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On Some New Inequalities of Hermite–Hadamard Midpoint and Trapezoid Type for Preinvex Functions in (p,q)-Calculus

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Abstract: In this paper, we establish some new Hermite–Hadamard type inequalities for preinvex functions and left-right estimates of newly established inequalities for (p,q)-differentiable preinvex functions in the context of (p,q)-calculus. We also show that the results established in this paper are generalizations of comparable results in the literature of integral inequalities. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Keywords: Hermite–Hadamard inequality; (p,q)-integral; post quantum calculus; convex function

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1. Introduction

The Hermite–Hadamard (H-H) inequality, which was independently found by C. Hermite and J. Hadamard, is particularly important in convex function theory (see, [1–3], and also [4], p. 137).

$$\Pi\left(\frac{\pi_1+\pi_2}{2}\right) \leq \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \leq \frac{\Pi(\pi_1)+\Pi(\pi_2)}{2} \tag{1}$$

where Π is a convex mapping. The aforementioned inequality is true in reverse order for concave maps. Jensen's inequality for convex functions can easily capture this inequality [5]. Several generalizations and extensions to classical convex functions have been proposed in recent years. In [6], the notions about invex function was given that is a significant generalization of convex functions. Weir and Mond introduced the concept of preinvex functions in [7], and it is used in optimization theory in a variety of ways. Prequasi-invex functions are a generalization of the invex functions introduced by Pini in [8]. Following that, the authors looked at some fundamental properties of generalized preinvex functions in [9]. Noor established H-H integral inequalities for preinvex functions in [10–12]. The authors of [13,14] used the ordinary and fractional integrals to calculate the left and right bounds of the H-H inequalities for preinvex functions. More recent results on the integral inequalities for various types of preinvexities can be found in [15–24].

On the other hand, beginning with Euler, various efforts in the subject of q-analysis have been implemented in order to master the mathematics that underpins quantum computing. The phrase q-calculus binds mathematics and physics together. It is employed in subjects including combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and others, as well as relativity theory, mechanics, and quantum theory [25,26]. It has numerous applications in quantum information theory [27,28]. Euler is believed to be the creator of this crucial branch of mathematics since he employed the q-parameter in Newton's work on infinite series. Jackson [29,30] introduced the concept of q-calculus, sometimes known as calculus without limits, for the first time in a proper manner. Al-Salam [31] introduced the concepts of q-fractional integral and q-Riemann–Liouville fractional integral in 1996. Because study in this subject is gradually increasing, Tariboon and Ntouyas [32] proposed the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Bermudo et al. [33] published their ideas regarding the ${}^{\pi_2}D_q$-difference operator and q^{π_2}-integral in 2020. By presenting the principles of (p,q)-calculus, Sadjang [34] broadened the concept of q-calculus. Tunç and Göv [35] introduced the (p,q)-variant of the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Chu et al. established the concepts of ${}^{\pi_2}D_{p,q}$-derivative and $(p,q)^{\pi_2}$-integral in [36], in 2021.

Quantum and post-quantum integrals have been used to study a variety of integral inequalities for a variety of functions. For example, multiple authors in [37–45] gave the H-H inequalities and their right-left estimates for convex and co-ordinated convex functions via ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals. In the setting of q-calculus, Noor et al. [46] employed preinvexity to verify H-H inequalities. Nwaeze et al. [47] discovered several parameterized q-integral inequalities for generalized quasi-convex functions. In [48], Khan et al. used the concept of Green functions to develop some novel H-H type inequalities. In the context of q-calculus, Budak et al. [49], Ali et al. [50,51], and Vivas-Cortez et al. [52] demonstrated new boundaries for Simpson's and Newton's type inequalities for convex and coordinated convex functions. For q-Ostrowski inequality for convex and co-ordinated convex functions, see [53–55]. The authors used the ${}_{\pi_1}D_{p,q}$-difference operator and the $(p,q)_{\pi_1}$-integral to generalize the results of [39] and show H-H type inequalities and their left estimates [56]. The authors recently established the right estimates of H-H type inequalities shown by Kunt et al. [56] in [57]. Reference [36] can be used to solve (p,q)-Ostrowski type inequalities. The findings in [58] are a generalization of [33].

Inspired by the ongoing studies, we give the generalizations of the results proved in [33,39,41,59] by proving H-H trapezoid and midpoint type inequalities for preinvex functions using the concepts of (p,q)-difference operators and (p,q)-integral.

This paper is organized in the following way: Section 2 introduces the basics of q-calculus and discusses other related research in the field. (p,q)-derivatives and integrals are discussed in Section 3. In Section 4, we show that in the (p,q)-calculus setting, H-H type inequalities exist for preinvex functions. Sections 5 and 6 prove new midpoint and trapezoid type inequalities for differentiable preinvex functions via (p,q)-calculus. The link between the findings reported here and analogous findings in the literature is also taken into account. Section 7 summarize the findings and suggests research topics for the future.

2. Quantum Derivatives and Integrals

This section discusses the key concepts and findings that will be needed to prove our critical findings in the next sections.

Definition 1 ([7,9])**.** *A set $\omega \subseteq \mathbb{R}^n$ is known as invex with respect to the given $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if*

$$\varkappa + t\eta(\gamma, \varkappa) \in \omega, \forall\, \varkappa, \gamma \in \omega, t \in [0,1].$$

The η-connected set is a more frequent name for the invex set ω.

Definition 2 ([7,9]). *Consider an invex set $\omega \subseteq \mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A mapping $\Pi : \omega \to \mathbb{R}$ is called preinvex, if*

$$\Pi(\varkappa + t\eta(\gamma, \varkappa)) \leq t\Pi(\gamma) + (1-t)\Pi(\varkappa), \forall\, \varkappa, \gamma \in \omega, t \in [0,1]. \tag{2}$$

If $-\Pi$ is preinvex, the mapping Π is called preconcave.

Remark 1. *Definition 2 becomes the definition of convex functions if $\eta(\gamma, \varkappa) = \gamma - \varkappa$ is set in Definition 2:*

$$\Pi(\varkappa + t(\gamma - \varkappa)) \leq t\Pi(\gamma) + (1-t)\Pi(\varkappa), \forall\, \varkappa, \gamma \in \omega, t \in [0,1].$$

Condition C. [9] The function η satisfies the following condition if

$$\begin{aligned} \eta(\gamma, \gamma + t\eta(\varkappa, \gamma)) &= -t\eta(\varkappa, \gamma), \\ \eta(\varkappa, \gamma + \eta(\varkappa, \gamma)) &= (1-t)\eta(\varkappa, \gamma) \end{aligned} \tag{3}$$

for every $\varkappa, \gamma \in \omega$ and any $t \in [0,1]$. Note that for every $\varkappa, \gamma \in \omega$, $t_1, t_2 \in [0,1]$, and from Condition C, we have the following:

$$\eta(\gamma + t_2\eta(\varkappa, \gamma), \gamma + t_1\eta(\varkappa, \gamma)) = (t_2 - t_1)\eta(\varkappa, \gamma).$$

Theorem 1 ([60] (Jensen's inequality for preinvex functions)). *Let $\Pi : \omega \to \mathbb{R}$ be a preinvex function. Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in [0,1]$ be the coefficients such that $\sum_{i=0}^{n} \gamma_i = 1$, and let $t_1, t_2, \ldots, t_n \in [0,1]$ be the coefficients. Then, the inequality*

$$\Pi\left(\sum_{i=1}^{n} \gamma_i(\varkappa + t_i\eta(\gamma, \varkappa))\right) \leq \sum_{i=1}^{n} \gamma_i \Pi(\varkappa + t_i\eta(\gamma, \varkappa)) \tag{4}$$

holds for all $\varkappa, \gamma \in \omega$.

Set the following notation [26]:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \ \ q \in (0,1).$$

The q-Jackson integral of a mapping Π from 0 to π_2 is given by Jackson [30] which is defined as:

$$\int_0^{\pi_2} \Pi(\varkappa) \ d_q\varkappa \ = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n), \text{ where } 0 < q < 1 \tag{5}$$

assuming that the sum is absolutely convergent. Moreover, over the interval $[\pi_1, \pi_2]$, he gave the following integral of a mapping Π:

$$\int_{\pi_1}^{\pi_2} \Pi(\varkappa) \ d_q\varkappa \ = \int_0^{\pi_2} \Pi(\varkappa) \ d_q\varkappa \ - \int_0^{\pi_1} \Pi(\varkappa) \ d_q\varkappa \ .$$

Definition 3 ([32]). *The q_{π_1}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is defined as:*

$${}_{\pi_1}D_q\Pi(\varkappa) \ = \frac{\Pi(\varkappa) - \Pi(q\varkappa + (1-q)\pi_1)}{(1-q)(\varkappa - \pi_1)}, \ \varkappa \neq \pi_1. \tag{6}$$

For $\varkappa = \pi_1$, we state ${}_{\pi_1}D_q\Pi(\pi_1) = \lim_{\varkappa \to \pi_1} {}_{\pi_1}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 4 ([33]). *The q^{π_2}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$ {}^{\pi_2}D_q\Pi(\varkappa) = \frac{\Pi(q\varkappa + (1-q)\pi_2) - \Pi(\varkappa)}{(1-q)(\pi_2 - \varkappa)}, \; \varkappa \neq \pi_2. \tag{7} $$

For $\varkappa = \pi_2$, we state ${}^{\pi_2}D_q\Pi(\pi_2) = \lim_{\varkappa \to \pi_2} {}^{\pi_2}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 5 ([32]). *The q_{π_1}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is defined as:*

$$ \int_{\pi_1}^{\varkappa} \Pi(\tau) \; {}_{\pi_1}d_q\tau = (1-q)(\varkappa - \pi_1) \sum_{n=0}^{\infty} q^n \Pi(q^n\varkappa + (1-q^n)\pi_1), \; \varkappa \in [\pi_1, \pi_2]. \tag{8} $$

Bermudo et al. [33], on the other hand, state the following concept of the q-definite integral:

Definition 6 ([33]). *The q^{π_2}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is given as:*

$$ \int_{\varkappa}^{\pi_2} \Pi(\tau) \; {}^{\pi_2}d_q\tau = (1-q)(\pi_2 - \varkappa) \sum_{n=0}^{\infty} q^n \Pi(q^n\varkappa + (1-q^n)\pi_2), \; \varkappa \in [\pi_1, \pi_2]. \tag{9} $$

3. Post-Quantum Derivatives and Integrals

We will go over some basic (p, q)-calculus concepts and notations in this section.
The $[n]_{p,q}$ is said to be (p, q)-integers and expressed as:

$$ [n]_{p,q} = \frac{p^n - q^n}{p - q} $$

with $0 < q < p \leq 1$. The $[n]_{p,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}!$ are called (p, q)-factorial and (p, q)-binomial, respectively, and expressed as:

$$ \begin{aligned} [n]_{p,q}! &= \prod_{k=1}^{n} [k]_{p,q}, \; n \geq 1, \; [0]_{p,q}! = 1, \\ \begin{bmatrix} n \\ k \end{bmatrix}! &= \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}. \end{aligned} $$

Definition 7 ([34]). *The (p, q)-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$ D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa) - \Pi(q\varkappa)}{(p-q)\varkappa}, \; \varkappa \neq 0 \tag{10} $$

with $0 < q < p \leq 1$.

Definition 8 ([35]). *The $(p, q)_{\pi_1}$-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$ {}_{\pi_1}D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa + (1-p)\pi_1) - \Pi(q\varkappa + (1-q)\pi_1)}{(p-q)(\varkappa - \pi_1)}, \; \varkappa \neq \pi_1 \tag{11} $$

with $0 < q < p \leq 1$.

For $\varkappa = \pi_1$, we state ${}_{\pi_1}D_{p,q}\Pi(\pi_1) = \lim_{\varkappa \to \pi_1} {}_{\pi_1}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Definition 9 ([36]). *The $(p,q)^{\pi_2}$-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$^{\pi_2}D_{p,q}\Pi(\varkappa) = \frac{\Pi(q\varkappa + (1-q)\pi_2) - \Pi(p\varkappa + (1-p)\pi_2)}{(p-q)(\pi_2 - \varkappa)}, \ \varkappa \neq \pi_2. \tag{12}$$

For $\varkappa = \pi_2$, we state $^{\pi_2}D_{p,q}\Pi(\pi_2) = \lim_{\varkappa \to \pi_2} {}^{\pi_2}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Remark 2. *It is clear that if we use $p = 1$ in (11) and (12), then the equalities (11) and (12) reduce to (6) and (7), respectively.*

Definition 10 ([35]). *The definite $(p,q)_{\pi_1}$-integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is stated as:*

$$\int_{\pi_1}^{\varkappa} \Pi(\tau) \ {}_{\pi_1}d_{p,q}\tau = (p-q)(\varkappa - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\varkappa + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_1 \right) \tag{13}$$

with $0 < q < p \leq 1$.

Definition 11 ([36]). *The definite $(p,q)^{\pi_2}$-integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is stated as:*

$$\int_{\varkappa}^{\pi_2} \Pi(\tau) \ {}^{\pi_2}d_{p,q}\tau = (p-q)(\pi_2 - \varkappa) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\varkappa + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_2 \right) \tag{14}$$

with $0 < q < p \leq 1$.

Remark 3. *It is evident that if we pick $p = 1$ in (13) and (14), then the equalities (13) and (14) change into (8) and (9), respectively.*

Remark 4. *If we take $\pi_1 = 0$ and $\varkappa = \pi_2 = 1$ in (13), then we have*

$$\int_0^1 \Pi(\tau) \ {}_0d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}} \right).$$

Similarly, by taking $\varkappa = \pi_1 = 0$ and $\pi_2 = 1$ in (14), then we obtain that

$$\int_0^1 \Pi(\tau) \ {}^1d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(1 - \frac{q^n}{p^{n+1}} \right).$$

In [56], Kunt et al. proved the following H-H type inequalities for convex functions via the $(p,q)_{\pi_1}$-integral:

Theorem 2. *For a convex mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$, which is differentiable on $[\pi_1, \pi_2]$, the following inequalities hold for the $(p,q)_{\pi_1}$-integral:*

$$\Pi\left(\frac{q\pi_1 + p\pi_2}{[2]_{p,q}} \right) \leq \frac{1}{p(\pi_2 - \pi_1)} \int_{\pi_1}^{p\pi_2 + (1-p)\pi_1} \Pi(\varkappa) \ {}_{\pi_1}d_{p,q}\varkappa \leq \frac{q\Pi(\pi_1) + p\Pi(\pi_2)}{[2]_{p,q}} \tag{15}$$

where $0 < q < p \leq 1$.

Lemma 1 ([58]). *We have the following equalities:*

$$\int_{\pi_1}^{\pi_2} (\pi_2 - \varkappa)^{\alpha} \ {}^{\pi_2}d_{p,q}\varkappa = \frac{(\pi_2 - \pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2} (\varkappa - \pi_1)^{\alpha} \, {}_{\pi_1}d_{p,q}\varkappa = \frac{(\pi_2 - \pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

where $\alpha \in \mathbb{R} - \{-1\}$.

4. New H-H Type Inequalities for Post-Quantum Integrals

We present a new variant of the (p,q)-H-H inequality for preinvex functions in this section. It is also demonstrated that the results presented here are a generalization of some previously published results. For brevity, we use $I = [\pi_2 + \eta(\pi_1, \pi_2), \pi_2]$ and $J = [\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$.

Theorem 3. *For a differentiable preinvex mapping* $\Pi : I \to \mathbb{R}$*, the following inequality holds for the* $(p,q)^{\pi_2}$*-integral:*

$$\begin{aligned} \Pi\left(\frac{p\eta(\pi_1,\pi_2) + [2]_{p,q}\pi_2}{[2]_{p,q}}\right) &\leq \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa) \, {}^{\pi_2}d_{p,q}\varkappa \\ &\leq \frac{p\Pi(\pi_2 + p\eta(\pi_1,\pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}}, \end{aligned} \tag{16}$$

where $0 < q < p \leq 1$.

Proof. For preinvex functions, we can use Jensen's inequality

$$\Pi\left(\frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \varkappa \, {}^{\pi_2}d_{p,q}\varkappa\right) \leq \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa) \, {}^{\pi_2}d_{p,q}\varkappa$$

and from the Definition 11, one can easily observe that

$$\frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \varkappa \, {}^{\pi_2}d_{p,q}\varkappa = \frac{p\eta(\pi_1,\pi_2) + [2]_{p,q}\pi_2}{[2]_{p,q}}.$$

Thus, the first part of the inequality (16) is proved. To prove the second inequality in (16), first, we note that Π is preinvex function and we have

$$\Pi(\pi_2 + t\eta(\pi_1,\pi_2)) \leq t\Pi(\pi_1) + (1-t)\Pi(\pi_2). \tag{17}$$

Applying the $(p,q)^{\pi_2}$-integral on the both sides of (17), we have

$$\frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa) \, {}^{\pi_2}d_{p,q}\varkappa \leq \frac{p\Pi(\pi_2 + p\eta(\pi_1,\pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}}.$$

Hence, the proof is completed. □

Remark 5. *We obtain Theorem 5 in [59], by letting* $p = 1$ *in Theorem 3.*

Remark 6. *If we set* $p = 1$ *in Theorem 3 and later assume that* $\eta(\pi_2,\pi_1) = -\eta(\pi_1,\pi_2) = \pi_2 - \pi_1$*, then Theorem 3 becomes Theorem 12 in [33].*

Example 1. *Let* $\Pi(\varkappa) = -|\varkappa|$*. Then,* Π *is preinvex function with respect to the following bifunction:*

$$\eta(\varkappa, y) = \begin{cases} \varkappa - y, & \text{if } \varkappa y \geq 0 \\ y - \varkappa, & \text{if } \varkappa y \leq 0. \end{cases}$$

1. *Let us consider $\pi_1, \pi_2 > 0$, then $\eta(\pi_1, \pi_2) = \pi_1 - \pi_2$ and*

$$\begin{aligned} \Pi\left(\frac{p\eta(\pi_1, \pi_2) + [2]_{p,q}\pi_2}{[2]_{p,q}}\right) &= -\frac{p\pi_1 + q\pi_2}{[2]_{p,q}}, \\ \frac{1}{p\eta(\pi_2, \pi_1)} \int_{\pi_2 + p\eta(\pi_1, \pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa &= -\frac{p\pi_1 + q\pi_2}{[2]_{p,q}} \end{aligned}$$

and

$$\frac{p\Pi(\pi_2 + p\eta(\pi_1, \pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} = -\frac{p\pi_1 + q\pi_2}{[2]_{p,q}}.$$

2. *Let $\pi_1, \pi_2 < 0$. Then, $\eta(\pi_1, \pi_2) = \pi_1 - \pi_2$ and*

$$\begin{aligned} \Pi\left(\frac{p\eta(\pi_1, \pi_2) + [2]_{p,q}\pi_2}{[2]_{p,q}}\right) &= \frac{p\pi_1 + q\pi_2}{[2]_{p,q}}, \\ \frac{1}{p\eta(\pi_2, \pi_1)} \int_{\pi_2 + p\eta(\pi_1, \pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa &= \frac{p\pi_1 + q\pi_2}{[2]_{p,q}} \end{aligned}$$

and

$$\frac{p\Pi(\pi_2 + p\eta(\pi_1, \pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} = -\frac{p\pi_1 + q\pi_2}{[2]_{p,q}}.$$

3. *Finally, let $\pi_1 < 0 < \pi_2$. Then, $\eta(\pi_1, \pi_2) = \pi_2 - \pi_1$ and*

$$\begin{aligned} \Pi\left(\frac{p\eta(\pi_1, \pi_2) + [2]_{p,q}\pi_2}{[2]_{p,q}}\right) &= -\frac{(2p + q)\pi_2 - p\pi_1}{[2]_{p,q}}, \\ \frac{1}{p\eta(\pi_2, \pi_1)} \int_{\pi_2 + p\eta(\pi_1, \pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa &= -\frac{(2p + q)\pi_2 - p\pi_1}{[2]_{p,q}} \end{aligned}$$

and

$$\frac{p\Pi(\pi_2 + p\eta(\pi_1, \pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} = -\frac{(2p + q)\pi_2 - p\pi_1}{[2]_{p,q}}.$$

It is clear that the Theorem 3 is valid.

Theorem 4. *For a differentiable preinvex mapping $\Pi : J \to \mathbb{R}$, the following inequality holds for $(p, q)_{\pi_1}$-integral:*

$$\begin{aligned} \Pi\left(\frac{p\eta(\pi_2, \pi_1) + [2]_{p,q}\pi_1}{[2]_{p,q}}\right) &\leq \frac{1}{p\eta(\pi_2, \pi_1)} \int_{\pi_1}^{\pi_1 + p\eta(\pi_2, \pi_1)} \Pi(\varkappa)\ {}_{\pi_1}d_{p,q}\varkappa \\ &\leq \frac{p\Pi(\pi_1) + q\Pi(\pi_1 + \eta(\pi_2, \pi_1))}{[2]_{p,q}}, \end{aligned} \tag{18}$$

where $0 < q < p \leq 1$.

Proof. One can easily obtain the inequality (18) by following the methodology used in the proof of Theorem 3 and taking into account Definition 10 of the $(p, q)_{\pi_1}$-integral. □

Remark 7. *We obtain Theorem 6 in [59] by letting $p = 1$ in Theorem 4.*

Remark 8. *If we set $p = 1$ in Theorem 4 and later assume that $\eta(\pi_2, \pi_1) = \pi_2 - \pi_1$, then Theorem 4 reduces to Theorem 6 in [39].*

5. Midpoint Type Inequalities through $(p,q)^{\pi_2}$-Integral

We present some new midpoint type inequalities using the (p,q)-derivative and integral in this section.

The following crucial lemma is required to prove the main results of this section.

Lemma 2. *Let $\Pi : I \to \mathbb{R}$ be a differentiable function on I°. If ${}^{\pi_2}D_{p,q}\Pi$ is continuous and integrable on I, then we have the following identity:*

$$\begin{aligned}
&\eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}} qt\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right.\\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^1 (qt-1)\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right]\\
=\;&\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right), \qquad (19)
\end{aligned}$$

where $0<q<p\le 1$.

Proof. From Definition 9, we have

$$\begin{aligned}
&{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\\
=\;&\frac{\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)-\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)}{t\eta(\pi_2,\pi_1)(p-q)}. \qquad (20)
\end{aligned}$$

From the left side of equality (19), we have

$$\begin{aligned}
&\eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}} qt\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right.\\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^1 (qt-1)\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right]\\
=\;&\eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}} {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right.\\
&+\int_0^1 qt\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\\
&\left.-\int_0^1 {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\right]. \qquad (21)
\end{aligned}$$

By the equality (14), we have

$$\begin{aligned}
&\int_0^{\frac{p}{[2]_{p,q}}} {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t\\
=\;&\frac{1}{\eta(\pi_2,\pi_1)(p-q)}\int_0^{\frac{p}{[2]_{p,q}}}\frac{\begin{array}{c}\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)\\-\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)\end{array}}{t}d_{p,q}t\\
=\;&\frac{1}{\eta(\pi_2,\pi_1)}\left[\sum_{n=0}^{\infty}\Pi\left(\frac{p}{[2]_{p,q}}\frac{q^{n+1}}{p^{n+1}}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{p}{[2]_{p,q}}\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right)\right.\\
&\left.-\sum_{n=0}^{\infty}\Pi\left(\frac{p}{[2]_{p,q}}\frac{q^{n}}{p^{n}}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{p}{[2]_{p,q}}\frac{q^{n}}{p^{n}}\right)\pi_2\right)\right]
\end{aligned}$$

$$= \frac{\Pi(\pi_2)}{\eta(\pi_2,\pi_1)} - \frac{1}{\eta(\pi_2,\pi_1)}\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right), \tag{22}$$

$$\begin{aligned}
&\int_0^1 {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t \\
= &\frac{1}{\eta(\pi_2,\pi_1)(p-q)}\int_0^1 \frac{\begin{array}{c}\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)\\ -\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)\end{array}}{t}d_{p,q}t \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\begin{array}{c}\sum_{n=0}^{\infty}\Pi\left(\frac{q^{n+1}}{p^{n+1}}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right)\\ -\sum_{n=0}^{\infty}\Pi\left(\frac{q^n}{p^n}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^n}{p^n}\right)\pi_2\right)\end{array}\right] \\
= &\frac{\Pi(\pi_2)-\Pi(\pi_2+\eta(\pi_1,\pi_2))}{\eta(\pi_2,\pi_1)}
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\int_0^1 t\,{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)d_{p,q}t \\
= &\frac{1}{\eta(\pi_2,\pi_1)(p-q)}\int_0^1\left[\begin{array}{c}\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)\\ -\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)\end{array}\right]d_{p,q}t \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\begin{array}{c}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^{n+1}}{p^{n+1}}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right)\\ -\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^n}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^n}{p^n}\right)\pi_2\right)\end{array}\right] \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\begin{array}{c}\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}\Pi\left(\frac{q^{n+1}}{p^{n+1}}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right)\\ -\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\Pi\left(\frac{q^n}{p^n}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^n}{p^n}\right)\pi_2\right)\end{array}\right] \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\begin{array}{c}\left(\frac{1}{q}-\frac{1}{p}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^n}\Pi\left(\frac{q^n}{p^n}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^n}{p^n}\right)\pi_2\right)\\ -\frac{1}{q}\Pi((\pi_2+\eta(\pi_1,\pi_2)))\end{array}\right] \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\frac{p-q}{pq}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\Pi\left(\frac{q^n}{p^n}(\pi_2+\eta(\pi_1,\pi_2))+\left(1-\frac{q^n}{p^n}\right)\pi_2\right)\right. \\
&\left.-\frac{1}{q}\Pi(\pi_2+\eta(\pi_1,\pi_2))\right] \\
= &\frac{1}{\eta(\pi_2,\pi_1)}\left[\frac{1}{pq\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa-\frac{1}{q}\Pi(\pi_2+\eta(\pi_1,\pi_2))\right].
\end{aligned} \tag{24}$$

By using (22)–(24) in (21), we obtain the desired identity (19). Thus, the proof is completed. □

Remark 9. *We obtain Lemma 3 in [59] by letting $p = 1$ in Lemma 2.*

Remark 10. *If we use $p = 1$ in Lemma 2 and later consider $\eta(\pi_2,\pi_1) = -\eta(\pi_1,\pi_2) = \pi_2 - \pi_1$, then Lemma 2 becomes Lemma 2 in [41].*

Theorem 5. *Suppose that the assumptions of Lemma 2 hold. If* $\left|{}^{\pi_2}D_{p,q}\Pi\right|$ *is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned}
&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right| \\
\leq\ & \eta(\pi_2,\pi_1)\left[\left(\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|A_1(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|A_2(p,q)\right)\right. \\
&\left.+\left(\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|A_3(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|A_4(p,q)\right)\right], \qquad (25)
\end{aligned}$$

where

$$\begin{aligned}
A_1(p,q) &= \frac{qp^3}{[2]_{p,q}^3[3]_{p,q}}, \\
A_2(p,q) &= \frac{q\left(p^3(p^2+q^2-p)+p^2[3]_{p,q}\right)}{[2]_{p,q}^4[3]_{p,q}}, \\
A_3(p,q) &= \frac{q(q+2p)}{[2]_{p,q}}-\frac{q^2(q^2+3p^2+3pq)}{[2]_{p,q}^3[3]_{p,q}}, \\
A_4(p,q) &= \frac{q}{[2]_{p,q}}-\frac{q^2(q+2p)}{[2]_{p,q}^4}-A_3(p,q).
\end{aligned}$$

Proof. Taking the modulus in Lemma 2 and using the preinvexity of $\left|{}^{\pi_2}D_{p,q}\Pi\right|$, we obtain that

$$\begin{aligned}
&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right| \\
\leq\ & \eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}}qt\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right. \\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^1(1-qt)\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right] \\
\leq\ & \eta(\pi_2,\pi_1)\left[q\int_0^{\frac{p}{[2]_{p,q}}}t\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|\right)d_{p,q}t\right. \\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^1(1-qt)\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|\right)d_{p,q}t\right]. \qquad (26)
\end{aligned}$$

One can easily compute the integrals that appeared on the right side of the inequality (26)

$$\begin{aligned}
\int_0^{\frac{p}{[2]_{p,q}}}t^2d_{p,q}t &= \frac{p^3}{[2]_{p,q}^3[3]_{p,q}}, \qquad (27) \\
\int_0^{\frac{p}{[2]_{p,q}}}t(1-t)d_{p,q}t &= \frac{p^3(p^2+q^2-p)+p^2[3]_{p,q}}{[2]_{p,q}^4[3]_{p,q}}, \qquad (28) \\
\int_{\frac{p}{[2]_{p,q}}}^1 t(1-qt)d_{p,q}t &= \frac{q(q+2p)}{[2]_{p,q}}-\frac{q^2(q^2+3p^2+3pq)}{[2]_{p,q}^3[3]_{p,q}}, \qquad (29)
\end{aligned}$$

$$\int_{\frac{p}{[2]_{p,q}}}^1(1-t)(1-qt)d_{p,q}t = \frac{q}{[2]_{p,q}}-\frac{q^2(q+2p)}{[2]_{p,q}^3}$$

$$-\left(\frac{q(q+2p)}{[2]_{p,q}}-\frac{q^2(q^2+3p^2+3pq)}{[2]^3_{p,q}[3]_{p,q}}\right). \tag{30}$$

Making use of (27)–(30) in (26), gives us the required inequality (25). Hence, the proof is finished. □

Remark 11. *We obtain Theorem 9 in [59] by letting $p=1$ in Theorem 5.*

Remark 12. *If we take $p=1$ in Theorem 5 and later consider $\eta(\pi_2,\pi_1)=-\eta(\pi_1,\pi_2)=\pi_2-\pi_1$, then Theorem 5 reduces to Theorem 5 in [41].*

Theorem 6. *Suppose that the assumptions of Lemma 2 hold. If $\left|{}^{\pi_2}D_{p,q}\Pi\right|^r$, $r\geq 1$ is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned}&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right|\\
\leq\ &\eta(\pi_2,\pi_1)\left(\frac{p^2}{\left([2]_{p,q}\right)^3}\right)^{1-\frac{1}{r}}\left[\left\{\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^rA_1(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^rA_2(p,q)\right\}^{\frac{1}{r}}\right.\\
&\left.+\left\{\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^rA_3(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^rA_4(p,q)\right\}^{\frac{1}{r}}\right],\end{aligned} \tag{31}$$

where $A_1(p,q)-A_4(p,q)$ are given in Theorem 5.

Proof. Taking the modulus in Lemma 2, applying the well-known power mean inequality for (p,q)-integrals and the preinvexity of $\left|{}^{\pi_2}D_{p,q}\Pi\right|^r$, $r\geq 1$, we have

$$\begin{aligned}&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right|\\
\leq\ &\eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}}qt\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right.\\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^{1}(1-qt)\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right]\\
\leq\ &\eta(\pi_2,\pi_1)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}}qtd_{p,q}t\right)^{1-\frac{1}{r}}\right.\\
&\times\left\{q\int_0^{\frac{p}{[2]_{p,q}}}t\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\right)d_{p,q}t\right\}^{\frac{1}{r}}\\
&+\left(\int_{\frac{p}{[2]_{p,q}}}^{1}(1-qt)d_{p,q}t\right)^{1-\frac{1}{r}}\\
&\left.\times\left\{q\int_{\frac{p}{[2]_{p,q}}}^{1}(1-qt)\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\right)d_{p,q}t\right\}^{\frac{1}{r}}\right]\\
=\ &\eta(\pi_2,\pi_1)\left(\frac{p^2}{\left([2]_{p,q}\right)^3}\right)^{1-\frac{1}{r}}\left[\left\{\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^rA_1(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^rA_2(p,q)\right\}^{\frac{1}{r}}\right.\end{aligned}$$

$$+\left\{\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r A_3(p,q)+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r A_4(p,q)\right\}^{\frac{1}{r}}\Bigg]$$

which completes the proof. □

Remark 13. *We obtain Theorem 10 in [59] by letting $p=1$ in Theorem 6.*

Remark 14. *If we put $p=1$ in Theorem 6 and later assume that $\eta(\pi_2,\pi_1)=-\eta(\pi_1,\pi_2)$ $=\pi_2-\pi_1$, then Theorem 6 reduces to Theorem 6 in [41].*

Theorem 7. *Suppose that the assumptions of Lemma 2 hold. If $\left|{}^{\pi_2}D_{p,q}\Pi\right|^r$, $r>1$ is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned}
&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right|\\
\leq\ & q\eta(\pi_2,\pi_1)\left[\left(\left(\frac{p}{[2]_{p,q}}\right)^{s+1}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)\right)^{\frac{1}{s}}\left\{\begin{array}{c}\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r\left(\frac{p^2}{[2]_{p,q}^3}\right)\\+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\left(\frac{p^3+pq^2+2p^2q-p^2}{[2]_{p,q}^3}\right)\end{array}\right\}^{\frac{1}{r}}\right.\\
&\left.+\left(\int_{\frac{p}{[2]_{p,q}}}^{1}\left(\frac{1}{q}-t\right)^s d_{p,q}t\right)^{\frac{1}{s}}\left\{\begin{array}{c}\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r\left(\frac{[2]_{p,q}-p^2}{[2]_{p,q}^3}\right)\\+\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\left(\frac{q[2]_{p,q}^2+p^2-p-q}{[2]_{p,q}^3}\right)\end{array}\right\}^{\frac{1}{r}}\right],
\end{aligned}\tag{32}$$

where $s+r=sr$.

Proof. Taking the modulus in Lemma 2, by applying the well-known Hölder inequality for definite (p,q)-integrals and the preinvexity of $\left|{}^{\pi_2}D_{p,q}\Pi\right|^r$, $r>1$, we obtain that

$$\begin{aligned}
&\left|\frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\eta(\pi_1,\pi_2)+[2]_{p,q}\pi_2}{[2]_{p,q}}\right)\right|\\
\leq\ & q\eta(\pi_2,\pi_1)\left[\int_0^{\frac{p}{[2]_{p,q}}}t\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right.\\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^{1}\left(\frac{1}{q}-t\right)\left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\right|d_{p,q}t\right]\\
\leq\ & q\eta(\pi_2,\pi_1)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}}t^s d_{p,q}t\right)^{\frac{1}{s}}\right.\\
&\left\{\int_0^{\frac{p}{[2]_{p,q}}}\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\right)d_{p,q}t\right\}^{\frac{1}{r}}\\
&+\left(\int_{\frac{p}{[2]_{p,q}}}^{1}\left(\frac{1}{q}-t\right)^s d_{p,q}t\right)^{\frac{1}{s}}\\
&\left.\times\left\{\int_{\frac{p}{[2]_{p,q}}}^{1}\left(t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r+(1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r\right)d_{p,q}t\right\}^{\frac{1}{r}}\right].
\end{aligned}\tag{33}$$

One can easily evaluate the integrals that appear on the right side of the inequality (33)

$$\begin{aligned}
\left(\int_0^{\frac{p}{[2]_{p,q}}} t^s d_{p,q}t\right)^{\frac{1}{s}} &= \left(\left(\frac{p}{[2]_{p,q}}\right)^{s+1}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)\right)^{\frac{1}{s}} && (34)\\
\int_0^{\frac{p}{[2]_{p,q}}} t d_{p,q}t &= \frac{p^2}{[2]_{p,q}^3}, && (35)\\
\int_0^{\frac{p}{[2]_{p,q}}} (1-t) d_{p,q}t &= \frac{p^3+pq^2+2p^2q-p^2}{[2]_{p,q}^3}, && (36)\\
\int_{\frac{p}{[2]_{p,q}}}^1 t d_{p,q}t &= \frac{[2]_{p,q}-p^2}{[2]_{p,q}^3}, && (37)\\
\int_{\frac{p}{[2]_{p,q}}}^1 (1-t) d_{p,q}t &= \frac{q[2]_{p,q}^2+p^2-p-q}{[2]_{p,q}^3}. && (38)
\end{aligned}$$

Making use of (34)–(38), gives us the required inequality (32). Hence, the proof is accomplished. □

6. Trapezoidal Type Inequalities through $(p,q)^{\pi_2}$-Integral

In this section, we give some new trapezoidal inequalities by using the (p,q)-derivative and integral.

To prove the main results of this section, we need the following crucial lemma.

Lemma 3. *Let $\Pi : I \to \mathbb{R}$ be a differentiable function on I°. If ${}^{\pi_2}D_{p,q}\Pi$ is continuous and integrable on I, then we have the following identity:*

$$\begin{aligned}
&\frac{p\Pi(\pi_2+p\eta(\pi_1,\pi_2))+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)}\int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa\\
=\ &\frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}}\int_0^1\left(1-[2]_{p,q}t\right)\ {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\ d_{p,q}t, \qquad (39)
\end{aligned}$$

where $0<q<p\le 1$.

Proof. From (20) and the right side of (39), we obtain that

$$\begin{aligned}
&\frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}}\int_0^1\left(1-[2]_{p,q}t\right)\ {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\ d_{p,q}t\\
=\ &\frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}}\left[\frac{1}{\eta(\pi_2,\pi_1)(p-q)}\int_0^1\frac{\begin{array}{c}\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)\\ -\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)\end{array}}{t}\,d_{p,q}t\right.\\
&\left.-\frac{[2]_{p,q}}{\eta(\pi_2,\pi_1)(p-q)}\int_0^1\left[\begin{array}{c}\Pi(qt(\pi_2+\eta(\pi_1,\pi_2))+(1-qt)\pi_2)\\ -\Pi(pt(\pi_2+\eta(\pi_1,\pi_2))+(1-pt)\pi_2)\end{array}\right]d_{p,q}t\right].
\end{aligned}$$

From (23) and (24), we have

$$\frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}}\int_0^1\left(1-[2]_{p,q}t\right)\ {}^{\pi_2}D_{p,q}\Pi(t(\pi_2+\eta(\pi_1,\pi_2))+(1-t)\pi_2)\ d_{p,q}t$$

$$= \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left[\begin{array}{c} \frac{\Pi(\pi_2)-\Pi(\pi_2+\eta(\pi_1,\pi_2))}{\eta(\pi_2,\pi_1)} \\ -\frac{[2]_{p,q}}{\eta(\pi_2,\pi_1)} \left\{ \frac{1}{pq\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa - \frac{1}{q}\Pi(\pi_2+\eta(\pi_1,\pi_2)) \right\} \end{array} \right]$$

where the identity (39) is obtained and the proof is accomplished. □

Remark 15. *We obtain Lemma 2 in [59] by letting $p = 1$ in Lemma 3.*

Remark 16. *If we adopt $p = 1$ in Lemma 3 and later we assume that $\eta(\pi_2, \pi_1) = -\eta(\pi_1, \pi_2) = \pi_2 - \pi_1$, then Lemma 3 becomes Lemma 1 in [41].*

Theorem 8. *Suppose that the assumptions of Lemma 3 hold. If $\left|{}^{\pi_2}D_{p,q}\Pi\right|$ is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned} &\left| \frac{p\Pi(\pi_2+p\eta(\pi_1,\pi_2))+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\ \leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left[\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right| A_5(p,q) + \left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right| A_6(p,q) \right], \end{aligned} \tag{40}$$

where

$$\begin{aligned} A_5(p,q) &= \int_0^1 t\left|\left(1-[2]_{p,q}t\right)\right| d_{p,q}t, \\ A_6(p,q) &= \int_0^1 (1-t)\left|\left(1-[2]_{p,q}t\right)\right| d_{p,q}t. \end{aligned}$$

Proof. Taking the modulus in Lemma 3 and using the preinvexity of $\left|{}^{\pi_2}D_{p,q}\Pi\right|$, we have

$$\begin{aligned} &\left| \frac{p\Pi(\pi_2+p\eta(\pi_1,\pi_2))+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\ \leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \int_0^1 t\left|\left(1-[2]_{p,q}t\right)\right| \left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right| d_{p,q}t \\ &+ \int_0^1 (1-t)\left|\left(1-[2]_{p,q}t\right)\right| \left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right| d_{p,q}t \\ =\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left[\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right| A_5(p,q) + \left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right| A_6(p,q) \right]. \end{aligned} \tag{41}$$

Thus, the proof is completed. □

Remark 17. *We obtain Theorem 7 in [59] by letting $p = 1$ in Theorem 8.*

Remark 18. *If we adopt $p = 1$ in Theorem 8 and later we assume that $\eta(\pi_2, \pi_1) = -\eta(\pi_1, \pi_2) = \pi_2 - \pi_1$, then Theorem 8 becomes Theorem 7 in [41].*

Theorem 9. *Suppose that the assumptions of Lemma 3 hold. If $\left|{}^{\pi_2}D_{p,q}\Pi\right|^r$, $r \geq 1$ is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned} &\left| \frac{p\Pi(\pi_2+p\eta(\pi_1,\pi_2))+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\ \leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1-[2]_{p,q}t\right| d_{p,q}t \right)^{1-\frac{1}{r}} \\ &\times \left[\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r A_5(p,q) + \left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r A_6(p,q) \right]^{\frac{1}{r}}, \end{aligned} \tag{42}$$

where $A_5(p,q)$ and $A_6(p,q)$ are given in Theorem 8.

Proof. Taking the modulus in Lemma 3 and applying the well-known power mean inequality for (p,q)-integrals and the preinvexity of $\left| {}^{\pi_2}D_{p,q}\Pi \right|^r$, $r \geq 1$, we obtain that

$$\begin{aligned}
&\left| \frac{p\Pi(\pi_2 + p\eta(\pi_1,\pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1 - [2]_{p,q}t\right| d_{p,q}t \right)^{1-\frac{1}{r}} \\
&\times \left[\int_0^1 \left|1 - [2]_{p,q}t\right| \left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2 + \eta(\pi_1,\pi_2)) + (1-t)\pi_2)\right|^r d_{p,q}t \right]^{\frac{1}{r}} \\
\leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1 - [2]_{p,q}t\right| d_{p,q}t \right)^{1-\frac{1}{r}} \left[\int_0^1 t\left|\left(1 - [2]_{p,q}t\right)\right| \left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r d_{p,q}t \right. \\
&\left. + \int_0^1 (1-t)\left|\left(1 - [2]_{p,q}t\right)\right| \left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r d_{p,q}t \right]^{\frac{1}{r}} \\
=\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1 - [2]_{p,q}t\right| d_{p,q}t \right)^{1-\frac{1}{r}} \\
&\times \left[\left| {}^{\pi_2}D_{p,q}\Pi(\pi_1) \right|^r A_5(p,q) + \left| {}^{\pi_2}D_{p,q}\Pi(\pi_2) \right|^r A_6(p,q) \right]^{\frac{1}{r}}.
\end{aligned} \tag{43}$$

Thus, the proof is finished. □

Remark 19. *We obtain Theorem 8 in [59] by letting $p = 1$ in Theorem 9.*

Remark 20. *If we adopt $p = 1$ in Theorem 9 and later we assume that $\eta(\pi_2,\pi_1) = -\eta(\pi_1,\pi_2) = \pi_2 - \pi_1$, then Theorem 9 becomes Theorem 4 in [41].*

Theorem 10. *Suppose that the assumptions of Lemma 3 hold. If $\left| {}^{\pi_2}D_{p,q}\Pi \right|^r$, $r > 1$ is a preinvex function over I, then we have the following new inequality:*

$$\begin{aligned}
&\left| \frac{p\Pi(\pi_2 + p\eta(\pi_1,\pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1 - [2]_{p,q}t\right|^s d_{p,q}t \right)^{\frac{1}{s}} \left[\frac{\left| {}^{\pi_2}D_{p,q}\Pi(\pi_1) \right|^r + \left([2]_{p,q} - 1\right) \left| {}^{\pi_2}D_{p,q}\Pi(\pi_2) \right|^r}{[2]_{p,q}} \right]^{\frac{1}{r}},
\end{aligned} \tag{44}$$

where $s + r = sr$.

Proof. Taking the modulus in Lemma 3 and applying the well-known Hölder inequality for (p,q)-integrals and the preinvexity of $\left| {}^{\pi_2}D_{p,q}\Pi \right|^r$, $r > 1$, we obtain that

$$\begin{aligned}
&\left| \frac{p\Pi(\pi_2 + p\eta(\pi_1,\pi_2)) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{p\eta(\pi_2,\pi_1)} \int_{\pi_2+p\eta(\pi_1,\pi_2)}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\leq\ & \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}} \left(\int_0^1 \left|1 - [2]_{p,q}t\right|^s d_{p,q}t \right)^{\frac{1}{s}} \\
&\times \left[\int_0^1 \left|{}^{\pi_2}D_{p,q}\Pi(t(\pi_2 + \eta(\pi_1,\pi_2)) + (1-t)\pi_2)\right|^r d_{p,q}t \right]^{\frac{1}{r}}
\end{aligned}$$

$$\leq \frac{q\eta(\pi_2,\pi_1)}{[2]_{p,q}}\left(\int_0^1 \left|1-[2]_{p,q}t\right|^s d_{p,q}t\right)^{\frac{1}{s}}$$
$$\times\left[\int_0^1 t\left|{}^{\pi_2}D_{p,q}\Pi(\pi_1)\right|^r d_{p,q}t+\int_0^1 (1-t)\left|{}^{\pi_2}D_{p,q}\Pi(\pi_2)\right|^r d_{p,q}t\right]^{\frac{1}{r}}. \tag{45}$$

We can calculate the integrals that occur on the right side of (45) as follows:

$$\int_0^1 t\, d_{p,q}t = \frac{1}{[2]_{p,q}}, \tag{46}$$

$$\int_0^1 (1-t)\, d_{p,q}t = \frac{[2]_{p,q}-1}{[2]_{p,q}}. \tag{47}$$

Making use of (46) and (47) in (45), gives the desired result. Hence, the proof is completed. □

Remark 21. *The left-right estimates of inequality (18) given in Theorem 4 that we left for the readers can be obtained by using the notions of $(p,q)_{\pi_1}$-derivative and integral, as well as the techniques used in the previous two sections.*

7. Conclusions

In this paper, we proved H-H type inequalities for preinvex functions using the (p,q)-calculus setup. For (p,q)-differentiable preinvex functions, we also proved some new midpoint-formula-type and trapezoid-formula-type inequalities. Furthermore, we demonstrated that the newly discovered inequalities are generalizations of the inequalities for convex functions in (p,q)-calculus. This study's conclusions can be used in symmetry. The results for symmetric functions can be reached by employing the notions of symmetric convex functions, which will be explored further in future work. It is an intriguing and novel problem, and future researchers will be able to obtain similar inequalities for co-ordinated preinvex functions in their studies.

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