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Surfaces and Curves Induced by Nonlinear Schrödinger-Type Equations and Their Spin Systems

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Abstract: In recent years, symmetry in abstract partial differential equations has found wide application in the field of nonlinear integrable equations. The symmetries of the corresponding transformation groups for such equations make it possible to significantly simplify the procedure for establishing equivalence between nonlinear integrable equations from different areas of physics, which in turn open up opportunities to easily find their solutions. In this paper, we study the symmetry between differential geometry of surfaces/curves and some integrable generalized spin systems. In particular, we investigate the gauge and geometrical equivalence between the local/nonlocal nonlinear Schrödinger type equations (NLSE) and the extended continuous Heisenberg ferromagnet equation (HFE) to investigate how nonlocality properties of one system are inherited by the other. First, we consider the space curves induced by the nonlinear Schrödinger-type equations and its equivalent spin systems. Such space curves are governed by the Serret–Frenet equation (SFE) for three basis vectors. We also show that the equation for the third of the basis vectors coincides with the well-known integrable HFE and its generalization. Two other equations for the remaining two vectors give new integrable spin systems. Finally, we investigated the relation between the differential geometry of surfaces and integrable spin systems for the three basis vectors.

Keywords: symmetry in nonlinear integrable equation; nonlinear Schrödinger equation; Heisenberg ferromagnet equation; Chen–Lee–Liu equation; derivative spin system; isomorphism of Lie algebras; soliton solution; soliton surfaces; nonlocal integrable equations

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1. Introduction

The paper proposes an algebraic-geometric approach, which enables a universal description of symmetric nonlinear integrable equations. The method is based on the theory of isomorphism of the $su(2)$ and $so(3)$ Lie algebras. The proposed scheme is twisted, starting from the previously known results in [1,2], where geometric and gauge equivalences are established, respectively, between the nonlinear Schrodinger equation (NLSE)

$$iq_t + q_{xx} + 2q^*q^2 = 0, \tag{1}$$

and the Heisenberg ferromagnet equation (HFE)

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx}. \tag{2}$$

Here, $q(x,t)$ is a complex-valued wave function, the asterisk $*$ means the complex conjugation, $\mathbf{S}(x,t) = (S_1, S_2, S_3)$ is a three-component spin vector, and $\mathbf{S}^2 = 1$. The equivalent matrix form of HFE (2) is given by

$$S_t = \frac{1}{2}[S, S_{xx}], \tag{3}$$

where

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^2 = I, \quad S^\pm = S_1 \pm iS_2. \tag{4}$$

The solutions of these two equations (NLSE and HFE) are related by the Hasimota transformation

$$q(x,t) = \frac{\kappa}{2} e^{i\int \tau dy}, \tag{5}$$

where κ and τ are the curvature and torsion of the space curve, respectively. The equations of motion for κ and τ are derived from the following Serret–Frenet equation (SFE) [3],

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{7}$$

where

$$C = \begin{pmatrix} 0 & \kappa & \sigma \\ -\kappa & 0 & \tau \\ -\sigma & -\tau & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{8}$$

Here, κ and σ are the geodesic and normal curvatures of the of the space curve, τ is its torsion, and ω_j $(j = 1,2,3)$ are some real functions. The later functions must be expressed in terms of κ, σ, τ and their derivatives, when identifying spin vector $\mathbf{S}$ with basis vector $\mathbf{e}_1$ ($\mathbf{S} \equiv \mathbf{e_1}$) [1].

As a second example of the application of the approach described in Section 2 to other nonlinear integrable equations, we demonstrate it to the derivative NLSE [4,5]

$$iq_t + q_{xx} + iqq^* q_x = 0, \tag{9}$$

which is also called the Chen–Lee–Liu equation (CLLE) and to the derivative spin system

$$iS_t + \frac{1}{2}[S, S_{xx}] - \frac{i}{8\beta^2} tr(S_x^2) S_x = 0. \tag{10}$$

The last equation is also known as the derivative HFE (dHFE).

The paper is organized as follows. Section 2 provides information on an algebraic-geometric approach to establishing geometric equivalence between integrable nonlinear equations based on the isomorphism of the $su(2)$ and $so(3)$ Lie algebras. Section 3 applies this method for NLSE (1) and HFE (3). A demonstration of this approach for derivative-type NLSE (9) and dHFE (10) is given in Section 4. Section 5 is devoted to solving dHFE (10). The soliton surface approach is presented in Section 6. The nonlocal NLSE and CLLE with their nonlocal dHFE was studied in Section 7. The conclusion of the work is given in Section 8.

2. Isomorphism of the $su(2) \approx so(3)$ Lie Algebras and Integrable Equations

The Lax pair for the NLSE (1) is given by [2]

$$U_1 = -i\lambda\sigma_3 + Q, \tag{11}$$

$$V_1 = -2i\lambda^2\sigma_3 + \lambda V_1 + V_0, \tag{12}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad V_1 = 2Q, \quad V_0 = i\begin{pmatrix} -rq & q_x \\ -r_x & rq \end{pmatrix}. \tag{13}$$

At the same, the Lax pair for the HFE (3) has the form

$$U_2 = -i\lambda S, \tag{14}$$

$$V_2 = -2i\lambda^2 S + \lambda S S_x, \tag{15}$$

where S has the form as (4). Then, the linear systems corresponding to the NLSE (1)

$$\Phi_{1x} = U_1 \Phi_1, \tag{16}$$

$$\Phi_{1t} = V_1 \Phi_1 \tag{17}$$

and to the HFE (3)

$$\Phi_{2x} = U_2 \Phi_2, \tag{18}$$

$$\Phi_{2t} = V_2 \Phi_2 \tag{19}$$

are gauge equivalent to each other through the transformation $\Phi_2 = g^{-1}\Phi_1$ [2], where the function $g(x,t)$ is a solution of the system (16) and (17) for $\lambda = \lambda_0$ and $U_1, V_1, U_2, V_2 \in su(2)$.

Let us give some information on the isomorphism $su(2) \approx so(3)$ Lie algebras [6]. We expand the matrix $C \in so(3)$ from the SFE (6)–(8) in the form

$$C = \begin{pmatrix} 0 & \kappa & \sigma \\ -\kappa & 0 & \tau \\ -\sigma & -\tau & 0 \end{pmatrix} = -\tau L_1 + \sigma L_2 - \kappa L_3. \tag{20}$$

Here, $L_j (j = 1,2,3)$ are basis of the $so(3)$ algebra and

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

These basis matrices satisfy the following commutation relations

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.$$

Similarly, we have

$$[l_1, l_2] = l_3, \quad [l_2, l_3] = l_1, \quad [l_3, l_1] = l_2,$$

where l_j are basis of the $su(2)$ algebra

$$l_j = \frac{1}{2i}\sigma_j.$$

Here, σ_j are Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

The matrix $U \in su(2)$ can be expanded in the basis matrices as

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix} = a_1 l_1 + a_2 l_2 + a_3 l_3 = \tfrac{a_1}{2i}\sigma_1 + \tfrac{a_2}{2i}\sigma_2 + \tfrac{a_3}{2i}\sigma_3 = \tfrac{1}{2i}\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \tag{23}$$

From Equation (23), we have

$$a_3 = 2iu_{11}, \quad a^+ = a_1 + ia_2 = 2iu_{21}, \quad a^- = a_1 - ia_2 = 2iu_{12} \tag{24}$$

or

$$a_1 = i(u_{21} + u_{12}), \quad a_2 = u_{21} - u_{12}, \quad a_3 = 2iu_{11}. \tag{25}$$

Similarly, the matrix $D \in so(3)$ can be expanded in basis matrices as

$$D = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix} = -\omega_1 L_1 + \omega_2 L_2 - \omega_3 L_3. \tag{26}$$

At the same time, for the matrix $V \in su(2)$ we have

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix} = b_1 l_1 + b_2 l_2 + b_3 l_3 = \tfrac{b_1}{2i}\sigma_1 + \tfrac{b_2}{2i}\sigma_2 + \tfrac{b_3}{2i}\sigma_3 = \tfrac{1}{2i}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}. \tag{27}$$

Thus, we obtain

$$b_3 = 2iv_{11}, \quad b^+ = b_1 + ib_2 = 2iv_{21}, \quad b^- = b_1 - ib_2 = 2iv_{12} \tag{28}$$

or

$$b_1 = i(v_{21} + v_{12}), \quad b_2 = v_{21} - v_{12}, \quad b_3 = 2iv_{11}. \tag{29}$$

Finally, we get the following connections between the elements of the matrices U, V and C, D:

$$a_1 = -\tau, \quad a_2 = \sigma, \quad a_3 = -\kappa, \tag{30}$$

$$b_1 = -\omega_1, \quad b_2 = \omega_2, \quad b_3 = -\omega_3 \tag{31}$$

or

$$\tau = -i(u_{21} + u_{12}), \quad \sigma = u_{21} - u_{12}, \quad \kappa = -2iu_{11}, \tag{32}$$

$$\omega_1 = -i(v_{21} + v_{12}), \quad \omega_2 = v_{21} - v_{12}, \quad \omega_3 = -2iv_{11}. \tag{33}$$

From the compatibility condition $\mathbf{e}_{jxt} = \mathbf{e}_{jtx}$ of the SFE (6)–(8) it is easy to write equations for κ, τ, and σ as

$$\kappa_t = \omega_{3x} - \tau\omega_2 + \sigma\omega_1, \tag{34}$$

$$\sigma_t = \omega_{2x} - \kappa\omega_1 + \tau\omega_3, \tag{35}$$

$$\tau_t = \omega_{1x} - \sigma\omega_3 + \kappa\omega_2. \tag{36}$$

For our convenience, let us rewrite the SFE (6)–(8) in components as

$$\mathbf{e}_{1x} = \kappa\mathbf{e}_2 + \sigma\mathbf{e}_3, \tag{37}$$

$$\mathbf{e}_{2x} = -\kappa\mathbf{e}_1 + \tau\mathbf{e}_3, \tag{38}$$

$$\mathbf{e}_{3x} = -\sigma\mathbf{e}_1 - \tau\mathbf{e}_2 \tag{39}$$

and

$$\mathbf{e}_{1t} = \omega_3\mathbf{e}_2 + \omega_2\mathbf{e}_3, \tag{40}$$

$$\mathbf{e}_{2t} = -\omega_3\mathbf{e}_1 + \omega_1\mathbf{e}_3, \tag{41}$$

$$\mathbf{e}_{3t} = -\omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2. \tag{42}$$

Calculating the vector product $\mathbf{e}_3 \times \mathbf{e}_{3xx}$ from (39) and taking into account (37) and (38), we get

$$\tau_x\mathbf{e}_1 - \sigma_x\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_{3xx} + \kappa\mathbf{e}_{3x}.$$

Now from (42), taking into account the last relation, we can always obtain the following generalized HFE in the form

$$\mathbf{e}_{3t} + \mathbf{e}_3 \times \mathbf{e}_{3xx} + 2\kappa\mathbf{e}_{3x} = 0. \tag{43}$$

The specific form of the spin system depends on the accepted value κ.

3. NLSE and HFE

Take into account the object of research the Lax pair (7) and (8), we expand $U_1(x,t)$ in the basis matrices as

$$U_1 = \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3. \tag{44}$$

We get

$$U_1 = i(r+q)l_1 + (r-q)l_2 - 2\lambda l_3. \tag{45}$$

Moreover, expanding $V_1(x,t)$ in these basis matrices as

$$V_1 = z_1 l_1 + z_2 l_2 + z_3 l_3, \tag{46}$$

we obtain that

$$V_1 = (2i\lambda(r+q) + (r-q)_x)l_1 + (2\lambda(r-q) - i(r+q)_x)l_2 + (4\lambda^2 + 2rq)l_3. \tag{47}$$

Now, we move from $l_j \in su(2)$ to $L_j \in so(3)$ and from $U_1, V_1 \in su(2)$ to $C, D \in so(3)$. Then, we have

$$\kappa = -2\lambda, \tag{48}$$

$$\sigma = r - q, \tag{49}$$

$$\tau = -i(r+q). \tag{50}$$

Similar transformation for the matrices V_1 and D

$$V_1 \to D = z_1 L_1 + z_2 L_2 + z_3 L_3 \tag{51}$$

gives us the following expressions for the functions ω_j:

$$\omega_3 = \frac{1}{2}(\tau^2 + \sigma^2) - \kappa^2, \tag{52}$$

$$\omega_2 = \tau_x - \kappa\sigma, \tag{53}$$

$$\omega_1 = -\sigma_x - \kappa\tau. \tag{54}$$

In this case, from the integrability condition $\mathbf{e}_{jxt} = \mathbf{e}_{jtx}$ taking into account the equations for κ, σ, τ (48)–(50) we derive the following equation:

$$\mathbf{e}_{3t} + \mathbf{e}_3 \times \mathbf{e}_{3xx} + 2\kappa\mathbf{e}_{3x} = 0. \tag{55}$$

This equation in the case when $\lambda = 0$ and $\mathbf{S} \equiv \mathbf{e}_3$ goes to the HFE (2).

Next, we consider the case $\lambda = 0$, then $\kappa = 0, \quad \sigma = r - q, \quad \tau = -i(r+q)$. Then, we have

$$\mathbf{e}_{1x} = \sigma\mathbf{e}_3, \tag{56}$$

$$\mathbf{e}_{2x} = -\tau\mathbf{e}_3 \tag{57}$$

and

$$\mathbf{e}_{1t} = \frac{1}{2}(\tau^2 + \sigma^2)\mathbf{e}_2 + \tau_x\mathbf{e}_3, \tag{58}$$

$$\mathbf{e}_{2t} = -\frac{1}{2}(\tau^2 + \sigma^2)\mathbf{e}_1 - \sigma_x\mathbf{e}_3. \tag{59}$$

Finding $\mathbf{e}_3$ from the Equation (56) and vector multiplying on the left by $\mathbf{e}_1$ we have

$$\mathbf{e}_2 = \frac{1}{\sigma}\mathbf{e}_1 \times \mathbf{e}_{1x} \tag{60}$$

and also

$$\mathbf{e}_{1xx} = \sigma_x \mathbf{e}_3 - \sigma(\sigma \mathbf{e}_1 + \tau \mathbf{e}_2). \tag{61}$$

Scalar multiplying (61) by $\mathbf{e}_1$, we get

$$\sigma = \sqrt{-\mathbf{e}_1 \cdot \mathbf{e}_{1xx}}. \tag{62}$$

Thus,

$$\frac{\sigma_x}{\sigma^2}\mathbf{e}_1 \cdot \mathbf{e}_{1x}\frac{1}{\sigma}\mathbf{e}_1 \times \mathbf{e}_{1xx} = \frac{\tau}{\sigma}\mathbf{e}_{1x}. \tag{63}$$

Therefore, we obtain

$$\tau = -\frac{\mathbf{e}_{1x} \cdot (\mathbf{e}_1 \times \mathbf{e}_{1xx})}{\mathbf{e}_{1x}^2}, \tag{64}$$

as well as

$$\mathbf{e}_1 \times \mathbf{e}_{1x} = \frac{\sigma}{\sigma_x}(\mathbf{e}_1 \times \mathbf{e}_{1xx} + \tau \mathbf{e}_{1x}). \tag{65}$$

Now, for $\mathbf{e}_1$ in (58), taking into consideration Equations (62)–(64), we get the equation

$$\mathbf{e}_{1t} = -\frac{\tau^2 + \sigma^2}{2\sigma_x}\mathbf{e}_1 \times \mathbf{e}_{1xx} + \left(\frac{\tau_x}{\sigma} - \frac{(\tau^2+\sigma^2)\tau}{2\sigma_x}\right)\mathbf{e}_{1x}. \tag{66}$$

Similarly, for $\mathbf{e}_2$ we get

$$\mathbf{e}_3 = \frac{1}{\tau}\mathbf{e}_{2x}, \quad \mathbf{e}_1 = \frac{1}{\tau}\mathbf{e}_2 \times \mathbf{e}_{2x}, \tag{67}$$

$$\tau = \sqrt{-\mathbf{e}_2 \cdot \mathbf{e}_{2xx}}, \quad \sigma = \frac{\mathbf{e}_{2x} \cdot (\mathbf{e}_2 \times \mathbf{e}_{2xx})}{\mathbf{e}_{2x}^2} \tag{68}$$

and

$$\mathbf{e}_2 \times \mathbf{e}_{2x} = \frac{\tau}{\tau_x}(\mathbf{e}_2 \times \mathbf{e}_{2xx} - \sigma \mathbf{e}_{2x}). \tag{69}$$

From these equations, we obtain the following equation for $\mathbf{e}_2$:

$$\mathbf{e}_{2t} = -\frac{\tau^2 + \sigma^2}{2\tau_x}\mathbf{e}_2 \times \mathbf{e}_{2xx} - \left(\frac{\sigma_x}{\tau} - \frac{(\tau^2+\sigma^2)\sigma}{2\tau_x}\right)\mathbf{e}_{2x}. \tag{70}$$

Thus, the well-known isomorphism $su(2) \approx so(3)$ of two Lie algebras gives the transformation from U_1, V_1 to C, D. Thus, we have obtained three integrable vector equations for three unit vectors $\mathbf{e}_j$. Note that the equation for the vector $\mathbf{e}_3$ coincides with Equation (55), which is the well-known integrable HFE (2) that corresponds to the case $\lambda = 0$ and the identification $\mathbf{S} = \mathbf{e}_3$. At the same time, for the $\kappa = 0, \sigma \neq 0, \tau \neq 0$ case we obtain the following two other integrable equations for the remaining two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{e}_{1t} = a_1 \mathbf{e}_1 \times \mathbf{e}_{1xx} + b_1 \mathbf{e}_{1x},$$

$$\mathbf{e}_{2t} = a_2 \mathbf{e}_2 \times \mathbf{e}_{2xx} + b_2 \mathbf{e}_{2x},$$

where

$$a_1 = -\frac{\tau^2+\sigma^2}{2\sigma_x}, \quad b_1 = \frac{\tau_x}{\sigma} - \frac{(\tau^2+\sigma^2)\tau}{2\sigma_x}$$

and

$$a_2 = -\frac{\tau^2+\sigma^2}{2\tau_x}, \quad b_2 = -\frac{\sigma_x}{\tau} + \frac{(\tau^2+\sigma^2)\sigma}{2\tau_x}.$$

This concludes the demonstration of the application of our proposed algebra-geometric approach to NLSE (1) and HFE (2). Thus, in this section, we presented the geometrical formulation of the two fundamental integrable equations: the NLSE and the HFE. Using this approach, we have found three integrable spin systems which are equivalent to the

NLSE. One of these equations, namely, the equation for the vector function $\mathbf{e}_3$, coincides with the original HFE. It is recovered the well-known geometrical equivalence between the NLSE and HFE.

4. Chen–Lee–Liu Equation and Its Equivalent Derivative Spin System

In this section, we will apply the algebraic-geometric approach of establishing geometrical equivalence between nonlinear integrable equations to the derivative NLSE, namely, to the so-called Chen–Lee–Liu equation (CLLE) [5]. The standard (local) CLLE is given by

$$iq_t + q_{xx} + 2qq^* q_x = 0. \tag{71}$$

Its equivalent spin system, the local derivative spin system reads as

$$iS_t + \frac{1}{2}[S, S_{xx}] - \frac{i}{8\beta^2} tr(S_x^2) S_x = 0. \tag{72}$$

The CLLE (71) is associated with the following linear system [5]:

$$\Phi_x = U_3 \Phi, \tag{73}$$
$$\Phi_t = V_3 \Phi. \tag{74}$$

Here, the Lax matrices U_3 and V_3 have the forms

$$U_3 = \left(-i\lambda^2 - \frac{i}{4} rq\right)\sigma_3 + \lambda Q, \tag{75}$$
$$V_3 = \left[-2i\lambda^4 - irq\lambda^2 - \frac{1}{4}(r_x q - r q_x) - \frac{i}{8} r^2 q^2\right]\sigma_3 + 2\lambda^3 Q + \lambda P, \tag{76}$$

where

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & iq_x + \frac{1}{2} r q^2 \\ -ir_x + \frac{1}{2} r^2 q & 0 \end{pmatrix}.$$

The compatibility condition of the linear Equations (73) and (74)

$$U_{3t} - V_{3x} + [U_3, V_3] = 0$$

gives the CLLE

$$iq_t + q_{xx} + 2rqq_x = 0, \tag{77}$$
$$ir_t - r_{xx} - 2rqr_x = 0, \tag{78}$$

where in the local case we have the following reduction $r(t, x) = kq^*(t, x)$ with $k = \pm 1$.

The set of the linear equations associated with the dHFE (72) reads as

$$\Psi_x = U_4 \Psi, \tag{79}$$

$$\Psi_t = V_4 \Psi. \tag{80}$$

To find the matrices U_4 and V_4, let us consider the transformation

$$\Psi = h^{-1} \Phi, \tag{81}$$

where Ψ is the solution of the required spectral problem, Φ is the solution of the linear system (73) and (74), $h = \Phi|_{\lambda=\beta}$.

The derivative of (81) with respect to x yields

$$\begin{aligned} \Psi_x = \left(h^{-1}\Phi\right)_x &= h^{-1}\Phi_x - h^{-1} h_x h^{-1} \Phi = h^{-1} U_3 \Phi - h^{-1} U_{03} \Phi \\ &= h^{-1}[U_3 - U_{03}]\Phi = h^{-1}[U_3 - U_{03}] h \Psi = U_4 \Psi, \end{aligned} \tag{82}$$

where $U_{03} = U_3|_{\lambda=\beta}$. From (75) in the case $\lambda = 0$, we get

$$\begin{aligned} U_3 - U_{03} &= \left(-i\lambda^2 - \tfrac{i}{4}rq\right)\sigma_3 + \lambda Q - \left(-i\beta^2 - \tfrac{i}{4}rq\right)\sigma_3 - \beta Q \\ &= -i(\lambda^2 - \beta^2)\sigma_3 + (\lambda - \beta)Q. \end{aligned} \tag{83}$$

Let us now introduce the notation

$$S = h^{-1}\sigma_3 h = \frac{1}{\Delta}\begin{pmatrix} |h_1|^2 - |h_2|^2 & -2h_1^* h_2^* \\ -2h_1 h_2 & |h_2|^2 - |h_1|^2 \end{pmatrix} = \frac{1}{1+|w|^2}\begin{pmatrix} 1 - |w|^2 & 2w^* \\ 2w & |w|^2 - 1 \end{pmatrix}, \tag{84}$$

where

$$h = \begin{pmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{pmatrix}, \quad \Delta = |h_1|^2 + |h_2|^2, \quad w = -\frac{h_2}{h_1^*}. \tag{85}$$

Here, $H = (h_1, h_2)^T$ is a solution of the system (79) and (80), and components of the spin matrix $S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}$ are written as

$$S^+ = -\frac{2h_1 h_2}{\Delta}, \quad S^- = -\frac{2h_1^* h_2^*}{\Delta}, \quad S_3 = \frac{|h_1|^2 - |h_2|^2}{\Delta} \tag{86}$$

and

$$S^+ = \frac{2w}{1+|w|^2}, \quad S^- = \frac{2w^*}{1+|w|^2}, \quad S_3 = \frac{1-|w|^2}{1+|w|^2}. \tag{87}$$

Let us here we present the angle presentation of the components of the spin vector (matrix). We have

$$S^+ = e^{i\varphi}\sin\theta, \quad S^- = e^{-i\varphi}\sin\theta, \quad S_3 = \cos\theta \tag{88}$$

so that

$$w = e^{i\varphi}\tan\frac{\theta}{2}. \tag{89}$$

Then,

$$S_x = \left(h^{-1}\sigma_3 h\right)_x = h^{-1}\left[\sigma_3, h_x h^{-1}\right]h = \beta h^{-1}[\sigma_3, Q]h = 2\beta S h^{-1} Q h, \tag{90}$$

and

$$S_x^2 = 4\beta^2 h^{-1}\begin{pmatrix} -rq & 0 \\ 0 & -rq \end{pmatrix}h = -4\beta^2 rq I = \mathbf{S}_x^2 I. \tag{91}$$

The trace of the last equation gives

$$tr\left(S_x^2\right) = -8\beta^2 rq = 2\mathbf{S}_x^2 \tag{92}$$

or

$$rq = -\frac{1}{8\beta^2}tr\left(S_x^2\right) = -\frac{1}{4\beta^2}\mathbf{S}_x^2. \tag{93}$$

From the equation (90), we obtain

$$2\beta h^{-1}Qh = SS_x \tag{94}$$

or

$$h^{-1}Qh = \frac{1}{2\beta}SS_x = \frac{1}{4\beta}[S, S_x]. \tag{95}$$

Now taking into account (82) and (87), we can finally write the matrix U_4 in the following form:

$$U_4 = h^{-1}(U_3 - U_{03})h = -i\left(\lambda^2 - \beta^2\right)S + \frac{\lambda - \beta}{2\beta}SS_x. \tag{96}$$

For convenience of further calculations, we represent U_4 as a polynomial of the second degree in λ as

$$U_4 = \lambda^2 A_2 + \lambda A_1 + A_0, \tag{97}$$

where

$$A_2 = -iS,$$

$$A_1 = \frac{1}{4\beta}[S, S_x] = \frac{1}{2\beta}SS_x,$$

$$A_0 = i\beta^2 S - \frac{1}{2}SS_x.$$

Similarly, we obtain

$$V_4 = h^{-1}(V_3 - V_{03})h = \left[-2i\left(\lambda^4 - \beta^4\right) - irq\left(\lambda^2 - \beta^2\right)\right]S +$$

$$+ 2\left(\lambda^3 - \beta^3\right)h^{-1}Qh + (\lambda - \beta)h^{-1}Ph. \tag{98}$$

The last term of this relation must also be expressed in terms of the matrix S. It is not difficult to verify that

$$h^{-1}Ph = iSh^{-1}Q_x h + \frac{rq}{4\beta}SS_x. \tag{99}$$

Next, in order to express $h^{-1}Q_x h$ in terms of S, we find the derivative of S_x in (90) with respect to x as

$$S_{xx} = -S_x^2 S + 2\left(i\beta^2 + \frac{i}{4}rq\right)SS_x + 2\beta S\left(h^{-1}Q_x h\right). \tag{100}$$

Therefore, we get

$$h^{-1}Q_x h = \frac{1}{2\beta}\left(SS_{xx} + S_x^2 - 2\left(i\beta^2 + \frac{i}{4}rq\right)S_x\right) \tag{101}$$

Substituting (101) into (99), we obtain the relation

$$h^{-1}Ph = \frac{rq}{4\beta}SS_x + \frac{i}{2\beta}\left(S_{xx} + S_x^2 S - 2i\left(\beta^2 + \frac{rq}{4}\right)SS_x\right) =$$

$$= \frac{i}{2\beta}\left(S_{xx} + SS_x^2\right) + \left(\beta + \frac{rq}{2\beta}\right)SS_x. \tag{102}$$

Finally, we have the following expression for the matrix V_4:

$$V_4 = \left[-2i\left(\lambda^4 - \beta^4\right) - irq\left(\lambda^2 - \beta^2\right)\right]S + \frac{\lambda^3 - \beta^3}{\beta}SS_x +$$

$$+ \frac{i}{2\beta}(\lambda - \beta)\left(S_{xx} + S_x^2 S\right) + (\lambda - \beta)\left(\beta + \frac{rq}{2\beta}\right)SS_x. \tag{103}$$

Equation (103), in short, can be rewritten as

$$V_4 = \lambda^4 B_4 + \lambda^3 B_3 + \lambda^2 B_2 + \lambda B_1 + B_0, \tag{104}$$

where

$$\begin{aligned} B_4 &= -2iS, \\ B_3 &= \frac{1}{\beta} SS_x, \\ B_2 &= -irqS, \\ B_1 &= \frac{i}{2\beta}\left(S_{xx} + S_x^2 S\right) + \left(\beta + \frac{rq}{2\beta}\right) SS_x, \\ B_0 &= \left(2i\beta^4 + 3i\beta^2 rq\right) S - \frac{i}{2} S_{xx} - \left(2\beta^2 + \frac{rq}{2}\right) SS_x. \end{aligned}$$

The left side of the zero curvature condition

$$U_{4t} - V_{4x} + [U_4, V_4] = 0 \tag{105}$$

is a sixth degree polynomial in λ. The coefficients at the corresponding powers of λ have the form

$$\begin{aligned} \lambda^6 &: \quad [A_2, B_4] = 0, \\ \lambda^5 &: \quad [A_2, B_3] + [A_1, B_4] = 0, \\ \lambda^4 &: \quad B_{4x} - [A_2, B_2] - [A_1, B_3] - [A_0, B_4] = 0, \\ \lambda^3 &: \quad B_{3x} - [A_2, B_1] - [A_1, B_2] - [A_0, B_3] = 0, \\ \lambda^2 &: \quad A_{2t} - B_{2x} + [A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0, \\ \lambda^1 &: \quad A_{1t} - B_{1x} + [A_1, B_0] + [A_0, B_1] = 0, \\ \lambda^0 &: \quad A_{0t} - B_{0x} + [A_0, B_0] = 0. \end{aligned}$$

The coefficients of powers λ^6, λ^5, and λ^4 satisfy identically, and the coefficient at powers λ^3 gives the expression

$$(SS_x)_x = \frac{1}{2}[S, S_{xx}] - irq(1 + \beta^2) S_x.$$

The coefficient of the degree λ^2 generates the dHFE (72). The coefficient of the constant term with the coefficient of λ^1 also gives equation (72). Thus, we have shown that there is a gauge equivalence between the local CLLE (71) and dHFE (72).

Next, we illustrate the geometrical formalism presented in Section 2 to the local CLLE (71) and dHFE (72). In this case, for the Lax matrices U_3, V_3 we have

$$U_3 = i(r+q) l_1 + (r-q) l_2 - 2\lambda l_3 \tag{106}$$

and

$$V_3 = (2i\lambda(r+q) + (r-q)_x) l_1 + (2\lambda(r-q) - i(r+q)_x) l_2 + (4\lambda^2 + 2rq) l_3. \tag{107}$$

Then, passing from $U_3, V_3 \in su(2)$ to $C, D \in so(3)$ through an isomorphism of Lie algebras $su(2) \approx so(3)$, for the functions $\kappa, \sigma, \tau, \omega_j$ we obtain the following expressions:

$$\kappa = -(2\lambda^2 + \frac{qr}{2}), \tag{108}$$

$$\sigma = \lambda(r-q), \tag{109}$$

$$\tau = -i\lambda(r+q) \tag{110}$$

and

$$\begin{aligned}\omega_3 &= -4\lambda^4 - 2rq\lambda^2 + \frac{i}{2}(r_x q - rq_x) + \frac{i}{4}r^2q^2, &(111)\\ \omega_2 &= 2\lambda^3(r-q) - i\lambda(r_x + q_x) + \frac{\lambda}{2}rq(r-q), &(112)\\ \omega_1 &= -\left(2i\lambda^3(r+q) + \lambda(r_x - q_x) + \frac{i\lambda}{2}rq(r+q)\right) &(113)\end{aligned}$$

with

$$r = \frac{\sigma + i\tau}{2\lambda}, \quad q = -\frac{\sigma - i\tau}{2\lambda}. \tag{114}$$

Now from SFE (6) and (7), using (108)–(113), for any $\lambda = \beta, \beta = const$, we get the following equation:

$$\mathbf{e}_{3t} + \mathbf{e}_3 \times \mathbf{e}_{3xx} - \left(2\lambda^2 - \frac{1}{8\lambda^2}\mathbf{e}_{3x}^2\right)\mathbf{e}_{3x} = 0. \tag{115}$$

Equation (115) at $rq = -\frac{1}{4\beta^2}\mathbf{S}_x^2 = -\frac{1}{8\beta^2}tr(S_x^2)$ and $\mathbf{S} \equiv \mathbf{e}_3$ takes the form

$$\mathbf{S}_t + \mathbf{S} \times \mathbf{S}_{3xx} - \left(2\lambda^2 - \frac{1}{8\beta^2}\mathbf{S}_x^2\right)\mathbf{S}_x = 0, \tag{116}$$

which in matrix form becomes exactly the same as (72). This confirms that the method used in the Section 2 works for any integrable equations.

5. Soliton Solution

Now we would like to construct, for example, the 1-soliton solution of the dHFE. To construct the 1-soliton solution of the dHFE (72), we consider the seed solution of the CLLE (71) of the form $r = q = 0$. Then, the associated linear system (73) and (74) takes the form

$$\begin{aligned}\Phi_{0x} &= -i\lambda^2\sigma_3\Phi_0, &(117)\\ \Phi_{0t} &= -2i\lambda^4\sigma_3\Phi_0, &(118)\end{aligned}$$

where

$$\Phi_0 = \begin{pmatrix}\phi_{01} & -\phi_{02}^* \\ \phi_{02} & \phi_{01}^*\end{pmatrix}, \quad \Phi_0^{-1} = \frac{1}{\det\Phi_0}\begin{pmatrix}\phi_{01}^* & \phi_{02}^* \\ -\phi_{02} & \phi_{01}\end{pmatrix}, \quad \det\Phi_0 = |\phi_{01}|^2 + |\phi_{02}|^2. \tag{119}$$

The corresponding solution of the linear Equations (117) and (118) has the form

$$\begin{aligned}\phi_{01} &= c_1 e^{-\chi} = c_1 e^{-i(\lambda^2 x + 2\lambda^4 t + \delta_1)}, &(120)\\ \phi_{02} &= c_2 e^{\chi + i\delta_{21}} = c_2 e^{i(\lambda^2 x + 2\lambda^4 t + \delta_2)}, &(121)\end{aligned}$$

where c_j are complex constants, and $\chi = \chi_1 + i\chi_2 = i(\lambda^2 x + 2\lambda^4 t + \delta_1)$, $\delta_{21} = \delta_2 - \delta_1$, $\lambda = \alpha + i\beta$ and δ_j, α, β are real constants. For the spin matrix S, we have

$$S = \begin{pmatrix}S_3 & S^- \\ S^+ & -S_3\end{pmatrix} = \Phi_0^{-1}\sigma_3\Phi_0 = \begin{pmatrix}|\phi_{01}|^2 - |\phi_{02}|^2 & -2\phi_{01}^*\phi_{02}^* \\ -2\phi_{01}\phi_{02} & |\phi_{02}|^2 - |\phi_{01}|^2\end{pmatrix}. \tag{122}$$

For the components of the spin matrix S, we obtain the following expressions:

$$S_3 = \frac{|\phi_{01}|^2 - |\phi_{02}|^2}{\det\Phi_0}, \quad S^+ = -\frac{2\phi_{01}\phi_{02}}{\det\Phi_0}. \tag{123}$$

Substituting the expressions for the functions ϕ_{oj} into the Formula (123), we obtain the 1-soliton solution of the spin system (72) as

$$S_3 = \frac{|c_1|^2 e^{-2\chi_1} - |c_2|^2 e^{2\chi_1}}{|c_1|^2 e^{-2\chi_1} + |c_2|^2 e^{2\chi_1}}, \quad S^+ = -\frac{2c_1 c_2 e^{i\delta_{21}}}{|c_1|^2 e^{-2\chi_1} + |c_2|^2 e^{2\chi_1}} \tag{124}$$

or

$$S_3 = -\tanh(2\chi_1) = 1 - \frac{e^{2\chi_1}}{|c_1|\cosh(2\chi_1)}, \quad S^+ = -\frac{e^{i(\delta_{21}+\epsilon_1+\epsilon_2)}}{\cosh(2\chi_1)}, \quad S^- = S^{+*}, \tag{125}$$

where $c_j = |c_j|e^{i\epsilon_j}$. Thus, using the gauge equivalence between the local CLLE and the local dHFE, we have constructed the 1-soliton solution of the dHFE.

6. Soliton Surface

In this section, our aim is to present the soliton surfaces induced by the local CLLE and its equivalent spin system. To do that, let us recall that the position vector $\mathbf{r} = (r_1, r_2, r_3)$ of the soliton surface satisfies the certain two equations. In terms of the matrix form of the position vector

$$r = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix}$$

these two equations have the following forms:

$$r_x = \Phi^{-1}U_\lambda \Phi, \tag{126}$$

$$r_t = \Phi^{-1}V_\lambda \Phi, \tag{127}$$

where U, V are the Lax pair of the corresponding integrable nonlinear differential equation. Therefore, we obtain the well-known Sym–Tafel formula

$$r = \Phi^{-1}\Phi_\lambda = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix}. \tag{128}$$

Using the following expressions

$$\Phi_\lambda = \begin{pmatrix} \phi_{1\lambda} & -\phi^*_{2\lambda} \\ \phi_{2\lambda} & \phi^*_{1\lambda} \end{pmatrix}, \quad \Phi^{-1} = \frac{1}{\det\Phi}\begin{pmatrix} \phi^*_1 & \phi^*_2 \\ -\phi_2 & \phi_1 \end{pmatrix}, \quad \det\Phi = |\phi_1|^2 + |\phi_2|^2, \tag{129}$$

we finally have

$$r = \frac{1}{\det\Phi}\begin{pmatrix} \phi^*_1\phi_{1\lambda} + \phi^*_2\phi_{2\lambda} & -\phi^*_1\phi^*_{2\lambda} + \phi^*_2\phi^*_{1\lambda} \\ -\phi_2\phi_{1\lambda} + \phi_1\phi_{2\lambda} & \phi_2\phi^*_{2\lambda} + \phi_1\phi^*_{1\lambda} \end{pmatrix}. \tag{130}$$

Thus, for the components of the position vector $\mathbf{r} = (r_1, r_2, r_3)$, we obtain

$$r^+ = r_1 + ir_2 = \frac{-\phi_2\phi_{1\lambda} + \phi_1\phi_{2\lambda}}{\det\Phi}, \quad r^- = \frac{\phi^*_1\phi^*_{2\lambda} + \phi^*_2\phi^*_{1\lambda}}{\det\Phi}, \quad r_3 = \frac{\phi^*_1\phi_{1\lambda} + \phi^*_2\phi_{2\lambda}}{\det\Phi}. \tag{131}$$

or

$$r_1 = \frac{-\phi_2\phi_{1\lambda} + \phi_1\phi_{2\lambda} - \phi^*_1\phi^*_{2\lambda} + \phi^*_2\phi^*_{1\lambda}}{2\det\Phi}, \tag{132}$$

$$r_2 = \frac{-\phi_2\phi_{1\lambda} + \phi_1\phi_{2\lambda} + \phi^*_1\phi^*_{2\lambda} - \phi^*_2\phi^*_{1\lambda}}{2i\det\Phi}, \tag{133}$$

$$r_3 = \frac{\phi^*_1\phi_{1\lambda} + \phi^*_2\phi_{2\lambda}}{\det\Phi}. \tag{134}$$

Let us now we construct the soliton surface corresponding to the 1-soliton solution of the dHFE (72) which we presented in the previous section. In this case, the components of the position vector are given by (132)–(134), where

$$\begin{aligned} \phi_1 &= c_1 e^{-\chi}, \quad \phi_2 = c_2 e^{\chi}, \quad \phi_1^* = c_1^* e^{-\chi^*}, \quad \phi_2^* = c_2^* e^{\chi^*} && (135)\\ \phi_{1\lambda} &= -2i(\lambda x + 4\lambda^3 t)\phi_{01}, \quad \phi_{2\lambda} = 2i(\lambda x + 4\lambda^3 t)\phi_{01} && (136)\\ \phi_{1\lambda}^* &= 2i(\lambda^* x + 4\lambda^{*3} t)\phi_{01}^*, \quad \phi_{2\lambda}^* = -2i(\lambda^* x + 4\lambda^{*3} t)\phi_{01}^*. && (137) \end{aligned}$$

Thus, in this section, we have presented the soliton surface given by the position vector $\mathbf{r}$ corresponding to the 1-soliton solution of the dHFE.

7. Nonlocal Versions of the Nonlinear Schrödinger-Type Equations and Related Integrable Spin Systems

In the previous sections, we have considered the local NLSE, local CLLE, and their spin counterparts, the HFE (3) and the dHFE (72). In this section we are going to study the nonlocal nonlinear Schrödinger-type equations and their spin equivalents namely the nonlocal Heisenberg ferromagnet type equations.

7.1. The Nonlocal NLSE and Nonlocal HFE

Let us start from the nonlocal NLSE in more general form as (see, for example, in [7–29])

$$\begin{aligned} \nu q_t - q_{xx} + q^2 r &= 0, && (138)\\ \nu r_t + r_{xx} - r^2 q &= 0, && (139) \end{aligned}$$

where $\nu = \alpha + i\beta$ is a complex number in general, and α, β are real constants. Now, we introduce the following reduction:

$$r(t,x) = kq^*(\epsilon_1 t, \epsilon_2 x), \quad (140)$$

where $\epsilon_j^2 = 1$ and $k = \pm 1$. In this case, the generalized NLSE (138) and (139) takes the form

$$\nu q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q^*(\epsilon_1 t, \epsilon_2 x) = 0, \quad (141)$$

where $\nu^* = -\epsilon_1 \nu$. This equation admits the following four reductions:
(i) $\epsilon_1 = \epsilon_2 = 1$ (standard (local) case):

$$i\beta q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q^*(t,x) = 0. \quad (142)$$

(ii) $\epsilon_1 = -1, \epsilon_2 = 1$ (T-symmetric case):

$$\alpha q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q^*(-t,x) = 0. \quad (143)$$

(iii) $\epsilon_1 = 1, \epsilon_2 = -1$ (S-symmetric case):

$$\beta q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q^*(t,-x) = 0. \quad (144)$$

(iv) $\epsilon_1 = -1, \epsilon_2 = -1$ (ST-symmetric case):

$$\alpha q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q^*(-t,-x) = 0. \quad (145)$$

Similarly, we can consider the reduction $r = kq(\epsilon_1 t, \epsilon_2 x)$ with $k \in R$ and a suitable adaptation of the two parameters α and β. In this case, we have the following equation:

$$\nu q_t(t,x) - q_{xx}(t,x) + kq^2(t,x)q(\epsilon_1 t, \epsilon_2 x) = 0. \quad (146)$$

that gives us a new four equations. Note that we must add also the equations for the functions $q(-t,x), q(t,-x), q(-t,-x)$, respectively. We do not present them here, as they are obtained from (138)–(146) by $t \to -t; x \to -x; (t \to -t, x \to -x)$ reflections respectively. As all of these equations contain fields that depend simultaneously on x and $-x$, and/or t and $-t$, they are referred to as *nonlocal*. However, in what follows, we will exclusively focus on the complex parity extended version corresponding to the choice $r(x,t) = kq^*(-x,t)$. The other cases can be investigated in the same lines, but we will not considered here. Note that all of these nonlocal NLS equations have the focusing $(k=1)$ and defocusing $(k=-1)$ cases. All these equations are integrable that is they possess Lax pairs, recursion operators, n-soliton solutions, infinite number integrals of motion, and so on.

It is well known that the gauge equivalent counterpart of the nonlocal NLSE (146) is the following nonlocal HFE [24]:

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx}, \tag{147}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the complex-valued vector. The complex-valued spin vector $\mathbf{S}$ induced that the unit vectors $\mathbf{e}_j$ become also complex-valued. This means that the curvature $\kappa(t,x)$, the torsion $\tau(t,x)$ and ω_j are complex-valued functions [30–57]. As result in the nonlocal case, we will lost the isomorphism $su(2) \approx so(3)$. But all geometrical formalism presented in Section 2 will works also in the nonlocal case, at least, for the examples which we consider in this paper.

As in the nonlocal case the spin vector $\mathbf{S}$ is no longer real and is the complex-valued vector function, we may decompose it as $\mathbf{S} = \mathbf{m} + i\mathbf{l}$. Now, $\mathbf{m}$ and $\mathbf{l}$ are already real valued vector functions which satisfy the following relations:

$$\mathbf{m}^2 - \mathbf{l}^2 = 1, \quad \mathbf{m} \cdot \mathbf{l} = 0. \tag{148}$$

As result instead of the HFE (147) we obtain the following set of coupled equations for the real valued vector functions $\mathbf{m}$ and $\mathbf{l}$ [24]:

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{xx} - \mathbf{l} \wedge \mathbf{l}_{xx}, \tag{149}$$
$$\mathbf{l}_t = \mathbf{m} \wedge \mathbf{l}_{xx} + \mathbf{l} \wedge \mathbf{m}_{xx}. \tag{150}$$

7.2. The Nonlocal CLLE and Nonlocal Derivative HFE

The nonlocal CLLE we write in the form

$$iq_t + q_{xx} + 2rqq_x = 0, \tag{151}$$
$$ir_t - r_{xx} - 2rqr_x = 0. \tag{152}$$

As in the previous subsection, we can consider the different reductions as

$$r = kq^*(\epsilon_1 x, \epsilon_2 t), \quad r = kq(\epsilon_1 x, \epsilon_2 t) \tag{153}$$

or

$$r = kq^*(-x,t), \quad r = kq^*(x,-t), \quad r = kq^*(-x,-t), \tag{154}$$
$$r = kq(-x,t), \quad r = kq(x,-t), \quad r = kq(-x,-t), \tag{155}$$

where $k = \pm 1$ and $\epsilon_j^2 = 1$. Using the standard procedure, we can show that the gauge equivalent spin system corresponding to the CLLE has the form

$$iS_t + \frac{1}{2}[S, S_{xx}] - \frac{i}{4\beta^2}\mathbf{S}_x^2 S_x = 0. \tag{156}$$

which is in fact an integrable generalized nonlocal dHFE. Its Lax representation is given by (73) and (74). To find the geometrical equivalent spin system of the nonlocal CLLE (151) and (152), we use the same geometrical formalism as in the Section 2. However, here we must note that in contrast to the local case, in our nonlocal case, in the Serret–Frenet Equations (6) and (7), the curvature $\kappa(t,x)$, the torsion $\tau(t,x)$, $\sigma(t,x)$, and $\omega_j(t,x)$ are complex-valued functions [57]. As results, in the nonlocal case, the spin matrix S is not Hermitian and has PT-symmetry $S(t,x) = \sigma_3 S^+(t,-x)\sigma_3$. The corresponding spin vector $\mathbf{S}(t,x) = (S_1(t,x), S_2(t,x), S_3(t,x))$ is complex-valued vector. At the same time, the geometrical equivalent of the nonlocal CLLE is given by

$$\mathbf{e}_{3t} + \mathbf{e}_3 \times \mathbf{e}_{3xx} - \left(2\lambda^2 - \frac{1}{8\lambda^2}\mathbf{e}_{3x}^2\right)\mathbf{e}_{3x} = 0. \tag{157}$$

As $rq = -\frac{1}{4\beta^2}\mathbf{S}_x^2 = -\frac{1}{8\beta^2}tr(S_x^2)$ and after the identification $\mathbf{S} \equiv \mathbf{e}_3$, this equation takes the form

$$\mathbf{S}_t + \mathbf{S} \times \mathbf{S}_{3xx} - \frac{1}{4\beta^2}\mathbf{S}_x^2\mathbf{S}_x = 0. \tag{158}$$

As we mentioned above, in the nonlocal case, the spin matrix $S(t,x)$ is not Hermitian. However, we can decompose it as the sum of a Hermitian matrix and a skew-Hermitain matrix as

$$S = M + iL, \tag{159}$$

where

$$M = \frac{1}{2}(S^+ + S), \quad L = \frac{i}{2}(S^+ - S). \tag{160}$$

Next, we use the standard Pauli matrix representation of these matrices: $M = m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3, \quad L = l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3$, where $\mathbf{m} = (m_1, m_2, m_3)$ and $\mathbf{l} = (l_1, l_2, l_3)$ are real valued vector functions. From $\mathbf{S} = \mathbf{m} + i\mathbf{l}$ and $\mathbf{S}^2 = 1$ we obtain

$$\mathbf{m}^2 - \mathbf{l}^2 = 1, \quad \mathbf{m} \cdot \mathbf{l} = 0. \tag{161}$$

Therefore, and from Equation (149) and (150), we get the following set of the vector equations:

$$\mathbf{m}_t + \mathbf{m} \wedge \mathbf{m}_{xx} - \mathbf{l} \wedge \mathbf{l}_{xx} - \frac{1}{4\lambda^2}[(\mathbf{m}_x^2 - \mathbf{l}^2)\mathbf{m}_x - 2(\mathbf{m}_x \cdot \mathbf{l}_x)\mathbf{l}_x] = 0, \tag{162}$$

$$\mathbf{l}_t + \mathbf{m} \wedge \mathbf{l}_{xx} + \mathbf{l} \wedge \mathbf{m}_{xx} - \frac{1}{4\lambda^2}[(\mathbf{m}_x^2 - \mathbf{l}^2)\mathbf{l}_x - 2(\mathbf{m}_x \cdot \mathbf{l}_x)\mathbf{m}_x] = 0. \tag{163}$$

This is one of forms of the desired nonlocal dHFE. This coupled generalized dHFE is gauge and geometrical equivalent spin system corresponding to the nonlocal CLLE (151) and (152).

8. Conclusions

In this paper, we have developed a method for establishing geometric equivalence based on the isomorphism of the Lie algebras $su(2) \approx so(3)$. The advantage of this geometrical method in comparison with the other approach, for example, the Lakshmanan method is that here in our case, the identification condition $\mathbf{S} \equiv \mathbf{e}_1$ is not required in advance, and the equation of motion for $\mathbf{e}_3$ (43), which gives the general form of spin systems for constant values of λ, is derived in a natural way. The form of a particular spin system differs depending on the accepted value of κ. Moreover, note that in [1], where consider case $\sigma = 0, \kappa \neq 0, \tau \neq 0$ and the connection between the solution of geometrically equivalent equations is given by the Hasimota transformation (5). In our case, in this study with $\sigma \neq 0, \kappa = 0, \tau \neq 0$ and solutions of NLSE (1) and HFE (3) are related by the formula $q = \frac{1}{2}(\sigma - i\tau)$. One of main results of this paper is the extension of the geometrical method for the local integrable equations to the nonlocal ones. We have shown that for the nonlocal equations, at least, for the nonlocal NLSE, the nonlocal CLLE and their related equivalent nonlocal spin systems (nonlocal Heisenberg ferromagnet type equations) the considered geometrical formalism works and fruitful. We have constructed two new integrable spin systems which are equivalent to the local and nonlocal versions of the NLSE and CLLE.

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