mu_0 \in H$, from the iterative scheme

$$\rho \mathcal{D}(\mu_{n+1}, \zeta(v, \mu_{n+1})) + \langle \varphi'_k(\mu_{n+1} - \varphi'_k(\mu_n, \zeta(v, \mu_{n+1}))) \rangle \geq 0, \quad \forall v \in \Omega_{k\zeta},$$

computing method of the approximate solution μ_{n+1} is obtained to solve the bifunction variational-like inequality.

From the proper selections of the spaces and the operators, some known and new algorithms are obtained to solve variational inequalities and related issues.

Definition 21. The bifunction $\mathcal{D}(\cdot, \cdot)$ is defined as

(a) monotone, iff,

$$\mathcal{D}(\mu, \eta(v, \mu)) + \mathcal{D}(v, \zeta(\mu, v)) \leq 0, \quad \forall \mu, v \in \Omega_{k\zeta}.$$

(b) pseudomonotone with $\beta \|\zeta(v, \mu)\|^p$, iff,

$$\begin{aligned} \mathcal{D}(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \geq 0 \\ \implies \\ -\mathcal{D}(v, \zeta(\mu, v)) - \beta \|\zeta(v, \mu)\|^p \geq 0, \quad \forall \mu, v \in \Omega_{k\zeta}. \end{aligned}$$

Theorem 9. Assuming that the bifunction $\mathcal{D}(\cdot, \cdot)$ is pseudomonotone about $\beta \|\zeta(v, \mu)\|^p$, when φ is a differentiable higher order strongly preinvex function when $\beta > 0$, the approximate solution μ_{n+1} which is obtained from Algorithm a, which converges to $\mu \in \Omega_{k\zeta}$, which satisfies the equilibrium-like problem Equation (23).

Proof. Assuming that $\mu \in \Omega_{k\zeta}$ is a solution of Equation (23), thus,

$$\mathcal{D}(\mu, \zeta(v, \mu)) + \beta \|\zeta(v, \mu)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta}.$$

The above equation becomes

$$-\mathcal{D}(v, \zeta(\mu, v)) - \beta \|\zeta(v, \mu)\|^p \geq 0, \quad \forall v \in \Omega_{k\zeta}, \quad (26)$$

since $\mathcal{D}(\cdot, \cdot)$ is pseudomonotone about $\beta\|\zeta(\nu, \mu)\|^p$, and, bringing $\nu = \mu$ into Equation (25) and $\nu = \mu_{n+1}$ into Equation (26), we obtain:

$$\rho\mathcal{D}(\mu_{n+1}, \zeta(\mu, \mu_{n+1})) + \langle \varphi'_k(\mu_{n+1}) - \varphi'_k(\mu_n, \zeta(\mu, \mu_{n+1})) \rangle \geq -\rho\beta\|\zeta(\mu, \mu_{n+1})\|^p, \tag{27}$$

and

$$-\mathcal{D}(\mu_{n+1}, \zeta(\mu, \mu_{n+1})) - \beta\|\zeta(\mu, \mu_{n+1})\|^p \geq 0. \tag{28}$$

Next, the Bregman function is discussed as follows:

$$\mathcal{B}(\mu, w) = E(\mu) - E(w) - \langle \varphi'_k(w, \zeta(\mu, w)) \rangle \geq \beta\|\zeta(\mu, w)\|^p, \tag{29}$$

from higher order strongly preinvexity of E .

Through Equations (27)–(29), we obtain:

$$\begin{aligned} \mathcal{B}(\mu, \mu_n) - \mathcal{B}(\mu, \mu_{n+1}) &= \varphi(\mu_{n+1}) - \varphi(\mu_n) - \langle \varphi'_k(\mu_n, \zeta(\mu, \mu_n)) \rangle \\ &\quad + \langle \varphi'_k(\mu_{n+1}, \zeta(\mu, \mu_{n+1})) \rangle \\ &= \varphi(\mu_{n+1}) - \varphi(\mu_n) - \langle \varphi'_k(\mu_n) - \varphi'_k(\mu_{n+1}, \zeta(\mu, \mu_{n+1})) \rangle \\ &\quad - \langle \varphi'_k(\mu_n, \mu_{n+1} - \mu_n) \rangle \\ &\geq \beta\|\zeta(\mu_{n+1}, \mu_n)\|^p + \langle \varphi'_k(\mu_{n+1}) - \varphi'_k(\mu_n, \zeta(\mu, \mu_{n+1})) \rangle \\ &\geq \beta\|\zeta(\mu_{n+1}, \mu_n)\|^p - \rho\mathcal{D}(\mu_{n+1}, \zeta(\mu, \mu_{n+1})) \\ &\quad - \rho\beta\|\zeta(\mu, \mu_{n+1})\|^p \\ &\geq \beta\|\zeta(\mu_{n+1}, \mu_n)\|^p. \end{aligned}$$

When $\mu_{n+1} = \mu_n$, μ_n is a solution of Equation (23). However, when $\mathcal{B}(\mu, \mu_n) - \mathcal{B}(\mu, \mu_{n+1})$ is nonnegative, the following result is obtained:

$$\lim_{n \rightarrow \infty} \zeta(\zeta(\mu_{n+1}, \mu_n)) = 0.$$

From the above equation, we obtain:

$$\lim_{n \rightarrow \infty} \|\mu_{n+1} - \mu_n\| = 0.$$

Therefore, the $\{\mu_n\}$ is a bounded sequence. Assuming that $\bar{\mu}$ is a cluster point of the subsequence $\{\mu_{n_i}\}$, and $\{\mu_{n_i}\}$ is a subsequence converging to $\bar{\mu}$, according to the method of Zhu and Marcotte [13], the sequence $\{\mu_n\}$ converges to the cluster point $\bar{\mu}$, which satisfies Equation (23). □

From the auxiliary principle technique, another method is considered to solve the higher order directional equilibrium-like problem Equation (23).

For $\mu \in \Omega_{k\zeta}$, $w \in \Omega_{k\zeta}$,

$$\rho\mathcal{D}(\mu, \zeta(\nu, w)) + \langle \varphi'_k(w) - \varphi'_k(\mu), \zeta(\nu, w) \rangle + \rho\beta\|\zeta(\nu, \mu)\|^p \geq 0, \quad \forall \nu \in \Omega_{k\zeta}, \tag{30}$$

where $\varphi'_k(\mu)$ is the differential which is based on a strongly k -preinvex function $\varphi(\mu)$ at $\mu \in \Omega_{k\zeta}$. Because φ is strongly k -preinvex function, Equation (23) has a unique solution. Equations (30) and (25) represent different problems. When $w = \mu$, the solution of Equation (23) is w . This result helps us research another iterative method to solve the higher order directional equilibrium-like problem Equation (23).

Algorithm c. For $\mu_0 \in H$, from the iterative scheme

$$\begin{aligned} \rho \mathcal{D}(\mu_n, \zeta(v, \mu_{n+1})) + \langle \varphi'_k(\mu_{n+1}) - \varphi'_k(\mu_n), \zeta(v, \mu_{n+1}) \rangle \\ \geq -\rho\beta \|\zeta(v, \mu_n)\|^p, \quad \forall v \in \Omega_{k\zeta}, \end{aligned} \quad (31)$$

the approximate solution μ_{n+1} is obtained for solving Equation (23).

If $D(\mu, \zeta(v, \mu)) = \langle \mathcal{A}\mu, \zeta(v, \mu) \rangle$, the Algorithm c becomes:

Algorithm d. For $u_0 \in H$, from the iterative scheme

$$\begin{aligned} \rho \langle \mathcal{A}\mu_n, \zeta(v, \mu_{n+1}) \rangle + \langle \varphi'(\mu_{n+1}) - \varphi'(\mu_n), \zeta(v, \mu_{n+1}) \rangle \\ \geq -\rho\beta \|\zeta(v, \mu_{n+1})\|^p, \quad \forall v \in \Omega_{k\zeta}, \end{aligned}$$

the approximate solution μ_{n+1} is obtained for solving the higher order directional variational-like inequalities.

In order to properly select the spaces and the operators, we obtain some new algorithms to solve higher order directional equilibrium-like problem Equation (23) and optimization problems. From analytical and numerical perspectives, it is an interesting problem.

6. Conclusions

Several new categories of higher order generalized preinvex functions are investigated. We discuss the new characterizations of the generalized preinvex functions, especially their relations with previously results. We derive some parallelograms laws of inner product spaces and Banach spaces. Optimality conditions of the differentiable k -preinvex functions are characterized by a category of directional variational-like inequalities. The result drives us to consider higher order equilibrium-like problems. Some iterative methods to solve higher order directional equilibrium-like problem are investigated from the auxiliary principle technique under the Bregman functions. Several Bregman distance functions are symmetric and are used to discuss the convergence criteria of proposed methods, These concepts highlight the role of symmetry. Some efficient computing methods to solve higher order directional equilibrium-like problem are also proposed and discussed.

In this paper, we have considered the theoretical aspects of the $k\zeta$ -preinvex functions and variational problems. The numerical results for the the preinvex equilibrium problems and k -equilibrium are interesting problems for the future research.

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