symmetry

MDPI

Article

Coefficient Estimates for a Subclass of Meromorphic Multivalent q-Close-to-Convex Functions

Lei Shi [1,*], Bakhtiar Ahmad [2], Nazar Khan [3], Muhammad Ghaffar Khan [4], Serkan Araci [5,*], Wali Khan Mashwani [4] and Bilal Khan [6,*]

1 School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China
2 Government Degree College Mardan, Mardan 23200, Pakistan; pirbakhtiarbacha@gmail.com
3 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; nazarmaths@aust.edu.pk
4 Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat 26000, Pakistan; ghaffarkhan020@gmail.com (M.G.K.); walikhan@kust.edu.pk (W.K.M.)
5 Department of Economics, Faculty of Economics Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep 27410, Turkey
6 School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China
* Correspondence: shimath@aynu.edu.cn (L.S.); serkan.araci@hku.edu.tr (S.A.); bilalmaths789@gmail.com (B.K.)

Citation: Shi, L.; Ahmad, B.; Khan, N.; Khan, M.G.; Araci, S.; Mashwani, W.K.; Khan, B. Coefficient Estimates for a Subclass of Meromorphic Multivalent q-Close-to-Convex Functions. *Symmetry* **2021**, *13*, 1840. https://doi.org/10.3390/sym13101840

Academic Editors: Paolo Emilio Ricci and Sergei D. Odintsov

Received: 6 August 2021
Accepted: 29 September 2021
Published: 1 October 2021

Abstract: By making use of the concept of basic (or q-) calculus, many subclasses of analytic and symmetric q-starlike functions have been defined and studied from different viewpoints and perspectives. In this article, we introduce a new class of meromorphic multivalent close-to-convex functions with the help of a q-differential operator. Furthermore, we investigate some useful properties such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikeness, and radius of convexity for this new subclass.

Keywords: meromorphic functions; Janowski functions; q-calculus; q-differential operator

1. Introduction, Definitions and Motivation

Calculus without notion of limits is called q-calculus or quantum calculus. This relatively new and advanced field of study became a core of attractions for many well-known mathematicians and physicists due to its various applications in applied mathematics, physics and various engineering areas, see details in [1,2]. The start of this field can be connected with the research of Jackson [3,4], who gave various applications of q-calculus and introduced the q-analogue of derivative and integral, while the most recent and trending work can be viewed in the study of Aral and Gupta [1,2,5]. Later, Aral and Anastassiu [6–9] gave the q-generalization of complex operators, which are known as q-Picard and q-Gauss–Weierstrass singular integral operators.

In Geometric Function Theory, the role of the q-Deference operator is quite significant. Historically speaking, it was Srivastava [10] who used the basic (or q-) hypergeometric functions:

$$_{\nabla}\Phi_{\int} \quad (\nabla, \int \in \mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\})$$

in Geometric Function Theory (GFT) of Complex Analysis (see, for details, [10]). Very recently, Srivastava's published review article [11] gave another flavor to this subject. In their published review article [11], Srivastava highlighted the triviality of the so-called (p, q)-calculus.

The aforementioned works of Srivastava [10,11] motivated a number of mathematicians to give their findings. In the above-cited work by Srivastava [11], many well-known convolution and fractional q-operators were surveyed. For example, in their published article [12], by use of the concept of convolution, a q-analogue of Ruscheweyh differential operator was defined by Kanas and Răducanu. More applications of this operator can be seen in the paper [13]. Very recently, by using q-Deference operator, Srivastava et al. [14] studied a certain subclass of analytic function with symmetric points. Several other authors (see, for example, [15–22]) have studied and generalized the classes of symmetric and other q-starlike functions from different viewpoints and perspectives. For some more recent investigation about q-calculus, we may refer the interested reader to [23,24]. In this article, we introduce a new family of meromorphic multivalent functions associated with Janowski domain using a differential operator and study some of its properties.

By the notation $\mathcal{A}_p$, we denote the family of all meromorphic multivalent functions that are analytic and invariant (or symmetric) under rotations, in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and with the following normalization conditions

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \ (z \in \mathbb{D}). \tag{1}$$

Furthermore, a function f is said to be in the class $\mathcal{MS}_p^*(\alpha)$ $(0 \leq \alpha < 1)$ of meromorphic p-valent starlike functions, if

$$f(z) \in \mathcal{MS}_p^*(\alpha) \Leftrightarrow \Re \frac{zf'(z)}{pf(z)} < -\alpha.$$

Next, a function f is said to be in the class $\mathcal{MC}_p(\alpha)$ $(0 \leq \alpha < 1)$ of meromorphic p-valent convex functions, if

$$f(z) \in \mathcal{MC}_p(\alpha) \Leftrightarrow \Re \frac{(zf'(z))'}{pf'(z)} < -\alpha$$

Clearly, we see that $\mathcal{MS}_p^*(0) = \mathcal{MS}_p^*$, the class of meromorphic p-valent starlike functions. Similarly $\mathcal{MK}_p^*(\alpha)$ denote the class of meromorphic p-valent close-to-convex functions and defined as

$$f(z) \in \mathcal{MK}_p^*(\alpha) \Leftrightarrow \Re \frac{zf'(z)}{pg(z)} < -\alpha$$

where $g(z) \in \mathcal{MS}_p^*$.

The q-derivative (or q-difference) operator of a function f, where $0 < q < 1$, is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \ (z \neq 0). \tag{2}$$

It can easily be seen that

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n,q] a_n z^{n-1}, \qquad (n \in \mathbb{N}, \ z \in \mathbb{D}) \tag{3}$$

where

$$[n,q] = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n} q^l, \text{ and } [0,q] = 0.$$

For any non-negative integer n, the q-number shift factorial is defined by

$$[n,q]! = \begin{cases} 1, \ n = 0, \\ [1,q][2,q][3,q] \cdots [n,q], \ n \in \mathbb{N}. \end{cases}$$

Furthermore, the q-generalized Pochhammer symbol for $x \in \mathbb{R}$ is given by

$$[x,q]_n = \begin{cases} 1, \; n = 0, \\ [x,q][x+1,q]\ldots[x+n-1,q], \; n \in \mathbb{N}, \end{cases}$$

The differential operator $\mathcal{D}_{\mu,q} : \mathcal{A}_p \to \mathcal{A}_p$ is defined in [25] as

$$\mathcal{D}_{\mu,q}f(z) = (1+[p,q]\mu)f(z) + \mu q^p z\partial_q f(z). \tag{4}$$

where $\mu \geq 0$.

Now, using Equation (1), one can easily find that

$$\mathcal{D}_{\mu,q}f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty}(1+[p,q]\mu + \mu q^p[n,q])\, a_{p+n}z^{p+n}.$$

where

$$\mathcal{D}^0_{\mu,q}f(z) = f(z).$$

and

$$\begin{aligned}\mathcal{D}^2_{\mu,q}f(z) &= \mathcal{D}_{\mu,q}(\mathcal{D}_{\mu,q}f(z)) \\ &= \frac{1}{z^p} + \sum_{n=1}^{\infty}(1+[p,q]\mu + \mu q^p[n,q])^2\, a_{p+n}z^{p+n}\end{aligned}$$

in a similar way for $m \in N$, we have

$$\mathcal{D}^m_{\mu,q}f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty}(1+[p,q]\mu + \mu q^p[n,q])^m\, a_{p+n}z^{p+n}. \tag{5}$$

In this article, we are essentially motivated by the recently published paper of Hu et al. in *Symmetry* (see [26]) and some other related works as discussed above (see for example [27–31]), we now define a subclass $\mathcal{MK}_{\mu,q}(p,m,A,B)$ of $\mathcal{A}_p$ by using the operator $\mathcal{D}^m_{\mu,q}$ as follows.

Definition 1. *A function $f \in \mathcal{A}_p$ is said to be in the functions class $\mathcal{MK}_{\mu,q}(p,m,A,B)$, if the following condition is satisfied*

$$\frac{-q^p z\partial_q \mathcal{D}^m_{\mu,q}f(z)}{[p,q]g(z)} \prec \frac{1+Az}{1+Bz}, \tag{6}$$

where $g(z) \in \mathcal{MS}^*_p$, $-1 \leq B < A \leq 1$ and $0 < q < 1$, here the notation "$\prec$" stands for the familiar concept of subordinations.

For particular values to parameters m, p, μ, A, B and q, we have some known and new consequences as follows.

1. If we put $m = 0$ and let $q \to 1^-$, we have $\mathcal{MK}^*_p[A,B]$, where $\mathcal{MK}^*_p[A,B]$, is the functions class of Janowski-type meromorphic multivalent close-to-convex functions.
2. If we put
$$A = 1, \quad B = -1 \quad \text{and} \quad m = 0$$
we have $\mathcal{MK}^*_{p,q}$, where $\mathcal{MK}^*_{p,q}$, is the class of meromorphic multivalent q-close-to-convex functions.
3. By putting
$$A = 1, \quad B = -1 \quad \text{and} \quad m = 0$$
and let $q \to 1^-$, we have $\mathcal{MK}^*_p$, where $\mathcal{MK}^*_p$, is functions class of meromorphic multivalent close-to-convex functions.

4. If we take
$$A = 1 = p, \quad B = -1 \quad \text{and} \quad m = 0$$
and let $q \to 1^-$, we have $\mathcal{MK}^*$, where $\mathcal{MK}^*$, denotes the class of meromorphic close-to-convex functions.

The following equivalent condition for a function $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$ can be verified easily
$$\left| \frac{\frac{q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)} + 1}{A + B\frac{q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)}} \right| < 1. \tag{7}$$

For our main result, we need the following important result.

Lemma 1 ([32])**.** *An analytic function $h(z)$ having series representation*
$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n,$$
and another function $k(z)$ with the following series representation
$$k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n.$$
If $h(z) \prec k(z)$, then $|d_n| \le |k_1|$, for $n \in \mathbb{N}$.

2. A Set of Main Results

In this section, we give our main results.

Theorem 1. *If a function $f \in \mathcal{A}_p$ has a series representation given by Equation (1), then $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$ if it satisfies the following inequality*
$$\sum_{n=1}^{\infty} \left(q^p[p+n,q](1+B)(1+[p,q]\mu + \mu q^p[p+n,q])^m |a_{p+n}| + \tfrac{2p[p,q](1+A)}{p+n} \right) \le [p,q](A-B). \tag{8}$$
it holds true.

Proof. For f to be in the class $\mathcal{MK}_{\mu,q}(p,m,A,B)$, we only need to prove the inequality (7). For this, we consider
$$\left| \frac{\frac{q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)} + 1}{A + B\frac{q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)}} \right| = \left| \frac{q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z) + [p,q]g(z)}{A[p,q]g(z) + Bq^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)} \right|$$

If the series representation of $g(z)$ is given by
$$g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \ (z \in \mathbb{D}),$$

Now, by using Equation (4), and then with the help of Equations (2) and (5), we find, after some simplification, that the above is equal to

$$\begin{aligned} & \left| \frac{\sum_{n=1}^{\infty}(1+[p,q]\mu+\mu q^p[p+n,q])^m q^p[p+n,q]a_{p+n}z^{p+n}+\sum_{n=1}^{\infty}[p,q]b_{p+n}z^{p+n}}{\frac{(A-B)[p,q]}{z^p}+\sum_{n=1}^{\infty}A[p,q]b_{p+n}z^{p+n}+B\sum_{n=1}^{\infty}q^p(1+[p,q]\mu+\mu q^p[p+n,q])^m[p+n,q]a_{p+n}z^{p+n}} \right| \\ = & \left| \frac{\sum_{n=1}^{\infty}(1+[p,q]\mu+\mu q^p[p+n,q])^m q^p[p+n,q]a_{p+n}z^{n+2p}+\sum_{n=1}^{\infty}[p,q]b_{p+n}z^{p+2n}}{(A-B)[p,q]+\sum_{n=1}^{\infty}A[p,q]b_{p+n}z^{p+2n}+B\sum_{n=1}^{\infty}q^p(1+[p,q]\mu+\mu q^p[p+n,q])^m[p+n,q]a_{p+n}z^{p+2n}} \right| \\ \leq & \frac{\sum_{n=1}^{\infty}(1+[p,q]\mu+\mu q^p[p+n,q])^m q^p[p+n,q]|a_{p+n}|+\sum_{n=1}^{\infty}[p,q]|b_{p+n}|}{(A-B)[p,q]-\sum_{n=1}^{\infty}A[p,q]|b_{p+n}|-B\sum_{n=1}^{\infty}q^p(1+[p,q]\mu+\mu q^p[p+n,q])^m[p+n,q]|a_{p+n}|} \end{aligned}$$

Now, for $g(z) \in \mathcal{MS}_p^*$, we have

$$|b_{p+n}| = \frac{2p}{p+n}, \tag{9}$$

and using the inequality (11) we find that the above is less than 1. The direct part of the proof of our theorem is now completed.

Conversely, let $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$ and is given by Equation (1), then from Equation (7), we have for $z \in \mathbb{D}$,

$$\left| \frac{\frac{q^p z\partial_q \mathcal{D}_{\mu,q}^m f(z)}{[p,q]g(z)}+1}{A+B\frac{q^p z\partial_q \mathcal{D}_{\mu,q}^m f(z)}{[p,q]g(z)}} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty}(1+[p,q]\mu+\mu q^p[p+n,q])^m q^p[p+n,q]a_{p+n}z^{n+2p}+\sum_{n=1}^{\infty}[p,q]b_{p+n}z^{p+2n}}{(A-B)[p,q]+\sum_{n=1}^{\infty}A[p,q]b_{p+n}z^{p+2n}+B\sum_{n=1}^{\infty}q^p(1+[p,q]\mu+\mu q^p[p+n,q])^m[p+n,q]a_{p+n}z^{p+2n}} \right|$$

Since $|\Re(z)| \leq |z|$, we have

$$\begin{aligned} & \Re\left\{ \frac{\sum_{n=1}^{\infty}(1+[p,q]\mu+\mu q^p[p+n,q])^m q^p[p+n,q]a_{p+n}z^{n+2p}+\sum_{n=1}^{\infty}[p,q]b_{p+n}z^{p+2n}}{(A-B)[p,q]+\sum_{n=1}^{\infty}A[p,q]b_{p+n}z^{p+2n}+B\sum_{n=1}^{\infty}q^p(1+[p,q]\mu+\mu q^p[p+n,q])^m[p+n,q]a_{p+n}z^{p+2n}} \right\} \\ & < 1 \end{aligned} \tag{10}$$

Now, choose values of z on the real axis so that $\frac{q^p z\partial_q \mathcal{D}_{\mu,q}^m f(z)}{[p,q]g(z)}$ is real. Upon clearing the denominator in Equation (10) and letting $z \to 1^-$ through real values, we obtain Equation (11). □

For an analytic case, the result is as follows:

Corollary 1. *If a function $f \in \mathcal{A}$, then $f \in \mathcal{MK}_{\mu,q}(1,m,A,B)$ if it satisfies the following inequality*

$$\begin{aligned} & \sum_{n=1}^{\infty}\left(q[n+1,q](1+B)(1+\mu+\mu q[n+1,q])^m|a_{n+1}| + \frac{2(1+A)}{n+1}\right) \\ & \leq \ (A-B). \end{aligned} \tag{11}$$

is holds true.

In the next theorem, we obtain the coefficient bounds of the functions of this class.

Theorem 2. *Let $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$ and be of the form (1). Then*

$$|a_{p+n}| \leq \frac{2p[p,q]}{l(n)q^p[p+n,q]}\left(\frac{1}{p+n}+(A-B)\sum_{i=0}^{n-1}\frac{1}{p+i}\right)$$

where

$$l(n) = (1+[p,q]\mu+\mu q^p[p+n,q])^m. \tag{12}$$

Proof. If $f \in \mathcal{A}$ and is in the class $\mathcal{MK}_{\mu,q}(p,m,A,B)$, then it satisfies

$$\frac{-q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)} \prec \frac{1+Az}{1+Bz}.$$

Now, if

$$h(z) = \frac{-q^p z\partial_q \mathcal{D}^m_{\mu,q} f(z)}{[p,q]g(z)}, \tag{13}$$

then it will have the representation

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

which implies that

$$h(z) \prec \frac{1+Az}{1+Bz}.$$

However, by simple calculations we get

$$\tfrac{1+Az}{1+Bz} = 1 + (A-B)z + \ldots.$$

with the help of Lemma 1 we obtain

$$|d_n| \leq (A-B) \tag{14}$$

we now put the series expansions of $h(z), g(z)$ and $f(z)$ in Equation (13), simplifying and comparing the coefficients of z^{n+p} on both sides

$$\begin{aligned} -q^p(1+[p,q]\mu + \mu q^p[n+p,q])^m[p+n,q]a_{p+n} &= [p,q]\, b_{p+n} \\ &+[p,q]\textstyle\sum_{i=0}^{n-1} b_{p+i}d_{n-i} \end{aligned}$$

Taking absolute on both sides of this equation and using the triangle inequility with the help of Equation (14), we obtain

$$\begin{aligned} (1+[p,q]\mu + \mu q^p[p+n,q])^m q^p[p+n,q]\left|a_{p+n}\right| &\leq \\ [p,q]\,\left|b_{p+n}\right| + [p,q](A-B)\textstyle\sum_{i=1}^{n-1}\left|b_{p+i}\right| \end{aligned}$$

Now, using Equations (9) and (15), we find that

$$\begin{aligned} (1+[p,q]\mu + \mu q^p[p+n,q])^m q^p[p+n,q]\left|a_{p+n}\right| &\leq \\ [p,q]\,\tfrac{2p}{p+n} + [p,q](A-B)\textstyle\sum_{i=1}^{n-1}\tfrac{2p}{p+i} \\ \Rightarrow \left|a_{p+n}\right| &\leq \tfrac{2p[p,q]}{l(n)(p+n)q^p[p+n,q]} + \tfrac{2p[p,q](A-B)}{l(n)q^p[p+n,q]}\textstyle\sum_{i=0}^{n-1}\tfrac{1}{p+i} \end{aligned}$$

we obtain the required result. □

Corollary 2. *If $f \in \mathcal{MK}_{\mu,q}(1,m,A,B)$ and is of the form (1). Then,*

$$\left|a_{n+1}\right| \leq \tfrac{2}{l(n)q[n+1,q]}\left(\tfrac{1}{n+1} + (A-B)\textstyle\sum_{i=0}^{n-1}\tfrac{1}{1+i}\right)$$

where

$$l(n) = (1+\mu+\mu q[n+1,q])^m. \tag{15}$$

Next, for our defined functions class, we give results related to growth and distortion.

Theorem 3. *If the function $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$. Then, for $|z| = r$, we have*

$$\frac{1}{r^p} - \Phi r^p \leq |f(z)| \leq \frac{1}{r^p} + \Phi r^p,$$

where

$$\Phi = \frac{(p+1)[p,q](A-B) - 2p[p,q](1+A)}{q^p(p+1)[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m}.$$

Proof. Since

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \right|, \\ &\leq \frac{1}{|z^p|} + \sum_{n=1}^{\infty} |a_{p+n}|\, |z|^{p+n} \\ &= \frac{1}{r^p} + \sum_{n=1}^{\infty} |a_{p+n}|\, r^{p+n} \end{aligned}$$

Since for $|z| = r < 1$ we have $r^{p+n} < r^p$ and

$$|f(z)| \leq \frac{1}{r^p} + r^p \sum_{n=p+1}^{\infty} |a_{p+n}| \tag{16}$$

Similarly,

$$|f(z)| \geq \frac{1}{r^p} - r^p \sum_{n=p+1}^{\infty} |a_{p+n}| \tag{17}$$

As by Equation (11), we know that

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(q^p[p+n,q](1+B)(1+[p,q]\mu + \mu q^p[p+n,q])^m |a_{p+n}| + \frac{2p[p,q](1+A)}{p+n} \right) \\ &\leq [p,q](A-B). \end{aligned}$$

However,

$$\begin{aligned} &\frac{2p[p,q](1+A)}{p+1} + q^p[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=1}^{\infty} |a_{p+n}| \\ \leq &\sum_{n=1}^{\infty} \left(q^p[p+n,q](1+B)(1+[p,q]\mu + \mu q^p[p+n,q])^m |a_{p+n}| + \frac{2p[p,q](1+A)}{p+n} \right). \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{2p[p,q](1+A)}{p+1} + q^p[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=1}^{\infty} |a_{p+n}| \\ &\leq [p,q](A-B), \end{aligned}$$

which can be written as

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{(p+1)[p,q](A-B) - 2p[p,q](1+A)}{q^p(p+1)[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m} \tag{18}$$

Now, the required result can easily be obtained if we make use of Equations (16)–(18). □

Corollary 3. *For a function $f \in \mathcal{MK}_{\mu,q}(p,1,A,B)$. Then, for $|z| = r$, we have*

$$\frac{1}{r^p} - \Phi r^p \leq |f(z)| \leq \frac{1}{r^p} + \Phi r^p,$$

where

$$\Phi = \frac{(p+1)[p,q](A-B)-2p[p,q](1+A)}{q^p(p+1)[p+1,q](1+B)(1+[p,q]\mu+\mu q^p[p+1,q])}.$$

Theorem 4. *If $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$ and is of the form (1). Then, for $|z| = r$*

$$\frac{[p,q]_m}{q^{mp+\zeta}r^{m+p}} - \Psi r^p \leq \left|\partial_q^m f(z)\right| \leq \frac{[p,q]_m}{q^{mp+\zeta}r^{m+p}} + \Psi r^p.$$

where

$$\Psi = \frac{(A-B)[p,q](p+1)-2p[p,q](1+A)}{(p+1)q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m} \text{ and } \zeta = \sum_{n=1}^{m} n.$$

Proof. With the help of Equation (2) and using Equation (3), we can easily find that

$$\partial_q^m f(z) = \frac{(-1)^m[p,q]_m}{q^{mp+\zeta}z^{p+m}} + \sum_{n=1}^{\infty}[p+n-(m-1),q]_{m+1}a_{p+n}z^{p+n-m}.$$

Since, for the parameters $m \leq n$, $n \geq p+1$ and also for $|z| = r < 1$ implies that $r^{n-m} \leq r^p$ hence, we have

$$\left|\partial_q^m f(z)\right| \leq \frac{[p,q]_m}{q^{mp+\zeta}r^{m+p}} + r^p\sum_{n=1}^{\infty}[p+n-(m-1),q]_{m+1}\left|a_{p+n}\right|, \tag{19}$$

Similarly,

$$\left|\partial_q^m f(z)\right| \geq \frac{[p,q]_m}{q^{mp+\zeta}r^{m+p}} - r^p\sum_{n=1}^{\infty}[p+n-(m-1),q]_{m+1}|a_n|. \tag{20}$$

Now, by using Equation (11), we obtain the following inequality

$$\left(\frac{2p[p,q](1+A)}{p+1} + q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m\sum_{n=1}^{\infty}[p+n,q]\left|a_{p+n}\right|\right)$$
$$\leq \quad [p,q](A-B).$$

which implies that

$$q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m\sum_{n=1}^{\infty}[p+n,q]|a_n|$$
$$\leq \quad \frac{(A-B)[p,q](p+1)-2p[p,q](1+A)}{p+1}.$$

so we have

$$\sum_{n=1}^{\infty}[p+n,q]|a_n| \leq \frac{(A-B)[p,q](p+1)-2p[p,q](1+A)}{(p+1)q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m},$$

but it can easily be seen that

$$\sum_{n=1}^{\infty}[p+n-(m-1),q]\left|a_{p+n}\right| \leq \sum_{n=1}^{\infty}[p+n,q]|a_n|,$$

which implies

$$\sum_{n=1}^{\infty}[p+n-(m-1),q]\left|a_{p+n}\right| \leq \frac{(A-B)[p,q](p+1)-2p[p,q](1+A)}{(p+1)q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}.$$

Now, using the last inequality in Equations (19) and (20), we can obtain our result. □

Corollary 4. *If $f \in \mathcal{MK}_{\mu,q}(p,1,A,B)$ and is of the form (1). Then, for $|z| = r$*

$$\frac{[p,q]}{q^{p+\zeta}r^{p+1}} - \Psi r^p \leq \left|\partial_q f(z)\right| \leq \frac{[p,q]}{q^{p+\zeta}r^{p+1}} + \Psi r^p.$$

where

$$\Psi = \frac{(A-B)[p,q](p+1)-2p[p,q](1+A)}{(p+1)q^p(1+B)(1+[p,q]\mu+\mu q^p[p+n,q])} \text{ and } \zeta = \frac{n(n+1)}{2}.$$

In the next two theorems for the functions class $\mathcal{M}_{\mu,q}(p,m,A,B)$, the radii of starlikeness and convexity is given.

Theorem 5. *Let $f \in \mathcal{MK}_{\mu,q}(p,m,A,B)$. Then, $f \in \mathcal{MC}_p(\alpha)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{p(p-\alpha)q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(p+n)(n+p+\alpha)([p,q](A-B)(p+1)-2p[p,q](1+A))}\right)^{\frac{1}{n+2p}}.$$

Proof. Let f be in the class $\mathcal{MK}_{\mu,q}(p,m,A,B)$, to prove that f is in the class $\mathcal{MC}_p(\alpha)$, it will be enough if we show

$$\left|\frac{zf''(z)+(p+1)f'(z)}{zf''(z)+(1+2\alpha-p)f'(z)}\right| < 1,$$

Using Equation (1) in conjunction with some elementary calculations, we have

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n+\alpha)}{p(p-\alpha)}\left|a_{n+p}\right||z|^{n+2p} < 1, \tag{21}$$

From Equation (11), we can easily find that

$$\sum_{n=1}^{\infty} q^p[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m\left|a_{p+n}\right|$$
$$\leq \frac{[p,q](A-B)(p+1)-2p[p,q](1+A)}{p+1}.$$

which gives, after some simplification, that

$$\sum_{n=1}^{\infty} \frac{q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{[p,q](A-B)(p+1)-2p[p,q](1+A)}\left|a_{p+n}\right| < 1.$$

Now, inequality (21) will be true, if the following holds

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p+\alpha)}{p(p-\alpha)}\left|a_{n+p}\right||z|^{n+2p} <$$
$$\sum_{n=1}^{\infty} \frac{q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{[p,q](A-B)(p+1)-2p[p,q](1+A)}\left|a_{n+p}\right|,$$

which implies that

$$|z|^{n+2p} < \frac{p(p-\alpha)q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(p+n)(n+p+\alpha)([p,q](A-B)(p+1)-2p[p,q](1+A))},$$

and so

$$|z| < \left(\frac{p(p-\alpha)q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(p+n)(n+p+\alpha)([p,q](A-B)(p+1)-2p[p,q](1+A))}\right)^{\frac{1}{n+2p}}$$
$$= r_1,$$

we get the required condition. □

Corollary 5. *Let $f \in \mathcal{MK}_{\mu,q}(1, m, A, B)$. Then, $f \in \mathcal{MC}_p(\alpha)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{(1-\alpha)2q[n+1,q](1+B)(1+\mu+\mu q[n+1,q])^m}{(n+1)(n+1+\alpha)(2(A-B)-2(1+A))} \right)^{\frac{1}{n+2}}.$$

Theorem 6. *Let $f \in \mathcal{MK}_{\mu,q}(p, m, A, B)$. Then, $f \in \mathcal{MS}^*_p(\alpha)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{(p-\alpha)(q^p[p+n,q](1+B)+(1+A)[p,q])(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(n+p+\alpha)(A-B)[p,q]} \right)^{\frac{1}{n+2p}}.$$

Proof. We know that f is in the class $\mathcal{MS}^*_p(\alpha)$, if and only if

$$\left| \frac{zf'(z) + pf(z)}{zf'(z) - (p-2\alpha)f(z)} \right| < 1.$$

Now, make use of Equation (1) and, after some simple calculations, we have

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\alpha}{p-\alpha} \right) |a_{n+p}| |z|^{n+2p} < 1. \tag{22}$$

Now, from Equation (11) we can easily obtain

$$\sum_{n=1}^{\infty} \frac{q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{[p,q](A-B)(p+1)-2p[p,q](1+A)} |a_n| \quad < \quad 1.$$

Inequality (22) will be true if

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\alpha}{p-\alpha} \right) |a_{n+p}| |z|^{n+2p} \quad <$$

$$\sum_{n=1}^{\infty} \frac{q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{[p,q](A-B)(p+1)-2p[p,q](1+A)} |a_n|.$$

This gives

$$|z|^{n+2p} < \frac{(p-\alpha)q^p(p+1)[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(n+p+\alpha)([p,q](A-B)(p+1)-2p[p,q](1+A))},$$

and hence

$$|z| < \left(\frac{(p-\alpha)(q^p[p+n,q](1+B)+(1+A)[p,q])(1+[p,q]\mu+\mu q^p[p+n,q])^m}{(n+p+\alpha)(A-B)[p,q]} \right)^{\frac{1}{n+2p}} = r_2 \cdot$$

The proof of our Theorem is now complete. □

Corollary 6. *Let $f \in \mathcal{MK}_{\mu,q}(1, m, A, B)$. Then $f \in \mathcal{MS}^*(\alpha)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{(1-\alpha)(q[n+1,q](1+B)+(1+A))(1+\mu+\mu q[n+1,q])^m}{(n+1+\alpha)(A-B)} \right)^{\frac{1}{n+2}}.$$

3. Concluding Remarks and Observations

Recently, the basic (or q-) calculus is a center of attraction for many well-known mathematicians, because of its diverse applications in many areas of Mathematics and Physics see for example [11,33]. In our present investigations, we were essentially motivated by the recent research going on in this field of study, and we have first introduced a new class of meromorphic multivalent q-close-to-convex function with the help of a q-differential operator. We next investigate some useful properties such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikness and radius of convexity for this new subclass of meromorphic multivalent q-close-to-convex functions.

Author Contributions: Writing—original draft, L.S., B.A., N.K., W.K.M. and B.K.; Writing—review and editing, L.S., B.A., N.K., M.G.K., S.A., W.K.M. and B.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Foundation of Excellent Youth Teachers of Colleges and Universities of Henan Province under grant No. 2019GGJS195.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Aral, A.; Gupta, V. On the Durrmeyer type modification of the q-Baskakov type operators. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1171–1180. [CrossRef]
2. Aral, A.; Gupta, V. On q-Baskakov type operators. *Demonstr. Math.* **2009**, *42*, 109–122.
3. Jackson, F.H. On q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
4. Jackson, F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
5. Aral, A.; Gupta, V. Generalized q-Baskakov operators. *Math. Slovaca* **2011**, *61*, 619–634. [CrossRef]
6. Aral, A. On the generalized Picard and Gauss Weierstrass singular integrals. *J. Comput. Anal. Appl.* **2006**, *8*, 249–261.
7. Anastassiu, G.A.; Gal, S.G. Geometric and approximation properties of generalized singular integrals. *J. Korean Math. Soc.* **2006**, *23*, 425–443. [CrossRef]
8. Anastassiu, G.A.; Gal, S.G. Geometric and approximation properties of some singular integrals in the unit disk. *J. Inequal. Appl.* **2006**, *2006*, 17231. [CrossRef]
9. Li, M.-L.; Shi, L.; Wang, Z.-G. On a Subclass of Meromorphic Close-to-Convex Functions. *Sci. World J.* **2014**, *2014*, 806168. [CrossRef]
10. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Halsted Press: Chichester, UK; John Wiley and Sons: New York, NY, USA, 1989; pp. 329–354.
11. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
12. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
13. Aldweby, H.; Darus, M. Some subordination results on q-analogue of Ruscheweyh differential operator. *Abstr. Appl. Anal.* **2014**, *2014*, 958563. [CrossRef]
14. Srivastava, H.M.; Khan, N.; Khan, S.; Ahmad, Q.Z.; Khan, B. A Class of k-Symmetric Harmonic Functions Involving a Certain q-Derivative Operator. *Mathematics* **2021**, *9*, 1812. [CrossRef]
15. Arif, M.; Ahmad, B. New subfamily of meromorphic multivalent starlike functions in circular domain involving q-differential operator. *Math. Slovaca* **2018**, *68*, 1049–1056. [CrossRef]
16. Ahmad, B.; Khan, M.G.; Frasin, B.A.; Aouf, M.K.; Abdeljawad, T.; Mashwani, W.K.; Arif, M. On q-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain. *AIMS Math.* **2021**, *6*, 3037–3052. [CrossRef]
17. Ahmad, B.; Arif, M. New subfamily of meromorphic convex functions in circular domain involving q-operator. *Int. J. Anal. Appl.* **2018**, *16*, 75–82.
18. Islam, S.; Khan, M.G.; Ahmad, B.; Arif, M.; Chinram, R. Q-Extension of Starlike Functions Subordinated with a Trigonometric Sine Function. *Mathematics* **2020**, *8*, 1676. [CrossRef]
19. Shi, L.; Khan, M.G.; Ahmad, B. Some Geometric Properties of a Family of Analytic Functions Involving a Generalized q-Operator. *Symmetry* **2020**, *12*, 291. [CrossRef]
20. Mahmood, S.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Khan, B.; Tahir, M. A certain subclass of meromorphically q-starlike functions associated with the Janowski functions. *J. Inequal. Appl.* **2019**, *2019*, 88. [CrossRef]
21. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q-starlike functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
22. Rehman, M.S.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Darus, M.; Khan, B. Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions. *AIMS Math.* **2021**, *6*, 1110–1125. [CrossRef]
23. Khan, B.; Srivastava, H.M.; Tahir, M.; Darus, M.; Ahmad, Q.Z.; Khan, N. Applications of a certain integral operator to the subclasses of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 1024–1039. [CrossRef]
24. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. *Bull. Sci. Math.* **2021**, *167*, 102942. [CrossRef]
25. Ahmad, B.; Khan, M.G.; Aouf, M.K.; Mashwani, W.K.; Salleh, Z.; Tang, H. Applications of a New q-Difference Operator in Janowski-Type Meromorphic Convex Functions. *J. Funct. Spaces* **2021**, *2021*, 5534357.

26. Hu, Q.; Srivastava, H.; Ahmad, B.; Khan, N.; Khan, M.; Mashwani, W.; Khan, B. A subclass of multivalent Janowski type q-Starlike functions and its consequences. *Symmetry* **2021**, *13*, 1275. [CrossRef]
27. Dziok, J.; Murugusundaramoorthy, G.; Sokoł, J. On certain class of meromorphic functions with positive coefcients. *Acta Math. Sci. B* **2012**, *32*, 1376–1390. [CrossRef]
28. Huda, A.; Darus, M. Integral operator defined by q-analogue of Liu-Srivastava operator. *Stud. Univ. Babes-Bolyai Math.* **2013**, *58*, 529–537.
29. Pommerenke, C. On meromorphic starlike functions. *Pac. J. Math.* **1963**, *13*, 221–235. [CrossRef]
30. Seoudy, T.M.; Aouf, M.K. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order. *J. Math. Inequal.* **2016**, *10*, 135–145. [CrossRef]
31. Uralegaddi, B.A.; Somanatha, C. Certain diferential operators for meromorphic functions. *Houst. J. Math.* **1991**, *17*, 279–284.
32. Rogosinski, W. On the coefficients of subordinate functions. *Proc. Lond. Math. Soc.* **1943**, *48*, 48–82. [CrossRef]
33. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformatioons. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.