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On 3-Rainbow Domination Number of Generalized Petersen Graphs $P(6k, k)$

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Abstract: We obtain new results on 3-rainbow domination numbers of generalized Petersen graphs $P(6k, k)$. In some cases, for some infinite families, exact values are established; in all other cases, the lower and upper bounds with small gaps are given. We also define singleton rainbow domination, where the sets assigned have a cardinality of, at most, one, and provide analogous results for this special case of rainbow domination.

Keywords: rainbow domination; rainbow domination number; generalized Petersen graphs

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1. Introduction

Inspired by several facility location problems, Brešar, Henning and Rall [1–3] initiated the study of the k-rainbow domination problem. The problem is proved to be NP-complete, even if the input graph is a chordal graph or a bipartite graph [2]. This variation of the general domination problem has already attracted considerable attention. The growing interest in domination problems [4] is based on a variety of practical applications on one hand and, on the other hand, expected (and usually proven) intractability on general graphs.

In [5], 2-rainbow domination of generalized Petersen graphs was extensively studied. Here, we continue this avenue of research and study the 3-rainbow domination of generalized Petersen graphs.

First, we recall the definitions and some other preliminary material from [5].

1.1. Graphs and Rainbow Domination

A (simple) graph $G = (V(G), E(G))$ is a combinatorial object, where $V = V(G)$ is a set whose elements are called vertices, and $E = E(G)$ is a set of edges. Edges are pairs of vertices: $e = \{u, v\}$. If $\{u, v\} \in E(G)$, then we say that vertices u and v are neighbors. The set of all neighbors of a vertex is its neighborhood. The degree of a vertex is the number of its neighbors. A graph is called 3-regular or *cubic* if all vertices are of degree three. Graph H is an *induced subgraph* of graph G if $V(H) \subseteq V(G)$ and for any pair of vertices $v, u \in v(H)$, $\{u, v\} \in E(G)$ implies $\{u, v\} \in E(H)$. The *interval* of integers is denoted by $[i, j] = \{k \in \mathbb{N} \mid i \leq k \leq j\}$. All subscripts in this paper are taken as modulo n.

Given a graph G and a positive integer t, the goal is to assign a subset of the color set $\{1, 2, \cdots, t\}$ to every vertex of G such that every vertex with the empty set assigned has all t colors in its neighborhood. Such an assignment is called a *t-rainbow dominating function* (tRD function) of the graph G. The weight of assignment g, a $tRDF$ of a graph G, is the value $w(g) = \sum_{v \in V(G)} |g(v)|$. We also say that G is *tRD-colored* (or simply, colored) by g. A vertex is tRD-dominated if either it is assigned a nonempty set of colors, or it has all colors in its neighborhood. A vertex is said to be *colored* if $g(v) \neq \emptyset$ and is not colored or *uncolored* otherwise. The *t-rainbow domination number* $\gamma_{rt}(G)$ is the minimum weight over all tRD functions in G. We also study a special case where vertices are colored by sets with, at most, one color. Such functions are called *singleton tRD functions* ($StRD$ functions) and the minimal weight obtained when considering only $StRD$ functions is *singleton t-rainbow domination number*, denoted by $\tilde{\gamma}_{rt}$.

1.2. Generalized Petersen Graphs

For $n \geq 3$ and k, $1 \leq k \leq n-1$, the generalized Petersen graph $P(n,k)$ is a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i \mid 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\} \mid 0 \leq i \leq n-1\}$. This standard notation was introduced by Watkins [6] (see Figure 1).

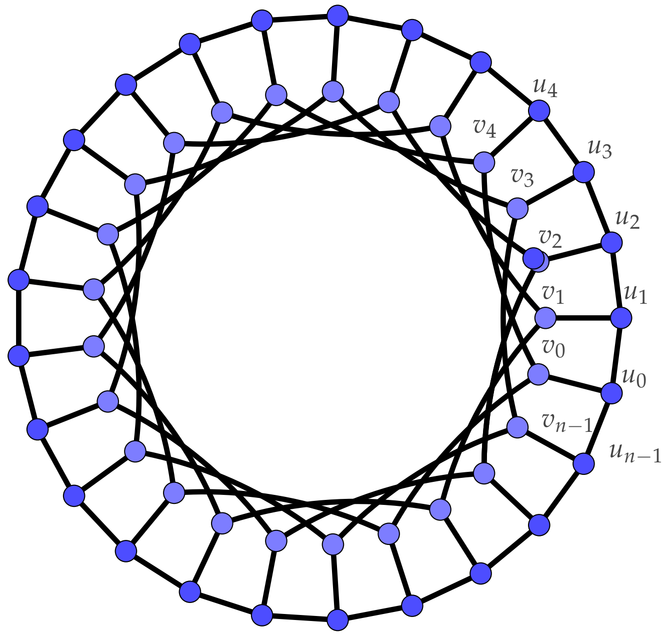

Figure 1. A generalized Petersen graph $P(n,k)$.

It is well known that the graphs $P(n,k)$ are 3-regular unless $k = \frac{n}{2}$ and that $P(n,k)$ are highly symmetric [6,7]. As $P(n,k)$ and $P(n,n-k)$ are isomorphic, it is natural to restrict attention to $P(n,k)$ with $n \geq 3$ and k, $1 \leq k < \frac{n}{2}$. In this work, we implicitly make use of another symmetry of Petersen graphs. It is well-known that the mapping which maps $v_i \rightarrow v_{i+1}$ and $u_i \rightarrow u_{i+1}$ is an automorphism. Hence, any rotation along the long cycle is an automorphism.

1.3. Related Previous Work

The early papers [1–3] already provide various results on k-rainbow domination. The special cases, 2-rainbow and 3-rainbow domination, have been studied in a number of works where the rainbow domination numbers of several graph classes were established; see [8–12] and the references therein. In particular, k-rainbow domination number of the Cartesian product of cycles, $C_n \Box C_m$, for $k \geq 4$ is considered in [13]. Among other things, based on the results in [14], it is shown that $\gamma_{rk}(C_n \Box C_m) = mn$ for $k \geq 8$. In [15], exact values of the 3-rainbow domination number of $C_3 \Box C_m$ and $C_4 \Box C_m$ and bounds on the 3-rainbow domination number of $C_n \Box C_m$ for $n \geq 5$ are given. In [16], sharp upper bounds on the k-rainbow domination number for all values of k are proved. Furthermore, they also consider the problem with minimum degree restrictions on the graph. In particular, for every connected graph G of order $n \geq 5$, $\gamma_{r3}(G)) \leq \dfrac{8n}{9}$. In [17], the authors proved that for every connected graph G of order $n \geq 8$ with $\delta(G) \geq 2$, $\gamma_{r3}(G) \leq \dfrac{5n}{6}$. Computing the rainbow domination of graphs is an NP-hard problem, as it holds for most of the variations of domination problems. An exact algorithm and a faster heuristic algorithm to obtain the

3-rainbow domination number is given in [18]. Therefore, while it is possible to compute exact 3-rainbow domination numbers for small or moderate size graphs, it is very hard or intractable to handle large graphs.

Generalized Petersen graphs have been studied extensively in the past, often as very interesting examples in research of various graph invariants. The interest seems to be even more intensive recently, the problems studied including domination [19], independent rainbow domination [20,21], Italian domination [22], Roman and double Roman domination [12,23,24], to name just a few. Many papers focus on subfamilies of Petersen graphs. Popular examples are $P(n,c)$, for fixed (and usually small) c, and $P(ck,k)$, for fixed c and arbitrary k (hence infinitely many $n = ck$). In [25] the exact values of $\gamma_{rt}(P(n,1))$ for any $t \geq 8$ and $t = 4$ are determined and it is proved that $\gamma_{rt}(P(2k,k)) = 4k$ for $t \geq 6$.

In [26] the 3-rainbow domination numbers of several classes of graphs, such as paths, cycles and the generalized Petersen graphs $P(n,k)$, were investigated. The 3-rainbow domination number of $P(n,k)$ for some cases are determined and the upper bounds for $P(n,2)$, $n \geq 5$, and $P(n,3)$, $n \geq 30$, were provided. In particular, the general lower bound was established, $\gamma_{r3}(P(n,k)) \geq n$, and it was proved that if $k \equiv 1(\text{mod}6)$, $n \equiv 0(\text{mod}6)$ and $n > 2k \geq 6$, then $\gamma_{r3}(P(n,k)) = n$. Additionally, it was determined that for $n \geq 6$, $\gamma_{r3}(P(n,1)) = n + \alpha$, where $\alpha = 0$ for $n \equiv 0(\text{mod}6)$, $\alpha = 1$ for $n \equiv 1,2,3,5(\text{mod}6)$, and $\alpha = 2$ for $n \equiv 4(\text{mod}6)$. The upper bound, $\gamma_{r3}(P(n,2)) \leq \lceil \frac{6n}{5} \rceil$ for $n \geq 5$ was provided. It follows that $\gamma_{r3}(P(6k,k)) \geq 6k$ for each $k \geq 1$, $\gamma_{r3}(P(6k,k)) = 6k$ if $k \equiv 1(\text{mod}6)$, and $12 \leq \gamma_{r3}(P(12,2)) \leq 15$.

The following two results are of particular importance for the present work. First, we recall the general bound $n \leq \gamma_{r3}(P(n,k))$, which was proved in [2] and directly implies the next proposition.

Proposition 1. $6k \leq \gamma_{r3}(P(6k,k))$.

In cases when $n = 6k$, on the other hand, we have the following upper bound for generalized Petersen graphs [26].

Proposition 2. *Let* $k = 1(\text{mod}6)$, $n = 0(\text{mod}6)$ *and* $n \geq 2k \geq 6$. *Then* $\gamma_{r3}(P(n,k)) \leq n$.

Propositions 1 and 2 together imply exact values for one infinite family.

Proposition 3. *Let* $k = 1(\text{mod}6)$, $n = 0(\text{mod}6)$ *and* $n \geq 2k \geq 6$. *Then* $\gamma_{r3}(P(n,k)) = n$.

1.4. Our Results

Recall that the tRD functions assign sets of colors to vertices. An interesting special case are tRD functions that assign only singletons or empty sets. We call such functions *singleton tRD functions* ($StRD$ functions) and the minimal weight obtained when considering only $StRD$ functions *singleton t-rainbow domination number* denoted by $\tilde{\gamma}_{rt}$. Clearly,

$$\gamma_{rt} \leq \tilde{\gamma}_{rt} .$$

In this paper, we study both γ_{rt} and $\tilde{\gamma}_{rt}$. In both cases, we give the exact values of 3-rainbow domination number for some, and bounds with a small gap for all other infinite subfamilies of generalized Petersen graphs, $P(6k,k)$. The main results are given in the following two theorems.

Theorem 1. *For the 3-rainbow domination number γ_{r3} of generalized Petersen graphs $P(6k,6)$ we have the following:*

- *If* $k = 1,5\ (\text{mod}6)$, *then* $\gamma_{r3}(P(6k,k)) = 6k$;
- *If* $k \equiv 0\ (\text{mod}2)$, *then* $6k < \gamma_{r3}(P(6k,k)) \leq 6k+3$;
- *If* $k \equiv 3\ (\text{mod}6)$, *then* $6k < \gamma_{r3}(P(6k,k)) \leq 6k+6$.

In the special case when only singleton $3RD$ functions are considered, we obtain bounds and in some cases exact values of singleton $3RD$ domination number.

Theorem 2. *For the singleton 3-rainbow domination number $\tilde{\gamma}_{r3}$ of generalized Petersen graphs $P(6k, 6)$ we have the following:*

- *If $k = 1, 5 \pmod 6$, then $\tilde{\gamma}_{r3}(P(6k, k)) = 6k$;*
- *If $k = 0 \pmod 2$, then $\tilde{\gamma}_{r3}(P(6k, k)) = 6k + 3$;*
- *If $k = 3 \pmod 6$, then $6k < \tilde{\gamma}_{r3}(P(6k, k)) \leq 6k + 6$.*

Finally, for a member of one family for which the exact values remain unknown we provide an example for which the general bounds are improved, we show that $21 \leq \tilde{\gamma}_{r3}(P(18, 3)) \leq 23$ (Proposition 9).

2. Constructions and Proofs

We start with a useful result that sheds some light in relation between the lower bounds for singleton $3RD$ functions and general $3RD$ functions.

Lemma 1. *Let G be a 3-regular graph on n vertices. Let f be a $3RD$ function that assigns empty sets to n_0 vertices, singletons to n_1 vertices, two element sets to n_2 vertices, and three element sets to n_3 vertices. Then*

$$|f(V)| \geq \frac{|V(G)|}{2} + \frac{n_2}{2} + n_3 .$$

Furthermore,

$$|f(V)| \geq \frac{|V(G)|}{2} + \frac{1}{6}n_\star + \frac{n_2}{2} + n_3 ,$$

where $n_\star$ is the number of vertices that are assigned one color and have a colored neighbor.

Proof. Observe that a vertex, which is assigned a singleton, fulfills the demand of one vertex and at most three thirds of the demands of its neighbors, thus summing up the fractions it serves at the greatest demand, which is two in total. Similarly, a vertex assigned set of two colors fulfills its demand plus three times two thirds of demands at most (serves demand ≤ 3). A vertex with three colors assigned can dominate four vertices. Furthermore, let $n_1 = n_1^{(\star)} + n_\star$ where $n_\star$ is the number of vertices that are assigned one color and have at least one neighbor that is colored.

As $|f(V)| = n_1 + 2n_2 + 3n_3 = n_1^{(\star)} + n_\star + 2n_2 + 3n_3$, and $2n_1 + 3n_2 + 4n_3 \geq 2n_1^{(\star)} + \frac{5}{3}n_\star + 3n_2 + 4n_3 \geq |V(G)|$, we have the following:

$$2(|f(V)| - n_\star - 2n_2 - 3n_3) + \frac{5}{3}n_\star + 3n_2 + 4n_3 \geq |V(G)| ,$$

and

$$|f(V)| \geq \frac{|V(G)|}{2} + \frac{1}{6}n_\star + \frac{n_2}{2} + n_3 ,$$

as claimed. □

For later reference, it is useful to note the following.

Corollary 1. *Let $G = P(n, k)$. If $\gamma_{r3}(G) = n = \frac{|V(G)|}{2}$, then $\gamma_{rt}(G) = \tilde{\gamma}_{rt}(G)$, and any minimal assignment is a singleton $3RD$ function.*

Next, we consider induced subgraphs, more precisely, paths. An induced *path* is a connected subgraph on $m \geq 2$ vertices of G such that $m - 2$ vertices have a degree of two, and two vertices have a degree of one.

Lemma 2. *Let f be a singleton 3RD function of a 3-regular graph G, and let P be an induced path of length ℓ on vertices $\{v_0, v_1, \ldots, v_\ell\}$ in G. Assume that one of the vertices v_0 and v_ℓ is uncolored and the other is assigned exactly one color, such as $f(v_\ell) = \emptyset$ and $|f(v_0)| = 1$. Then, $|f(P)| \geq \left\lceil \frac{\ell+1}{2} \right\rceil$.*

Proof. The edges of P are $\{v_i, v_{i+1}\}$ for $i = 0, 1, \ldots, \ell - 1$. Thus, v_0 and v_ℓ have two neighbors outside P and all other vertices have exactly one neighbor outside P. Assume that one of the vertices v_0 and v_ℓ is uncolored and the other is assigned exactly one color. Clearly, each of the uncolored vertices v_i, $i \in [1, \ell - 1]$ must have two neighbors in P that are colored with two (distinct) colors, and consequently, the total number of colored vertices must be at least $\left\lceil \frac{\ell+1}{2} \right\rceil$. □

Example 1. *It is even easier to handle the cycles because every vertex of a cycle has exactly one neighbor outside the cycle, and thus, needs at least two colors unless it is colored. Let C be a cycle of length ℓ in a 3-regular graph. Then, any singleton 3RD function assigns colors to at least $\lceil \frac{\ell}{2} \rceil$ vertices, and hence $|f(C)| \geq \lceil \frac{|V(C)|}{2} \rceil$.*

We use two different notations to outline the 3RD functions of generalized Petersen graphs. For smaller graphs, we use the following:

$$\begin{pmatrix} f(u_0) & f(u_1) & \ldots & f(u_{n-1}) \\ f(v_0) & f(v_1) & \ldots & f(v_{n-1}) \end{pmatrix}.$$

The elements in rows are written in triples for easier reading. For example, a 3*RD* function showing $\gamma_{r3}(P(6,1)) = \tilde{\gamma}_{r3}(P(6,1)) = 6$ is the following:

$$\begin{pmatrix} 102 & 030 \\ 030 & 102 \end{pmatrix}.$$

More often, we use the second notation that provides only the values on the outer cycle. The columns correspond to the sets U_i, and we assume that the inner cycles (sets V_i) are completed (with minimal number of colors) such that the whole assignment is a 3RD function. In Table 1 below, the first two and the last two columns provide the same information, namely the values at $U_0 = U_k$, and $U_1 = U_{k+1}$. This will be useful to observe when certain patterns give rise to optimal assignments—it will hold exactly when columns 0 and k will match, taking into account the shift of rows as indicated in Table 1.

Table 1. A 3RD-coloring of U_i for $P(6k, k)$.

$f(u_0)$	$f(u_1)$	$\ldots$	$f(u_i)$	$\ldots$	$f(u_{k-1})$	$f(u_k)$	$f(u_{k+1})$	$\ldots$
$f(u_k)$	$f(u_{k+1})$	$\ldots$	$f(u_{k+i})$	$\ldots$	$f(u_{2k-1})$	$f(u_{2k})$	$f(u_{2k+1})$	$\ldots$
$f(u_{2k})$	$f(u_{2k+1})$	$\ldots$	$f(u_{2k+i})$	$\ldots$	$f(u_{3k-1})$	$f(u_{3k})$	$f(u_{3k+1})$	$\ldots$
$f(u_{3k})$	$f(u_{3k+1})$	$\ldots$	$f(u_{3k+i})$	$\ldots$	$f(u_{4k-1})$	$f(u_{4k})$	$f(u_{4k+1})$	$\ldots$
$f(u_{4k})$	$f(u_{4k+1})$	$\ldots$	$f(u_{4k+i})$	$\ldots$	$f(u_{5k-1})$	$f(u_{5k})$	$f(u_{5k+1})$	$\ldots$
$f(u_{5k})$	$f(u_{5k+1})$	$\ldots$	$f(u_{5k+i})$	$\ldots$	$f(u_{6k-1})$	$f(u_{6k}) = f(u_0)$	$f(u_1)$	$\ldots$
0	1	$\ldots$	i	$\ldots$	$k-1$	k	$k+1$	$\ldots$

2.1. The Cases $k = 1 \pmod 6$ and $k = 5 \pmod 6$

Consider the pattern in Table 2 that provides 3RDFs for $P(42, 7)$ and $P(78, 13)$.

Table 2. An optimal 3RD-coloring of U_i for $P(6k,k)$. The first column provides a 3RD function for $P(6,1)$, the first 7 columns provide a 3RD function for $P(42,7)$, and the first 13 columns provide a 3RD function for $P(78,13)$.

1	0	2	0	3	0	1	0	2	0	3	0	1	0	...
0	2	0	3	0	1	0	2	0	3	0	1	0	2	...
2	0	3	0	1	0	2	0	3	0	1	0	2	0	...
0	3	0	1	0	2	0	3	0	1	0	2	0	3	...
3	0	1	0	2	0	3	0	1	0	2	0	3	0	...
0	1	0	2	0	3	0	1	0	2	0	3	0	1	...
0	1	2	3	4	5	6	7	8	9	10	11	12	13	...

First, consider closely the graph $P(42,7)$. Using the fact that the columns 0 and 7 correspond to the same set of vertices, $U_0 = U_7$, and that the column 7 equals column 0 shifted one row downwards (see Table 1), it is clear that the pattern is well defined on the outer cycle of $P(42,7)$. It is trivial to check that the vertices on the inner cycles can be colored by three additional colors each, so we can conclude that we have a 3RDF of $P(42,7)$ of weight 42. Similarly, regarding $P(78,13)$, we have $U_0 = U_{13}$, and the same reasoning applies. For a later reference, observe that the pattern in Table 2 repeats after six columns.

Using a symmetrical pattern, we find that there are 3RDFs of $P(30,5)$ and $P(66,11)$ and of weights 30 and 66, respectively (see Table 3).

Table 3. Optimal 3RD colorings. of U_i for $P(30,5)$ and $P(66,11)$.

1	0	3	0	2	0
0	1	0	3	0	2
2	0	1	0	3	0
0	2	0	1	0	3
3	0	2	0	1	0
0	3	0	2	0	1
0	1	2	3	4	5

1	0	3	0	2	0	1	0	3	0	2	0
0	1	0	3	0	2	0	1	0	3	0	2
2	0	1	0	3	0	2	0	1	0	3	0
0	2	0	1	0	3	0	2	0	1	0	3
3	0	2	0	1	0	3	0	2	0	1	0
0	3	0	2	0	1	0	3	0	2	0	1
0	1	2	3	4	5	6	7	8	9	10	11

The observed facts can be summarized in the next proposition.

Proposition 4. *For* $k = 1(\bmod 6)$ *and* $k = 5(\bmod 6)$ *we have* $\gamma_{r3}(P(6k,k)) = 6k$.

Proof. By previous arguments, the two patterns (Tables 2 and 3) both have a period of 6. Therefore, the assignments for $k = 5,7$ can, by induction, be extended to the provide assignments for families $k = 1(\bmod 6)$ and $k = 5(\bmod 6)$, as claimed. □

Remark 1. *The case* $k = 1(\bmod 6)$ *is also proved in [26]; see Proposition 3 above.*

2.2. The Case k Even

First, we consider the lower bound for $\tilde{\gamma}_{r3}(P(6k,k))$.

Lemma 3. *Let* $k > 2$ *be an even number. Then,* $\tilde{\gamma}_{r3}(P(6k,k)) \geq 6k + 3$.

Proof. Let $k \geq 4$. Consider the k inner cycles together with their neighbors, i.e., the subgraphs induced on vertices $V_i \cup U_i$, where $U_i = \{u_i, u_{i+k}, u_{i+2k}, u_{i+3k}, u_{i+4k}, u_{i+5k}\}$ and $V_i = \{v_i, v_{i+k}, v_{i+2k}, v_{i+3k}, v_{i+4k}, v_{i+5k}\}$. Choose i with $|f(V_i \cup U_i)| = 6$. (Such an i clearly exists as, otherwise, $\gamma_{r3}(P(6k,k)) \geq 7k > 6k+3$.) Consider Figure 2, and observe that there are six paths on the outer cycle between vertices of U_i. These paths (three are depicted on Figure 2) have length $k+1$. Three of them, together with vertices of $V_i \cup U_i$, form three disjoint cycles of length $k+3$. These cycles are odd, so we need to color $\frac{k}{2}+2$ of the vertices. For the vertices on the paths, the number of colored vertices is at least $\lceil \frac{k+1}{2} \rceil - 1 = \frac{k}{2}$ because k is even. As $|f(V_i \cup U_i)| = 6$, exactly one endpoint of the path is colored, and the other is uncolored. By Lemma 2, at least $\lceil \frac{k+1}{2} \rceil$ vertices need to be colored, but the endpoint is already used on the odd cycle.) Finally, we also have $k-1$ inner six cycles and need to color at least three vertices on each of them. In total, we need to color at least the following vertices:

$$3(\frac{k}{2}+2)+3\frac{k}{2}+(k-1)3 \geq 6k+3$$

and hence, $|f| \geq 6k+3$, as claimed. □

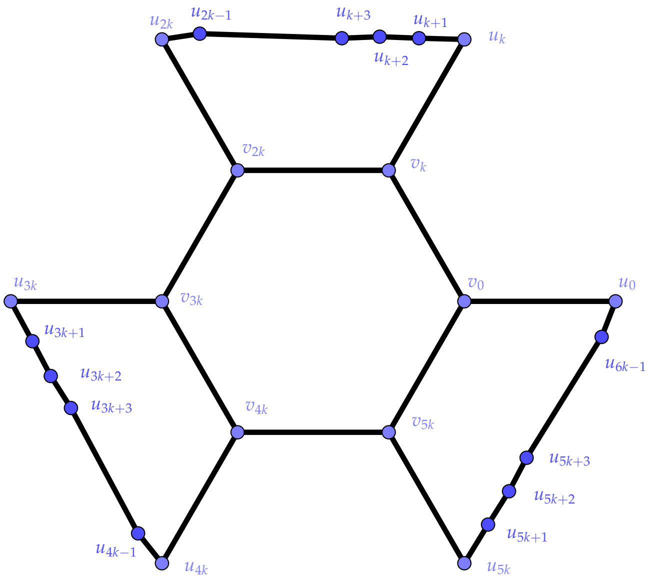

Figure 2. An inner cycle with neighbors on the outer cycle.

The lower bound is tight, as we can construct 3RD functions of weight $6k+3$. For small cases $k = 2,4,6$, see Table 4. The examples are based on the pattern shown in Table 2 Another set of examples can be obtained similarly from pattern in Table 1. Recall that both the patterns have a period of six.

Table 4. Optimal 3RD functions (**a**) of weight 15 for $P(12,2)$, (**b**) of weight 27 for $P(24,4)$, and (**c**) of weight 39 for $P(36,6)$.

(a)		(b)				(c)					
1	2	1	0	3	2	1	0	3	0	2	2
0	1	0	1	0	3	0	1	0	3	0	2
2	3	2	0	1	3	2	0	1	0	3	3
0	2	0	2	0	1	0	2	0	1	0	3
3	1	3	0	2	1	3	0	2	0	1	1
0	3	0	3	0	2	0	3	0	2	0	1

The only case not covered by Lemma 3 is $k = 2$. The graph $P(12,2)$ has only 24 vertices, so it can be shown by a trivial algorithm that $\gamma_{r3}(P(12,2)) = 15$. An alternative argument that $\gamma_{r3}(P(12,2)) \geq 15$ is given in Section 2.4. Summarizing, we have the following result.

Proposition 5. *Let k be an even number. Then $\tilde{\gamma}_{r3}(P(6k,k)) = 6k+3$.*

Proof. The examples in Table 4 are based on the two patterns with period 6. Hence, one of the columns, wlog column 0, can be replaced by seven columns (columns $0, 1, \dots, 6$ from Table 2) to obtain a 3RD function with 6 more columns and weight increased by 36. By induction, we obtain constructions for all even k of weight $6k+3$. Recalling the lower bound from Lemma 3, it follows that the constructions are the best possible. □

We conclude the subsection with the bounds for $\gamma_{r3}(P(6k,k))$.

Proposition 6. *Let k be an even number. Then, $6k < \gamma_{r3}(P(6k,k)) \le 6k+3$.*

Proof. First, recall that $\gamma_{r3}(G) \le \tilde{\gamma}_{r3}(G) = 6k+3$. To prove the lower bound, observe one of the cycles C of length $k+3$, say $V(C) = \{v_0, u_0, u_1, \dots, u_{k-1}, u_k, v_k\}$. The cycle is odd; hence, either there is a pair of adjacent colored vertices in $V(C)$, or at least one of the vertices of C has a neighbor that is assigned at least two colors. Hence, either $n_\star > 0$ or $n_2 > 0$ or $n_3 > 0$. By Lemma 1, $\gamma_{r3}(P(6k,k)) > 6k$. □

2.3. The Case $k = 3 (\bmod 6)$

First, we prove a property of any 3RD function f of $P(n,k)$ with $|f(V)| = n$.

Lemma 4. *Assume $\gamma_{r3}(P(6k,k)) = 6k$ and let f be a 3RD function with minimal weight $|f(V)| = n$. Then, exactly one half of the vertices on the outer cycle are colored. Wlog assumes that these are vertices with even indices. Then, the following holds: (1) $f(u_i) = \emptyset$, for all odd i, (2) $f(u_{i+6}) = f(u_i)$, $i \in [0, 6k-1]$, and (3) $f(u_0)$, $f(u_2)$, $f(u_4)$ are pairwise different.*

In other words, the lemma says that the outer cycle vertices are colored following the pattern $a - 0 - b - 0 - c - 0 - \cdots - a - 0 - b - 0 - c - 0$, where a, b, c are the three colors.

Proof. Recall that by Corollary 1, any minimal 3RDF must be a singleton 3RDF and hence that $\gamma_{r3}(P(6k,k)) = \tilde{\gamma}_{r3}(P(6k,k)) = 6k$. Consider an uncolored vertex on the outer cycle, and observe that its three neighbors must be colored by three distinct colors. Wlog, for example, u_i, is the vertex with $f(u_i) = \emptyset$. Then, the three neighbors of u_i must be colored by distinct colors, for example, $f(v_i) = c$, $f(u_{i-1}) = a$ and $f(u_{i+1}) = b$.

We show that we must have $f(u_{i+3}) = c$. First, because $f(v_i) = c$, we have $f(v_{i+2k}) \cup f(v_{i+4k}) = \{a, b\}$, and, consequently $f(u_{i+3k}) = c$. Furthermore, $f(u_{i+3}) \ne b$ because $f(u_{i+1}) = b$. If $f(u_{i+3}) = c$, we are done. Otherwise, $f(u_{i+3}) = a$ and by the same reasoning as above, we have to have $f(v_{i+2}) = c$ and in turn $f(u_{i+2+3k}) = c$. Now, the vertex u_{i+1+3k} has two neighbors colored by the same color, and hence, f must have weight $> 6k$. This is a contradiction, so $f(u_{i+3}) = c$.

By induction, we have $f(u_{i+5}) = a$, $f(u_{i+7}) = b$, and so on, which implies the statements of the lemma. □

The last lemma can be used to obtain a lower bound.

Lemma 5. *If $k = 3 (\bmod 6)$ then $\tilde{\gamma}_{r3}(P(6k,k)) \ge \gamma_{r3}(P(6k,k)) > 6k$.*

Proof. Let $k = 6\ell + 3$ and assume that $\gamma_{r3}(P(6k,k)) = 6k = 36\ell + 18$. Consider an inner cycle, say U_0. As $\gamma_{r3}(P(6k,k)) = 6k$, any inner cycle is colored by three distinct colors. If we assume wlog such that $f(u_0) = a$, then we know that we must have $f(v_{3k}) = a$ (as observed in the proof of Lemma 4). It follows that $f(u_{2k}) \ne a$ because $f(v_{2k})$ cannot have two neighbors colored by the same color, a. On the other hand, by Lemma 4 $f(u_{2k}) = f(u_{12\ell+6}) = f(u_0) = a$. This is a contradiction. □

The next table provides singleton 3RD function of weight 60 for $P(54,9)$.

Formally, Table 5 and an inductive argument imply the following.

Table 5. A singleton $3RD$ function of U_i for $P(54,9)$.

1	0	3	0	2	0	1	2	2	0	1
0	1	0	3	0	2	0	1	3	2	0
2	0	1	0	3	0	2	3	3	0	2
0	2	0	1	0	3	0	2	1	3	0
3	0	2	0	1	0	3	1	1	0	3
0	3	0	2	0	1	0	3	2	1	0
0	1	2	3	4	5	6	7	8	9	10

Lemma 6. *If* $k > 3$, $k = 3(\bmod 6)$ *then* $\tilde{\gamma}_{r3}(P(6k,k)) \leq 6k + 6$.

For case $k = 3$, see Proposition 9. Summarizing, we have the following result for $k = 3\ (\bmod 6)$.

Proposition 7. *If* $k = 3(\bmod 6)$, *then* $6k < \gamma_{r3}(P(6k,k)) \leq \tilde{\gamma}_{r3}(P(6k,k)) \leq 6k + 6$.

2.4. Two Examples

In this subsection, we consider two small examples that are not covered by the general arguments above.

Proposition 8. $\tilde{\gamma}_{r3}(P(12,2)) = 15$.

Proof. First, we prove that at least 15 vertices must be colored. Observe that at least 6 vertices must be colored on the outer cycle because it has 12 vertices, and the two inner cycles C_1 and C_2 on six vertices must have at least 3 colored vertices. Furthermore, the vertices of $U = \{u_0, u_1, \ldots, u_{11}\}$ with odd indices are adjacent to one, and the vertices with even indices are adjacent to the other six cycle. Now consider several cases:

(1) $|f(U)| = 6$. Clearly, all colored vertices of U have either odd or even indices. However, then one of the six cycles has no colored vertex in the neighborhood, and we need six colors for this six cycle. Thus, we need 6 + 6 + 3 = 15 colors.

(2) $|f(U)| = 7$. Taking into account that all vertices of U must have at least two colored neighbors within U, we observe that the set of uncolored vertices must be an independent set. On a 12 cycle, there are (up to isomorphism) three possibilities for an independent set of five vertices; their indices may be either (a) 0, 2, 4, 6, 8, or (b) 0, 2, 4, 6, 9, or (c) 0, 2, 4, 7, 9. In case (a), the inner cycle induced on vertices $C_2 = \{v_0, v_2, v_4, v_6, v_8, v_{10}\}$ has only one colored neighbor, v_{10}. Hence, $|f(C_2)| \geq 5$. In case (b), the inner cycle C_2 has two colored neighbors, u_8 and u_{10}. Again, we must have $|f(C_2)| \geq 5$, because v_8 and v_{10} are neighbors and one of them has to be colored. In case (c), the inner cycle C_2 has three colored neighbors, u_6, u_8 and u_{10}. This implies that $|f(C_2)| \geq 4$. Furthermore, the inner cycle C_1 has four colored neighbors, u_1, u_3, u_5 and u_{11}. Observe that it is not possible to complete the coloring of C_1 by coloring only three vertices, so $|f(C_1)| \geq 4$. In all cases, at least 15 vertices need to be assigned a color.

(3) $|f(U)| = 8$. In this case, it can be seen that we need to color at least four vertices of one six cycle (C_1 or C_2) to obtain a S3RDF. We omit the details.

(4) $|f(U)| \geq 9$. It is clear that we need at least three vertices on C_1 and on C_2.

Summarizing, we have the lower bound. To complete the proof, note that a singleton 3RDF can be constructed easily, for example, by applying the reasoning in case (1) above. □

Proposition 9. $21 \leq \tilde{\gamma}_{r3}(P(18,3)) \leq 23$.

Proof. Consider a 3RDF of weight 23 of $P(18,3)$: $\begin{pmatrix} 01p & 102 & 01p & 301 & 02p & 103 \\ 201 & 030 & 303 & p20 & 3p2 & 020 \end{pmatrix}$ where p can be any color. Hence, $\tilde{\gamma}_{r3}(P(18,3)) \le 23$.

Next, we show that $21 \le \tilde{\gamma}_{r3}(P(18,3))$. Let f be a 3RDF of $P(18,3)$ such that $|f| < 21$. Then, $|f(U_i \cup V_i)| = 6$ for at least one $i \in \{1,2,3\}$, and we can assume that $|f(V_i)| = 3$. Wlog., let $|f(U_1 \cup V_1)| = 6$, $f(v_1) = a$, $f(v_7) = b$, and $f(v_{13}) = c$. Then, $f(u_4) = c$, $f(u_{10}) = a$, $f(u_{16}) = b$. It follows that $\{f(u_0), f(u_2)\} = \{b,c\}$, $\{f(u_6), f(u_8)\} = \{a,c\}$, and $\{f(u_{12}), f(u_{14})\} = \{a,b\}$. There are eight possibilities:

(1) $f(u_2) = b$, $f(u_8) = c$, $f(u_{14}) = a$. Then, $f(u_0) = c$, $f(u_6) = a$, $f(u_{12}) = b$. Now consider $f(U_2 \cup V_2)$. The vertex v_{11} (or u_{11}) should be assigned color c because u_{11} has, so far, no neighbor with color c. On the other hand, v_{11} has a neighbor, v_8, that already has a neighbor with color c, namely, u_8. This is similarly the case for v_5 and v_{17}. Considering all possibilities to complete the coloring of $U_2 \cup V_2$, we conclude that we need nine colors. Hence, we need to color at least 21 vertices, contradicting the assumption. Analogous reasoning applies if we consider $U_3 \cup V_3$.

(2–8) In each case, we arrive at the conclusion that at least 21 vertices need to be colored. We omit the details. □

3. Conclusions

We have demonstrated exact values of 3-rainbow domination number for some infinite families of Petersen graphs $P(6k,k)$, and bounds with small gaps for others (Theorem 1). Moreover, we study a special case when only colorings that assign sets with, at most, one element are considered. In this case, we provide exact values for all but one infinite subfamily (Theorem 2). The largest gaps remain in the case $n = 6k$, where $k = 3 \pmod 6$. The example $P(18,3)$ (see Proposition 9) indicates that we may have a stronger lower bound, at least for singleton 3RDF. On the other hand, our construction of a S3RDF for $P(18,3)$ does not seem to generalize easily. Therefore, we propose the following conjecture.

Conjecture 1. *If* $k = 3 \pmod 6$, *then* $6k + 3 \le \tilde{\gamma}_{r3}(P(6k,k)) \le 6k + 6$.

Due to well-known symmetries of generalized Petersen graphs, it is straightforward that the results of this paper directly apply to the family $P(6k,5k)$ as $P(6k,5k) \approx P(6k,k)$ (where $\approx$ denotes graph isomorphism). The remaining cases among graphs with $n = 6k$ are Petersen graphs $P(6k,2k)$ and $P(6k,4k)$, where the inner cycles are triangles. Clearly, in both cases $\gamma_{r3} > n$, but it may be an interesting task for future research to obtain exact values or at least close bounds for these families.. Note that the case $P(6k,3k)$ is not interesting, as in this case, the graphs are not 3-regular, as all inner vertices are of degree two, and hence, must be colored. Then, it follows trivially that $\gamma_{r3}(P(6k,3k)) = 9k$.

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