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Fractional N-Laplacian Problems Defined on the One-Dimensional Subspace

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Abstract: The research of the fractional Orlicz-Sobolev space and the fractional N-Laplacian operators will give the development of nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium as well as Probability and Analysis as they proved to be accurate models to describe different phenomena in Physics, Finance, Image processing and Ecology. We study the number of weak solutions for one-dimensional fractional N-Laplacian systems in the product of the fractional Orlicz-Sobolev spaces, where the corresponding functionals of one-dimensional fractional N-Laplacian systems are even and symmetric. We obtain two results for these problems. One result is that these problems have at least one nontrivial solution under some conditions. The other result is that these problems also have infinitely many weak solutions on the same conditions. We use the variational approach, critical point theory and homology theory on the product of the fractional Orlicz-Sobolev spaces.

Keywords: fractional N-Laplacian systems; fractional Orlicz-Sobolev spaces; product of the fractional Orlicz-Sobolev spaces; variational approach; critical point theory; homology theory

MSC: 35A01; 35A15; 35A16; 35J75

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1. Introduction

Let $(a, b) \subset \mathbb{R}^1$ and $0 < s < 1$ be a fractional parameter. Let N denote the dimension of the domain Ω , N denote the left-hand side attached character of the N-function and n denote the dimension of the space $\{(u_1, \dots, u_n)\} \subset \mathbb{R}^n$. Let g be an odd, increasing homeomorphism from $[0, \infty)$ onto $[0, \infty)$. Let G be the function defined by

$$G(x) = \int_0^x g(t)dt \quad \text{for all } x \geq 0.$$

G is a Young function and also a N-function (The Young function G is defined as follows: G is the Young function if and only if G satisfies that $G(0) = 0$, $\lim_{x \rightarrow +\infty} G(x) = +\infty$ and G is convex. The N-function G is defined as follows: G is the N-function if and only if $\lim_{x \rightarrow 0} \frac{G(x)}{x} = 0$, $\lim_{x \rightarrow \infty} \frac{G(x)}{x} = +\infty$, $G(x) = 0$ if and only if $x = 0$). Let $\alpha(x)$, $p(x)$, $m(x)$, $\gamma(x) \in C([a, b], \mathbb{R})$ be measurable functions with $\alpha(x) > 0$, $1 < p(x) < \infty$, $m(x) > 0$ and $\gamma(x) > 1$. Let D be an open subset in $\mathbb{R}^n$, $n \geq 2$, with compact complement $C = \mathbb{R}^n \setminus D$ containing $\theta = (0, \dots, 0)$. Let $|\cdot|$ be denoted as a $\mathbb{R}^n$ norm.

We consider the number of weak solutions $u = (u_1, \dots, u_n) \in C^1([a, b], D)$ for one-dimensional N-Laplacian systems with measurable positive coefficient functions

$$\left\{ \begin{array}{l} (-\Delta)_g^s u_1(x) + \text{grad}_{u_1}(\alpha(x)|u(x)|^{p(x)}) + \text{grad}_{u_1}(m(x)\frac{1}{|u(x)|^{\gamma(x)}}) = 0 \text{ in } (a,b), \\ \vdots \\ (-\Delta)_g^s u_n(x) + \text{grad}_{u_n}(\alpha(x)|u(x)|^{p(x)}) + \text{grad}_{u_n}(m(x)\frac{1}{|u(x)|^{\gamma(x)}}) = 0 \text{ in } (a,b) \\ u_i(a) = u_i(b), \quad u'_i(a) = u'_i(b) \quad \text{for all } 1 \leq i \leq n, \end{array} \right. \quad (1)$$

where $\text{grad}_{u_i} = \frac{\partial}{\partial u_i}$ and $(-\Delta)_g^s u_i(x)$ is the fractional N-Laplacian operator defined as follows: for each $x \in R$ and any u_i in the fractional Orlicz-Sobolev space $W^s L_G((a,b), R)$,

$$(-\Delta)_g^s u_i(x) = \text{P.V.} \int_a^b g\left(\frac{|u_i(x) - u_i(y)|}{|x - y|^s}\right) \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{dy}{|x - y|^{1+s}}. \quad (2)$$

Here P.V. denotes the Cauchy principle value. For $0 < s < 1$, $(-\Delta)_g^s$ is called the fractional N-Laplacian operator and for $u = (u_1, \dots, u_n) \in W^s L_G((a,b), R^n)$,

$$(-\Delta)_g^s u(x) = ((-\Delta)_g^s u_1(x), \dots, (-\Delta)_g^s u_n(x)).$$

We assume that

- (i) $\sup\{|\alpha(x)|u(x)|^{p(x)} + m(x)\frac{1}{|u(x)|^{\gamma(x)}}| + \|\text{grad}_u(\alpha(x)|u(x)|^{p(x)}) + \text{grad}_u(m(x)\frac{1}{|u(x)|^{\gamma(x)}})\|_{R^n} \mid (x, u) \in (a,b) \times (R^n \setminus B_{R_0})\} < +\infty$
for some $R_0 > 0$.
- (ii) $\alpha(x)|u(x)|^{p(x)} + m(x)\frac{1}{|u(x)|^{\gamma(x)}} \geq \frac{A}{d^2(u,C)}$ for $(x, u) \in (a,b) \times U$
for some neighborhood U of C , the distance function $d(u, C)$ from u to C and a constant $A > 0$.

The corresponding functionals of (1)

$$\int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} + \int_a^b \alpha(x)|u(x)|^{p(x)} dx + \int_a^b m(x)\frac{1}{|u|^{\gamma(x)}} dx$$

are even and symmetric.

Our problems contain the Orlicz space, the fractional Orlicz-Sobolev space and the fractional N-Laplacian operators. The Orlicz space, the fractional Orlicz-Sobolev space and the fractional N-Laplacian operators arise in the non-linear elasticity theory, electro rheological fluids, non-Newtonian fluid theory, a porous medium and the context of stochastic Lvy processes with jumps. In recent years, Probability and Analysis in the context of Physics, Finance, Image processing and Ecology have been provided the mathematical models containing the Orlicz space, the fractional Orlicz-Sobolev space and the fractional N-Laplacian operators to describe different phenomena.

For The Orlicz space, the fractional Orlicz-Sobolev space and the fractional N-Laplacian operators, we refer to [1–14].

For the singular elliptic systems and the singular problems involving fractional N-Laplacian, we refer to [15–35].

Let $0 < s < 1$, $p : \Omega \times \Omega \rightarrow (1, \infty)$ be a continuous function with $1 < p(x, y) < \infty$. The fractional Sobolev spaces with variable exponent defined on one-dimensional subspace (a, b) are defined as

$$W^{s,p(x,y)}((a,b), R) = \{u \in L^{p(x,y)}((a,b), R) \mid \int_a^b \int_a^b \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{1+sp(x,y)}} dx dy < \infty, \\ \text{for some } \lambda > 0\}$$

with the norm

$$\|u\|_{s,p(x,y)} = \inf\{\lambda > 0 \mid \int_a^b \int_a^b \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{1+sp(x,y)}} dx dy \leq 1\}.$$

when $g(t) = |t|^{p(x,y)-2}t$ in (2), (2) reduced to the fractional $p(\cdot)$ -Laplacian operator with variable exponent

$$(-\Delta)_{p(\cdot)}^s u_i(x) = \text{P.V.} \int_a^b \frac{|u_i(x) - u_i(y)|^{p(x,y)-2} (u_i(x) - u_i(y))}{|x - y|^{1+sp(x,y)}} dy, \quad x \in (a, b). \quad (3)$$

Nonsingular $p(\cdot)$ -Laplacian boundary value problems like the following

$$-\text{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda f(x, u),$$

$$u = 0 \quad \partial\Omega,$$

we refer to [2,13,36–46]. In case of p -Laplacian with $p(x) = p$ a constant, we refer to [3,47–63]. When $0 < s < 1$, $g(t) = t$ in (2), (2) reduced to the usual fractional Laplacian operator $(-\Delta)^s$. For the fractional Laplacian operator, see [45,64–87].

We deal with the fractional Orlicz spaces with variable exponent, the fractional Orlicz-Sobolev spaces with variable exponent and their corresponding nonhomogeneous fractional N-Laplacian operators as the adequate function spaces. For the theory of Orlicz and Orlicz-Sobolev spaces and the fractional N-Laplacian operator, we also refer the readers to [1,10,60,88,89].

When $s = 1$, $g(t) = |t|^{p-2}t$ and $\Omega \subset \mathbb{R}^n$, (1) reduce to the p -Laplacian problems. We [90] proved the existence of at least one solution for the p -Laplacian boundary value problem with two singular and subcritical nonlinearities

$$\left\{ \begin{array}{l} -\text{div}(|\nabla u|^{p-2} \nabla u) = a|u|^{p-2}u - \text{grad}_u \frac{1}{(|u|^2 + |v|^2)^q} - \text{grad}_u \frac{1}{(|u-e_1|^2 + |v-e_2|^2)^r} + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-1} |v|^\beta \\ \quad \text{in } \Omega, \\ -\text{div}(|\nabla v|^{p-2} \nabla v) = b|v|^{p-2}v - \text{grad}_v \frac{1}{(|u|^2 + |v|^2)^q} - \text{grad}_v \frac{1}{(|u-e_1|^2 + |v-e_2|^2)^r} + \frac{2\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-1} \\ \quad \text{in } \Omega, \end{array} \right.$$

where a, b, p, q, r, α and β are real constants, and $1 < p < \infty$, $q, r > 1$ and $p < \alpha + \beta < p^*$, where p^* is critical exponent such that

$$p^* = \begin{cases} \frac{np}{n-p} & \text{if } n > p, \\ \infty & \text{if } n \leq p. \end{cases}$$

When $s = 1$, $g(t) = |t|^{p-2}t$, $n = 1$ and $\Omega = (a, b) \subset \mathbb{R}^1$, (1) reduce to the one-dimensional p -Laplacian problems. For the one-dimensional singular problems involving p -Laplacian

$$\begin{aligned} -(|u'|^{p-2} u')' &= m(x) u^{-\gamma} & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

we refer to [24,30,91–101]. For the one-dimensional fractional p -Laplacian problems, we refer to [102]. For the Kirchhoff equations involving fractional p -Laplacian, we refer to [60,76,77,103–105].

Let G^{-1} be the function defined by

$$G^{-1}(x) = \int_0^x g^{-1}(t) dt \quad \text{for all } x \geq 0.$$

The function G^{-1} is called the complementary function of G and satisfies

$$G^{-1}(x) = \sup\{yx - G(y) \mid y \geq 0\} \quad \text{for all } x \geq 0.$$

G^{-1} satisfies that

$$\lim_{x \rightarrow 0} \frac{G^{-1}(x)}{x} = 0, \quad \lim_{x \rightarrow \infty} \frac{G^{-1}(x)}{x} = +\infty,$$

i.e., G^{-1} is an N-function. Moreover, by the Young's inequality,

$$xy \leq G(x) + G^{-1}(y), \quad \text{for all } x, y \geq 0. \quad (4)$$

From now on we shall denote by $C^1(S^1, D)$ the subset of $C^1([a, b], D)$ satisfying the conditions

$$u(a) = u(b), \quad u'(a) = u'(b).$$

Orlicz space $L_G(S^1, R)$ defined by N-function G is defined as

$$L_G(S^1, R) = \{u \mid u : S^1 \rightarrow R \text{ is a measurable function with}$$

$$\|u\|_{L_G} = \sup \left\{ \int_a^b u v dx \mid \int_a^b G^{-1}(|v|) dx \leq 1 \right\} < \infty \}.$$

$L_G(S^1, R)$ is a Banach space with a norm $\|u\|_{L_G}$ equivalent to the Luxemburg norm

$$\|u\|_G = \inf \{v > 0 \mid \int_a^b G(|\frac{u(x)}{v}|) \leq 1\}.$$

Hölder inequality holds in the Orlicz space $L_G(S^1, R)$ (see [19]):

$$\int_a^b |uv| dx \leq 2 \|u\|_{L_G} \|v\|_{L_{G^{-1}}} \quad \text{for all } u, v \in L_G(S^1, R).$$

Orlicz-Sobolev space $W^s L_G(S^1, R)$ is defined as

$$W^s L_G(S^1, R) = \{u \in L_G(S^1, R) : \int_a^b \int_a^b G\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx dy}{|x - y|} < \infty\}$$

with the norm

$$\|u\|_{s,G} = \|u\|_G + [u]_{s,G},$$

where $[u]_{s,G}$ is the Gagliardo semi-norm defined by

$$[u]_{s,G} = \inf \{ \mu > 0 \mid \int_a^b \int_a^b G\left(\frac{|u(x) - u(y)|}{\mu |x - y|^s}\right) \frac{dx dy}{|x - y|} \leq 1 \}.$$

In [88], for any $0 < s < 1$ and G a Young function such that G and G^{-1} satisfy that

$$G(2t) \leq C_1 G(t) \text{ and } G^{-1}(2t) \leq C_2 G^{-1}(t), \quad \forall t \geq 0, C_1, C_2 > 0, \quad (5)$$

$W^s L_G(S^1, R)$ is reflexive and separable. Moreover $C_0^\infty(S^1, R)$ is dense in $W^s L_G(S^1, R)$ in the norm $\|\cdot\|_{s,G}$. Let the Orlicz-Sobolev space $W_0^s L_G(S^1, R)$ be the closure of $C_0^\infty(S^1, R)$ in $W^s L_G(S^1, R)$. The space $W_0^s L_G(S^1, R)$ is also reflexive. By Lemma 5.7 in [4], the norm $\|u\|_{s,G}$ is equivalent to $[u]_{s,G}$.

Let

$$L_G(S^1, R^n) = L_G(S^1, R) \times \cdots \times L_G(S^1, R)$$

be the n -cartesian product space of $L_G(S^1, R)$ equipped with the norm

$$\|u\|_{L_G(S^1, R^n)} = \sum_{i=1}^n \|u_i\|_{L_G(S^1, R)}, \quad u = (u_1, \dots, u_n)$$

and

$$H = W^s L_G(S^1, R^n) = W^s L_G(S^1, R) \times \cdots \times W^s L_G(S^1, R)$$

equipped with the norm

$$\|u\|_H = \sum_{i=1}^n \|u_i\|_{s,G}.$$

For any G a Young function such that G and G^{-1} satisfy (5), $W^s L_G(S^1, R^n)$ is reflexive and separable. Moreover $C_0^\infty(S^1, R^n)$ is dense in $W^s L_G(S^1, R^n)$ in the norm $\|\cdot\|_{s,G}$. Let $W_0^s L_G(S^1, R^n)$ be the closure of $C_0^\infty(S^1, R^n)$ in $W^s L_G(S^1, R^n)$.

Zhikov [106] observed that smooth functions are not dense in $W^{1,p(x)}(S^1, R)$ without additional assumptions on the exponent $p(x)$. However, when the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant B such that

$$|p(x) - p(y)| \leq \frac{B}{-\log|x - y|}$$

for every $x, y \in (a, b)$ with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in $W^{1,p(x)}(S^1, R)$, and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}$, as the completion of $C_0^\infty(S^1, R)$ with respect to the norm $\|u\|_{W^{1,p(x)}(S^1, R)}$ [39,93].

Let us set

$$g_0 = \inf_{t>0} \frac{tg(t)}{G(t)} \quad g^0 = \sup_{t>0} \frac{tg(t)}{G(t)}$$

and assume that

$$1 < g_0 \leq \frac{tg(t)}{G(t)} \leq g^0 < \infty \quad \forall t \geq 0. \quad (6)$$

By Proposition 2.3 of [6], it implies that each G satisfies the Δ_2 -condition, i.e., there exists a constant $C > 0$ such that

$$G(2t) \leq CG(t), \quad t \geq 0.$$

Assume that G is a function such that

$$G : t \in [0, \infty) \mapsto G(\sqrt{t}) \text{ is convex.} \quad (7)$$

Weak solutions $u = (u_1, \dots, u_n) \in W^s L_G(S^1, D)$ satisfy

$$\int_a^b [(-\Delta)_g^s u(x) \cdot \phi(x) dx + \text{grad}_u(\alpha(x)|u(x)|^{p(x)}) \cdot \phi(x) + \text{grad}_u(m(x) \frac{1}{|u|^{\gamma(x)}}) \cdot \phi(x)] dx = 0$$

$$\forall \phi \in W^s L_G(S^1, D).$$

Theorem 1. Let $0 < s < 1$, $sg_0 < 1$, $p \in C([a, b], R)$ with $p(x) > 1$ be log-Hölder continuous. Let $\alpha(x), p(x), m(x), \gamma(x) \in C([a, b], R)$ be measurable functions with $\alpha(x) > 0$, $1 < p(x) < \infty$, $m(x) > 0$ and $\gamma(x) > 1$. We assume that (5)–(7) and

$$1 < p(x) \leq p_+ < g_0^* = \frac{g_0}{1 - sg_0}$$

hold, where $p_+ = \sup_{x \in [a, b]} p(x)$ and $p_- = \inf_{x \in [a, b]} p(x)$. Then (1) has at least one nontrivial weak solution.

Theorem 2. Under the assumptions of Theorem 1, (1) has infinitely many weak solutions.

We use variational approach, minimax method in critical point theory on the loop space $W^s L_G(S^1, D)$ and homology theory. In Section 2, we introduce some preliminaries. In Section 3, we obtain some variational results on the potential and prove that the associated functional J of (1) satisfies the (P.S.) condition on the loop subspace $W^s L_G(S^1, D)$. In Section 4, we prove Theorems 1 and 2 by using minimax method, critical point theory and homology theory.

2. Fractional Orlicz-Sobolev Space

For the variational setting, we need some properties on $L^{p(x)}(S^1, R)$, $L^{p(x)}(S^1, R^n)$, $L_G(S^1, R)$, $L_G(S^1, R^n)$, $W^s L_G(S^1, R)$ and $W^s L_G(S^1, R^n)$ which are introduced in Section 1 and can be found in [107,108].

Lemma 1. ([107]) *The space $L^{p(x)}(S^1, R)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(S^1, R)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Let $u \in L^{p(x)}(S^1, R)$ and $v \in L^{p'(x)}(S^1, R)$. Then we have*

$$|\int_a^b uv dx| \leq (\frac{1}{p_-} + \frac{1}{(p')_-}) \|u\|_{L^{p(x)}(S^1, R)} \|v\|_{L^{p'(x)}(S^1, R)} \leq 2 \|u\|_{L^{p(x)}(S^1, R)} \|v\|_{L^{p'(x)}(S^1, R)}.$$

Lemma 2. *Let $p_+ < \infty$ and set*

$$\tau(u) = \int_a^b |u|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(S^1, R).$$

Then

- (i) $\tau(u) > 1 \Leftrightarrow \|u\|_{L^{p(x)}(S^1, R)} > 1$,
 $\tau(u) = 1 \Leftrightarrow \|u\|_{L^{p(x)}(S^1, R)} = 1$,
 $\tau(u) < 1 \Leftrightarrow \|u\|_{L^{p(x)}(S^1, R)} < 1$.
- (ii) *If $\|u\|_{L^{p(x)}(S^1, R)} > 1$, then $\|u\|_{L^{p(x)}(S^1, R)}^{p_-} \leq \tau(u) \leq \|u\|_{L^{p(x)}(S^1, R)}^{p_+}$.*
- (iii) *If $\|u\|_{L^{p(x)}(S^1, R)} < 1$, then $\|u\|_{L^{p(x)}(S^1, R)}^{p_+} \leq \tau(u) \leq \|u\|_{L^{p(x)}(S^1, R)}^{p_-}$.*

Lemma 3. ([88]) *Let $0 < s < 1$ and G be a N -function. Then $W^s L_G(S^1, R^n)$ is a reflexive and separable Banach space. Furthermore $C_0^\infty(S^1, R^n)$ is dense in $W^s L_G(S^1, R^n)$ in the norm $\|\cdot\|_{s,G}$.*

Let $u_i \in W^s L_G(S^1, R)$ and set

$$[u_i]_{s,G} = \inf\{\sigma > 0 \mid \int_a^b \int_a^b G(\frac{|u_i(x) - u_i(y)|}{\sigma |x - y|^s}) \frac{dx dy}{|x - y|} \leq 1\}, \quad 1 \leq i \leq n.$$

Lemma 4. ([109]) *(Generalized Poincar inequality on the Orlicz-Sobolev space)*
 Let $0 < s < 1$ and G be a Young function. Then

$$\|u_i\|_G \leq \nu_i [u_i]_{s,G}, \quad \forall u_i \in W^s L_G(S^1, R), \quad 1 \leq i \leq n \quad (8)$$

for some constants $\nu_i > 0$, $1 \leq i \leq n$. That is, the embedding

$$W^s L_G(S^1, R) \hookrightarrow L_G(S^1, R)$$

is continuous. Furthermore $[u_i]_{s,G}$ is a norm of $W^s L_G(S^1, R)$ equivalent to $\|u_i\|_{s,G}$.

Let $u = (u_1, \dots, u_n) \in W^s L_G(S^1, R^n)$, $u_i \in W^s L_G(S^1, R)$ and set

$$[u]_{s,G} = \sum_{i=1}^n [u_i]_{s,G}. \quad (9)$$

Combining Lemma 4 and (9), we obtain the following lemma:

Lemma 5. *(Generalized Poincar inequality on the product of the Orlicz-Sobolev space)*
 Let $0 < s < 1$ and G be a Young function. Then

$$\|u\|_G \leq \nu [u]_{s,G}, \quad \forall u \in W^s L_G(S^1, R^n)$$

for some constant $\nu > 0$. That is, the embedding

$$W^s L_G(S^1, R^n) \hookrightarrow L_G(S^1, R^n)$$

is continuous. Moreover $[u]_{s,G}$ is a norm of $W^s L_G(S^1, R^n)$ equivalent to $\|\cdot\|_{s,G}$.

Lemma 6. ([110]) Let $u_i \in W^s L_G(S^1, R)$. Then

$$\begin{aligned} \|u_i\|_{s,G}^{g_0} &\leq \int_a^b \int_a^b G\left(\frac{|u_i(x) - u_i(y)|}{|x - y|^s}\right) \frac{dx dy}{|x - y|} \leq \|u_i\|_{s,G}^{g_0}, \text{ if } \|u_i\|_{s,G} > 1, 1 \leq i \leq n, \\ \|u_i\|_{s,G}^{g_0} &\leq \int_a^b \int_a^b G\left(\frac{|u_i(x) - u_i(y)|}{|x - y|^s}\right) \frac{dx dy}{|x - y|} \leq \|u_i\|_{s,G}^{g_0}, \text{ if } \|u_i\|_{s,G} < 1. \end{aligned} \quad (10)$$

Proof. The proof follows from (6) and Theorem 3.11 of [110]. $\square$

Let us define the functional $Q_{s,G} : W^s L_G(S^1, R^n) \rightarrow R$ by

$$Q_{s,G}(u) = \int_a^b \int_a^b [\Sigma_{i=1}^n G\left(\frac{|u_i(x) - u_i(y)|}{|x - y|^s}\right)] \frac{dx dy}{|x - y|},$$

$$u = (u_1, \dots, u_n) \in W^s L_G(S^1, R^n), u_i \in W^s L_G(S^1, R).$$

Combining Lemma 6 and (9), we obtain that:

Lemma 7. Let $u \in W^s L_G(S^1, R^n)$. Then

$$\begin{aligned} \|u\|_{s,G}^{g_0} \leq Q_{s,G}(u) &\leq \|u\|_{s,G}^{g_0}, \quad \text{if } \|u\|_{s,G} > 1, \\ \|u\|_{s,G}^{g_0} \leq Q_{s,G}(u) &\leq \|u\|_{s,G}^{g_0}, \quad \text{if } \|u\|_{s,G} < 1. \end{aligned}$$

Lemma 8. ([109]) Let $0 < s < 1$, $sg_0 < 1$ and G be a N -function. Then the embeddings

$$W^{s,g_0}(S^1, R) \hookrightarrow L^{r(x)}(S^1, R)$$

and

$$W^s L_G(S^1, R) \hookrightarrow L^{r(x)}(S^1, R)$$

are continuous and compact for all $1 \leq r(x) < g_0^*$.

Moreover

$$\|u_i\|_{L^{r(x)}(S^1, R)} \leq C_i [u_i]_{s,G} \text{ for all } 1 \leq r(x) < g_0^*$$

for some constants C_i , $1 \leq i \leq n$.

Proof. By [109], the embedding $W^{s,g_0}(S^1, R) \hookrightarrow L^{r(x)}(S^1, R)$ is continuous and compact for all $1 \leq r(x) < g_0^*$. By (6), the embedding $W^s L_G(S^1, R) \hookrightarrow W^{s,g_0}(S^1, R)$ is continuous. Combining these facts, we obtain that the embedding $W^s L_G(S^1, R) \hookrightarrow L^{r(x)}(S^1, R)$ is continuous and compact for all $1 \leq r(x) < g_0^*$. $\square$

Lemma 9. Let $0 < s < 1$, $sg_0 < 1$ and G be a N -function. Then the embeddings

$$W^{s,g_0}(S^1, R^n) \hookrightarrow L^{r(x)}(S^1, R^n)$$

and

$$W^s L_G(S^1, R^n) \hookrightarrow L^{r(x)}(S^1, R^n)$$

are continuous and compact for all $1 \leq r(x) < g_0^*$.

Moreover

$$\|u\|_{L^{r(x)}(S^1, R^n)} \leq C[u]_{s,G} \text{ for all } 1 \leq r(x) < g_0^*. \quad (11)$$

for some constant $C > 0$.

Lemma 10. Weakly convergent sequence $\{u_k = (u_{k_1}, \dots, u_{k_n})\}$ converging to $u = (u_1, \dots, u_n)$ in $W^s L_G(S^1, R^n)$ satisfying

$$\lim_{k \rightarrow +\infty} \sup < Q'_{s,G}(u_k), u_k - u > \leq 0.$$

converges strongly to u in $W^s L_G(S^1, R^n)$.

Proof. Since the sequence $\{u_k\}$ converges weakly to u in $W^s L_G(S^1, R^n)$ and $\lim_{k \rightarrow +\infty} \sup < Q'(u_k), u_k - u > \leq 0$, by (5), we have

$$\begin{aligned} & \int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{u_{k_i}(x) - u_{k_i}(y)}{|u_{k_i}(x) - u_{k_i}(y)|} \frac{u_{k_i}(x) - u_{k_i}(y)}{|x-y|^s} \frac{dx dy}{|x-y|} \\ & \leq \int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{u_i(x) - u_i(y)}{|x-y|^s} \frac{dx dy}{|x-y|} \\ & \leq g^0 \int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|}. \end{aligned}$$

Sequence $\{\int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{u_{k_i}(x) - u_{k_i}(y)}{|u_{k_i}(x) - u_{k_i}(y)|} \frac{u_{k_i}(x) - u_{k_i}(y)}{|x-y|^s} \frac{dx dy}{|x-y|}\}$ is bounded and converges to

$\int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{u_i(x) - u_i(y)}{|x-y|^s} \frac{dx dy}{|x-y|}$. By (6), we have

$$\begin{aligned} & \int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{u_{k_i}(x) - u_{k_i}(y)}{|u_{k_i}(x) - u_{k_i}(y)|} \frac{u_{k_i}(x) - u_{k_i}(y)}{|x-y|^s} \frac{dx dy}{|x-y|} \\ & \geq g_0 \int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|}. \end{aligned}$$

Sequence $\{\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|}\}$ is bounded and converges to

$\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|}$. Thus the sequence $\{u_k\}$ is bounded and converges weakly to u in $W^s L_G(S^1, R^n)$. Since the embedding $W^s L_G(S^1, R^n) \hookrightarrow L_G(S^1, R^n)$ is continuous and compact, $\{u_k\}$ converges strongly to u in $W^s L_G(S^1, R^n)$. $\square$

Lemma 11. If $u_k = (u_{k_1}, \dots, u_{k_n})$, $u = (u_1, \dots, u_n) \in W^s L_G(S^1, R^n)$, $k = 1, 2, \dots$, then the following statement are equivalent to each other

- (i) $\lim_{k \rightarrow \infty} \|u_k - u\|_{s,G} = 0$, $k = 1, 2$,
- (ii) $\lim_{k \rightarrow \infty} \int_a^b [\Sigma_{i=1}^n (G(u_{k_i}(x) - u_i(x)))] dx = 0$ and $\lim_{k \rightarrow \infty} \|u_k - u\|_{s,G} = 0$,
- (iii) $u_k \rightarrow u$ in measure in $W^s L_G(S^1, R^n)$ and $\lim_{k \rightarrow \infty} \int_a^b [\Sigma_{i=1}^n G(u_{k_i}(x))] dx = \int_a^b [\Sigma_{i=1}^n G(u_i(x))] dx$.

Proof. By the definition of $\|\cdot\|_{s,G}$, (i) $\Leftrightarrow$ (ii) holds. We shall show that (i) implies (iii). We assume that (i) holds. Then

$$\begin{aligned} & \int_a^b [(\Sigma_{i=1}^n G(u_{k_i}(x))) - (\Sigma_{i=1}^n G(u_i(x)))] dx \\ & \leq \int_a^b [\Sigma_{i=1}^n (g(u_i + \lambda(u_{k_i} - u_i))(u_{k_i} - u_i))] dx \\ & \leq 2[\Sigma_{i=1}^n (\|g(u_i + \lambda(u_{k_i} - u_i))\|_{G^{-1}} \|u_{k_i} - u_i\|_G)] \\ & \leq C(\Sigma_{i=1}^n (\|g(u_i + \lambda(u_{k_i} - u_i))\|_{G^{-1}} \|u_{k_i} - u_i\|_{s,G})) \\ & \rightarrow 0 \end{aligned}$$

for $0 < \lambda < 1$ and some $C > 0$. It follows that (iii) holds. Assume that (iii) holds. Since $\lim_{k \rightarrow \infty} \int_a^b [\sum_{i=1}^n G(u_{k_i}(x))] dx = \int_a^b [\sum_{i=1}^n G(u_i(x))] dx$, $\{u_k\}$ converges weakly to u in $L_G(S^1, R^n)$. By assumption (iii), $u_k \rightarrow u$ in measure in $W^s L_G(S^1, R^n)$. It follows that $\{u_k\}$ is bounded in $W^s L_G(S^1, R^n) \subset L_G(S^1, R^n)$. By Lemma 9, the embedding $W^s L_G(S^1, R^n) \hookrightarrow L_G(S^1, R^n)$ is continuous and compact. Thus $u_k \rightarrow u$ strongly in $W^s L_G(S^1, R^n)$. Thus (i) holds. $\square$

3. Variational Results

Let us set an open set of the Hilbert space $W^s L_G(S^1, R^n)$ as

$$\Lambda_{s,G} D = \{u \in W^s L_G(S^1, R^n) \mid \int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} < \infty,$$

$$\forall u(x) = (u_1(x), \dots, u_n(x)) \in D = R^n \setminus C \subset R^n, \forall x \in S^1, \theta \in C, \theta = (0, \dots, 0)\}$$

the loop space on D . We note that $\text{grad}_u(\frac{m(x)}{|u(x)|^{\gamma(x)}})$ is not singular on $W^s L_G(S^1, D)$ because D is away from 0.

Let us define the functional $J_{s,G} : \Lambda D \subset W^s L_G(S^1, R^n) \rightarrow R$ by

$$J_{s,G}(u) = Q_{s,G}(u) + \int_a^b \alpha(x) |u(x)|^{p(x)} dx + \int_a^b m(x) \frac{1}{|u|^{\gamma(x)}} dx,$$

where

$$Q_{s,G}(u) = \int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|}$$

for $u = (u_1, \dots, u_n) \in \Lambda D$. Then the functional $Q_{s,G}$ is of class $C^1(W^s L_G(S^1, R^n), R)$ with

$$\begin{aligned} < Q'_{s,G}(u), v > = < (-\Delta)_g^s u, v > \\ &= \int_a^b \int_a^b [\sum_{i=1}^n g(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{v_i(x) - v_i(y)}{|x - y|^s} \frac{dx dy}{|x - y|} \end{aligned} \quad (12)$$

$\forall u = (u_1, \dots, u_n) \in \Lambda D$, which is proved in Proposition 3.3 in [89].

Lemma 12. $J_{s,G}(u)$ is continuous and C^1 on $\Lambda_{s,G} D$ with Frchet derivative

$$\begin{aligned} &DJ_{s,G}(u)v \\ &= \int_a^b \int_a^b [\sum_{i=1}^n g(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{v_i(x) - v_i(y)}{|x - y|^s} \frac{dx dy}{|x - y|} \\ &\quad + \int_a^b [(\text{grad}_u \alpha(x) |u(x)|^{p(x)}) \cdot v(x) + (\text{grad}_u (\frac{m(x)}{|u(x)|^{\gamma(x)}})) \cdot v(x)] dx, \end{aligned} \quad (13)$$

$\forall v = (v_1, \dots, v_n) \in \Lambda_{s,G} D$. Moreover $DJ_{s,G} \in C$. That is, $J_{s,G} \in C^1$.

Proof. By Proposition 3.3 in [89],

$$Q_{s,G}(u) = \int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|}$$

is C^1 with Frchet derivative

$$\begin{aligned} < Q'_{s,G}(u), v > &= < (-\Delta)_g^s u, v > \\ &= \int_a^b \int_a^b [\sum_{i=1}^n g(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{u_i(x) - u_i(y)}{|u_i(x) - u_i(y)|} \frac{v_i(x) - v_i(y)}{|x - y|^s} \frac{dx dy}{|x - y|}. \end{aligned}$$

It suffices to show that $\int_a^b \alpha(x) |u(x)|^{p(x)} dx + \int_a^b \frac{m(x)}{|u(x)|^{\gamma(x)}} dx$ is C^1 with Frchet derivative $\int_a^b (\text{grad}_u \alpha(x) |u(x)|^{p(x)}) \cdot v(x) + \int_a^b (\text{grad}_u (\frac{m(x)}{|u(x)|^{\gamma(x)}})) \cdot v(x) dx \forall v \in \Lambda_{s,G} D$. Let us set

$$\Psi(u) = \Psi_1(u) + \Psi_2(u),$$

where

$$\Psi_1(u) = \int_a^b \alpha(x) |u(x)|^{p(x)} dx, \quad \Psi_2(u) = \int_a^b \frac{m(x)}{|u(x)|^{\gamma(x)}} dx.$$

We shall prove that $\Psi(u)$ is continuous. For $v, w \in \Lambda_{s,G} D$, we have

$$\begin{aligned} & |\Psi_1(v+w) - \Psi_1(v)| \\ & \leq |\alpha(x)|_\infty \left| \int_a^b [(\text{grad}_v |v(x) + \tau_1 w(x)|^{p(x)}) \cdot w(x) + O(\|w\|_H)] dx \right| = O(\|w\|_H) \end{aligned} \quad (14)$$

for some $0 < \tau_1 < 1$. and

$$\begin{aligned} & |\Psi_2(v+w) - \Psi_2(v)| \\ & \leq |m(x)|_\infty \left| \int_a^b [(\text{grad}_v \frac{1}{|v(x) + \tau_2 w(x)|^{\gamma(x)}}) \cdot w(x) + O(\|w\|_H)] dx \right| = O(\|w\|_H) \end{aligned} \quad (15)$$

for some $0 < \tau_2 < 1$. Thus we have

$|\Psi(v+w) - \Psi(v)| \leq |\Psi_1(v+w) - \Psi_1(v)| + |\Psi_2(v+w) - \Psi_2(v)| = O(\|w\|_H)$. Next we shall prove that $\Psi(u)$ is C^1 in $\Lambda_{s,G} D$. Let $v, w \in \Lambda_{s,G} D$. By (14) and (15), we have

$$|\Psi_1(v+w) - \Psi_1(v) - D\Psi_1(v + \tau_1 w) \cdot w| = O(\|w\|_H).$$

for some $\tau_1 > 0$ and

$$|\Psi_2(v+w) - \Psi_2(v) - D\Psi_2(v + \tau_2 w) \cdot w| = O(\|w\|_H).$$

for some $\tau_2 > 0$. Thus $\Psi(u)$ is C^1 . $\square$

Now we shall investigate the boundary behaviour of $J_{s,G}$

Lemma 13. Let $\{u_k = (u_{k_1}, \dots, u_{k_n})\} \subset \Lambda_{s,G} D$ and $u_k \rightharpoonup u$ weakly in $\Lambda_{s,G} D$ with $u = (u_1, \dots, u_n) \in \partial \Lambda_{s,G} D$. Then $J(u_k) \rightarrow +\infty$.

Proof. Since $\int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|} + \int_a^b \alpha(x) |u_k(x)|^{p(x)} dx > 0$, it suffices to show that

$$\int_a^b \frac{m(x)}{|u_k(x)|^{\gamma(x)}} dx \rightarrow +\infty.$$

Since $\frac{m(x)}{|u_k(x)|^{\gamma(x)}} > 0$,

$$\int_{a_0}^{b_0} \frac{m(x)}{|u_k(x)|^{\gamma(x)}} dx \rightarrow +\infty. \quad (16)$$

for some subinterval (a_0, b_0) of (a, b) . Since $u \in \partial \Lambda_{s,G} D$, there exists $x_0 \in (a, b)$ and a neighbourhood $N_\epsilon(x_0) = \{x \in (a, b) \mid |x - x_0| < \epsilon\}$ of x_0 such that $u(x_0) \in \partial D$ and for any $x \in N_\epsilon(x_0)$,

$$\frac{m(x)}{|u(x)|^{\gamma(x)}} \geq \frac{c_1}{|u(x) - u(x_0)|} - c_2$$

for some $c_1 > 0$ and $c_2 > 0$. Thus we have

$$\int_{N_\epsilon(x_0)} \frac{m(x)}{|u(x)|^{\gamma(x)}} dx \geq \int_{N_\epsilon(x_0)} [\frac{c_1}{|u(x) - u(x_0)|} - c_2] dx \quad (17)$$

$\forall \epsilon > 0$. Thus

$$\begin{aligned} |u(x) - u(x_0)| &\leq \|u(x)\|_{W^s L_G(S^1, R^n)} \frac{|x - x_0|}{b - a} \\ &\leq \frac{|x - x_0|}{b - a} \left(\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \right)^{\frac{1}{s_0}} \\ &< \epsilon \left(\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \right)^{\frac{1}{s_0}}. \end{aligned} \quad (18)$$

From (17) and (18),

$$\begin{aligned} &\int_{N_\epsilon(x_0)} \frac{m(x)}{|u(x)|^{\gamma(x)}} dx \\ &\geq \int_{N_\epsilon(x_0)} \left[\frac{c_1}{\epsilon \left(\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \right)^{\frac{1}{s_0}}} - c_2 \right] dx. \end{aligned}$$

Then we have

$$\int_{N_\epsilon(x_0)} \frac{m(x)}{|u(x)|^{\gamma(x)}} dx \rightarrow \infty.$$

□

Now, we shall prove that $J_{s,G}(u)$ satisfies $(P.S.)_{\tau_{s,G}}$ condition for any $\tau_{s,G} > \tau_{s,G}^0$ in $\Lambda_{s,G}D$.

Lemma 14. Let $(u_k)_k$ be a sequence such that $J_{s,G}(u_k) \rightarrow \tau_{s,G}$ and $DJ_{s,G}(u_k) \rightarrow \theta$ for some $\tau_{s,G} > 0$. Then if $\|u_k\|_H \rightarrow \infty$, then $J_{s,G}(u_k) \leq \tau_{s,G}^0$ in $\Lambda_{s,G}D$ for some constant $\tau_{s,G}^0 > 0$.

Proof. Let $u_k = (u_{k_1}, \dots, u_{k_n}) \in \Lambda_{s,G}D \subset H$. Assume that $\|u_k\|_H \rightarrow \infty$. Since $DJ_{s,G}(u_k) = (-\Delta)_g^s u_k + \text{grad}_u(\alpha(x)|u_k(x)|^{p(x)}) + \text{grad}_u(m(x)\frac{1}{|u_k|^{\gamma(x)}}) \rightarrow \theta$, we have, for $u_k \in \Lambda_{s,G}D$,

$$\begin{aligned} \|u\|_H &\geq DJ_{s,G}(u_k) \cdot u_k \\ &= \int_a^b (-\Delta)_g^s u_k \cdot u_k(x) dx + \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})) \cdot u_k(x) dx \\ &\quad + \int_a^b (\text{grad}_u(m(x)\frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx. \end{aligned} \quad (19)$$

By (6), we have

$$\begin{aligned} &DJ_{s,G}(u_k) \cdot u_k \\ &= \int_a^b \int_a^b [\Sigma_{i=1}^n g(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x - y|^s})] \frac{u_{k_i}(x) - u_{k_i}(y)}{|u_{k_i}(x) - u_{k_i}(y)|} \frac{u_{k_i}(x) - u_{k_i}(y)}{|x - y|^s} \frac{dx dy}{|x - y|} \\ &\quad + \int_a^b [(\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})) \cdot u_k(x) + (\text{grad}_u(m(x)\frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x)] dx \\ &\geq g_0 \left[\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \right. \\ &\quad \left. + \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})) \cdot u_k(x) dx + \int_a^b (\text{grad}_u(m(x)\frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx \right]. \end{aligned} \quad (20)$$

for large k . By Lemma 7, (20) becomes that

$$\begin{aligned}
\|u_k\|_H &\geq DJ_{s,G}(u_k) \cdot u_k \\
&\geq g_0 \left[\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|} \right. \\
&\quad \left. + \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)}) \cdot u_k(x) dx + \int_a^b (\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx \right. \\
&\geq g_0 \|u_k\|_H^{g_0} + \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)}) \cdot u_k(x) dx \\
&\quad \left. + \int_a^b (\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx \right.
\end{aligned} \tag{21}$$

and

$$\|u_k\|_H \geq g_0 \|u_k\|_H^{g_0} + \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)}) \cdot u_k(x) dx + \int_a^b (\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx. \tag{22}$$

By (22) and Hölder inequality in $\Lambda_{s,G}D \subset W^s L_G(S^1, R^n)$, we have

$$\begin{aligned}
&g_0 \|u_k\|_H^{g_0} - \|u_k\|_H \\
&\leq - \int_a^b (\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)}) \cdot u_k(x) dx - \int_a^b (\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) dx \\
&\leq 2 \|\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})\|_{L_{G^*}} \|u_k\|_{L_G} + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}} \|u_k\|_{L_G} \\
&\leq 2 \|\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})\|_{L_{G^*}} \|u_k\|_H + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}} \|u_k\|_H.
\end{aligned} \tag{23}$$

From (23),

$$g_0 \|u_k\|_H^{g_0-1} \leq 1 + 2 \|\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)})\|_{L_{G^*}} + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}}. \tag{24}$$

$\|u_k\|_H \rightarrow \infty$ implies that there exists a constant $R_0 > 0$ such that $\|u_k\|_H \geq R_0$ for large k . Thus we have

$$\int_a^b \frac{m(x)}{|u_k|^{\gamma(x)}} \leq (b-a) |m(x)|_\infty \sup \{ \frac{1}{|u_k|^{\gamma(x)}} \mid x \in (a,b), u_k \in H \setminus B_{R_0} \}. \tag{25}$$

By Lemma 9, the embedding

$$W^s L_G(S^1, R^n) \hookrightarrow L^{r(x)}(S^1, R^n)$$

is continuous and compact for all $1 \leq r(x) < g_0^*$. Since $p(x) < g_0^*$, there exists a positive constant C such that

$$\|u_k\|_{L^{p(x)}} \leq C \|u_k\|_H. \tag{26}$$

By (24)–(26), we have

$$\begin{aligned}
J_{s,G}(u_k) &= \int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_{k_i}(x) - u_{k_i}(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \\
&\quad + \int_a^b \alpha(x) |u_k(x)|^{p(x)} dx + \int_a^b m(x) \frac{1}{|u_k|^{\gamma(x)}} dx \\
&\leq \|u_k\|^{g^0} + |\alpha(x)|_\infty \|u_k(x)\|_{L^{p(x)}(a,b), \mathbb{R}^n}^{p(x)} + (b-a) |m(x)|_\infty \sup\{\frac{1}{|u_k(x)|^{\gamma(x)}}\} \\
&\leq \|u_k\|^{g^0} + C |\alpha(x)|_\infty \|u_k(x)\|_H^{p(x)} + (b-a) |m(x)|_\infty \sup\{\frac{1}{|u_k(x)|^{\gamma(x)}}\} \\
&\leq [\frac{1}{g_0} (1 + 2 \|\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)})\|_{L_{G^*}} + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}})]^{\frac{g^0}{g_0-1}} \\
&\quad + C |\alpha(x)|_\infty ([\frac{1}{g_0} (1 + 2 \|\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)})\|_{L_{G^*}} \\
&\quad + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}})]^{\frac{p_+}{g_0-1}}) + (b-a) |m(x)|_\infty \sup\{\frac{1}{|u_k(x)|^{\gamma(x)}}\}.
\end{aligned}$$

If we set,

$$\begin{aligned}
\tau_{s,G}^0 &= [\frac{1}{g_0} (1 + 2 \|\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)})\|_{L_{G^*}} + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}})]^{\frac{g^0}{g_0-1}} \\
&\quad + C |\alpha(x)|_\infty ([\frac{1}{g_0} (1 + 2 \|\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)})\|_{L_{G^*}} \\
&\quad + 2 \|\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})\|_{L_{G^*}})]^{\frac{p_+}{g_0-1}}) + (b-a) |m(x)|_\infty \sup\{\frac{1}{|u_k(x)|^{\gamma(x)}}\},
\end{aligned}$$

we have $J_{s,G}(u_k) \leq \tau_{s,G}^0$. $\square$

Lemma 15. $J_{s,G}(u)$ satisfies the $(P.S.)_{\tau_{s,G}}$ condition for any $\tau_{s,G} > \tau_{s,G}^0$ in $\Lambda_{s,G}D$.

Proof. Let $\tau_{s,G} > 0$ and $(u_k)_k \subset \Lambda_{s,G}D$ be a sequence such that $J_{s,G}(u_k) \rightarrow \tau_{s,G}$ and $DJ_{s,G}(u_k) \rightarrow \theta$, $\theta = (0, \dots, 0)$ in $\Lambda_{s,G}D$. First we shall show that $\{u_k\}$ is bounded in $\Lambda_{s,G}D$. Suppose that $\|u_k\|_H \rightarrow \infty$ as $k \rightarrow \infty$. By Lemma 14, $J_{s,G}(u_k) \leq \tau_{s,G}^0$, where $\tau_{s,G}^0$ is introduced in the proof of Lemma 14. This leads to a contradiction because $J_{s,G}(u_k) \rightarrow \tau_{s,G} > \tau_{s,G}^0$. Thus $\{u_k\}$ is bounded in $\Lambda_{s,G}D$. Up to a subsequence, $\{u_k\}$ converges weakly to some $u \in \Lambda_{s,G}D$. We claim that $\{u_k\}$ converges strongly to $u \in \Lambda_{s,G}D$. Since $DJ_{s,G}(u_k) \rightarrow \theta$, we have

$$u_k(x) + ((-\Delta)_g^s)^{-1} (\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)}) + \text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \rightarrow \theta \quad (27)$$

and we also have

$$\begin{aligned}
DJ_{s,G}(u_k(x)) \cdot u_k(x) &= (-\Delta)_g^s u_k(x) \cdot u_k(x) + (\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)}) \cdot u_k(x) \\
&\quad + (\text{grad}_u(m(x) \frac{1}{|u_k|^{\gamma(x)}})) \cdot u_k(x) \rightarrow 0.
\end{aligned} \quad (28)$$

The second and third part of the right-hand side of (28) satisfy

$$\int_a^b [\text{grad}_u(\alpha(x) |u_k(x)|^{p(x)}) \cdot u_k(x)] dx \leq p_+ |\alpha(x)|_\infty \|u_k(x)\|_{L^{p(x)}(S^1, \mathbb{R}^n)}^{p_+}$$

and

$$\int_a^b (\text{grad}_u(m(x)) \frac{1}{|u_k|^{\gamma(x)}}) \cdot u_k dx \leq |m(x)|_\infty \gamma_+ \int_a^b \frac{1}{|u_k(x)|^{\gamma(x)}} dx.$$

Continuous and compact embedding $W^s L_G(S^1, R^n) \hookrightarrow L^{p(x)}(S^1, R^n)$ for $1 < p(x) < g_0^*$ implies that $\|u_k\|_{L^{p(x)}(S^1, R^n)} \leq C \|u_k\|_H$ for some constant $C > 0$. Thus if $\{u_k\}$ is bounded in $\Lambda_{s,G} D \subset W^s L_G(S^1, R^n)$, then $\{u_k\}$ is bounded in $L^{p(x)}(S^1, R^n)$. It follows that $\int_a^b [\text{grad}_u(\alpha(x)|u_k(x)|^{p(x)} \cdot u_k(x))] dx$ and $\int_a^b (\text{grad}_u(m(x)) \frac{1}{|u_k|^{\gamma(x)}}) \cdot u_k dx$ are bounded in $L^{p(x)}(S^1, R^n)$. By (28), $(-\Delta)_g^s u_k(x) \cdot u_k(x)$ is bounded. Since $\{u_k\}$ is bounded in $\Lambda_{s,G} D$,

$$(-\Delta)_g^s u_k(x) \text{ is bounded in } L^{p(x)}(S^1, R^n).$$

Lemma 9 leads that $\{u_k\}$ converges to u strongly in $\Lambda_{s,G} D$. $\square$

4. Proofs of Theorems 1 and 2

Let us set the level set

$$J_{s,G} \tau_{s,G} = \{u \in \Lambda_{s,G} D \mid J_{s,G}(u) \leq \tau_{s,G}\}.$$

Lemma 16. $H_{b_i}(\Lambda_{s,G} D) \neq 0$ for some strictly increasing sequence b_i .

Proof. Let $\bar{C}_\epsilon(\theta)$ be a neighbourhood θ with radius $\epsilon > 0$ and choose $R > 0$ such that $\bar{C}_\epsilon(\theta) \subset \text{int}(B_R(\theta))$. Then we have

$$R^n \setminus B_R(\theta) \subset (R^n \setminus \{\theta\})$$

and $R^n \setminus B_R(\theta)$ is a deformation retract of $R^n \setminus \{\theta\}$. Then $\Lambda_{s,G}(R^n \setminus B_R(\theta))$ is a deformation retract of $\Lambda_{s,G} D$. Thus we have

$$\begin{aligned} H_*(\Lambda_{s,G} D) &\cong H_*(\Lambda_{s,G}(R^n \setminus B_R(\theta))) \oplus H_*(\Lambda_{s,G} D, \Lambda_{s,G}(R^n \setminus B_R(\theta))) \\ &\cong H_*(\Lambda_{s,G}(S^{n-1})) \oplus H_*(\Lambda_{s,G} D, \Lambda_{s,G}(S^{n-1})). \end{aligned}$$

In [111], the Poincaré series of $\Lambda_{s,G}(S^{n-1})$ is written as

$$P_t(\Lambda_{s,G}(S^{n-1})) = (1 + t^n) + \frac{t^{n-1}}{1 - t^{2(n-1)}} (1 + t^n)(1 + t^{n-1}).$$

$\square$

Lemma 17. The level set $J_{s,G} \tau_{s,G}$ is deformed into $\Sigma_{s,G}$ for some finite dimensional singular complex $\Sigma_{s,G} = \Sigma_{\tau_{s,G}}$.

Proof. Let $u \in J_{s,G} \tau_{s,G}$. Then $u = (u_1, \dots, u_n) \in \Lambda_{s,G} D$ and

$$\begin{aligned} J_{s,G}(u) &= \int_a^b \int_a^b [\sum_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x - y|^s})] \frac{dx dy}{|x - y|} \\ &\quad + \int_a^b \alpha(x) |u(x)|^{p(x)} dx + \int_a^b m(x) \frac{1}{|u|^{\gamma(x)}} dx \leq \tau_{s,G}. \end{aligned} \quad (29)$$

If $(x, u(x)) \in [a, b] \times (R^n \setminus B_{R_0}(\theta))$, then

$$0 < \frac{m(x)}{|u(x)|^{\gamma(x)}} < +\infty, \text{ and } |\text{grad}_u(\frac{m(x)}{|u(x)|^{\gamma(x)}})| < +\infty \quad (30)$$

for some constant $R_0 > 0$. We also note that if $u(x) \in B_{R_0} \setminus \bar{C}_\epsilon(\theta)$, then

$$(b-a) \inf\{\alpha(x)\} \epsilon^{p-} + (b-a) \frac{\inf\{m(x)\}}{R_0^{\gamma+}} < \int_a^b \alpha(x) |u(x)|^{p(x)} dx + \int_a^b \frac{m(x)}{|u(x)|^{\gamma(x)}} dx < \tau_{s,G}. \quad (31)$$

(29)–(31) imply that

$$\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|} < \tau_{s,G}^1 \quad (32)$$

for some constant $\tau_{s,G}^1 > 0$. By Lemma 13, there exists $\epsilon_{s,G}^0 = \epsilon_{s,G}(\tau_{s,G}, \tau_{s,G}^1)$ such that

$$d(u, \bar{C}_{\epsilon_{s,G}}(\theta)) \geq \epsilon_{s,G}^0 \quad \forall u \in J_{s,G\tau_{s,G}}, \forall x \in [a, b].$$

Let us choose an integer $N_{s,G} = N_{\tau_{s,G}^1} > \frac{(\tau_{s,G}^1)^{\frac{1}{s_0}}}{\epsilon_{s,G}^0}$ and set

$$x_{s,G}^i = \frac{(b-a)i}{N_{s,G}} \quad i = 1, 2, \dots, N_{s,G}$$

and

$$\bar{u}_{s,G}(x) = (1 - \frac{1}{b-a} N_{s,G}(x - x_{s,G}^{i-1})) u(x_{s,G}^{i-1}) + \frac{1}{b-a} N_{s,G}(x - x_{s,G}^{i-1}) u(x_{s,G}^i)$$

$\forall x \in [x_{s,G}^{i-1}, x_{s,G}^i], i = 1, 2, \dots, N_{s,G}, \forall u \in J_{s,G\tau_{s,G}}$. Let

$$\Sigma_{s,G} = \{\bar{u}_{s,G}(x) | u \in J_{s,G\tau_{s,G}}\}.$$

Mapping $D \times D \times \dots \times_{N_{s,G}} D \ni \bar{u}_{s,G} \mapsto (u(x_{s,G}^1), u(x_{s,G}^2), \dots, u(x_{s,G}^{N_{s,G}}))$ is a homeomorphism. We shall show that $\Sigma_{s,G} \subset \Lambda_{s,G}D$. In fact, for $\forall u \in J_{s,G\tau_{s,G}}$ and $x_{s,G}^i > x_{s,G}^{i-1}$, by (32), we have

$$\begin{aligned} |u(x_{s,G}^i) - u(x_{s,G}^{i-1})| &\leq \|\bar{u}_{s,G}(x)\|_{W^s L_G(S^1, R^n)} \frac{|x_{s,G}^i - x_{s,G}^{i-1}|}{b-a} \\ &\leq \frac{|x_{s,G}^i - x_{s,G}^{i-1}|}{b-a} \left(\int_a^b \int_a^b [\Sigma_{i=1}^n G(\frac{|u_i(x) - u_i(y)|}{|x-y|^s})] \frac{dx dy}{|x-y|} \right)^{\frac{1}{s_0}} \\ &< \frac{\epsilon_{s,G}}{b-a} (\tau_{s,G}^1)^{\frac{1}{s_0}}. \end{aligned}$$

Thus

$$\begin{aligned} d(\bar{u}_{s,G}(x), \bar{C}_{\epsilon_{s,G}}(\theta)) &\geq d(u(x_{s,G}^i), \bar{C}_{\epsilon_{s,G}}(\theta)) \\ &\quad - (1 - \frac{1}{b-a} N_{s,G}(x - x_{s,G}^{i-1})) |u(x_{s,G}^i) - u(x_{s,G}^{i-1})| \\ &\geq \epsilon_{s,G}^0 - (N_{s,G})^{-1} (\tau_{s,G}^1)^{\frac{1}{s_0}} > 0 \end{aligned}$$

$\forall x \in [x_{s,G}^{i-1}, x_{s,G}^i]$. We shall show that there exists $\eta_{s,G} \in C([0, 1] \times J_{s,G\tau_{s,G}}, \Lambda_{s,G}D)$ such that $\eta_{s,G}(0, \cdot) = \text{id}$, and $\eta_{s,G}(1, J_{s,G\tau_{s,G}}) = \Sigma_{s,G}$. In fact, Let us choose $u(x) \in \Lambda_{s,G}D$ and let us define $\eta_{s,G}$ as follows:

$$\eta_{s,G}(y, u)(x) = \begin{cases} u(x) & \text{for } x \geq (b-a)y, \\ (1 - \frac{x - x_{s,G}^{i-1}}{(b-a)y - x_{s,G}^{i-1}}) u(x_{s,G}^{i-1}) + \frac{x - x_{s,G}^{i-1}}{(b-a)y - x_{s,G}^{i-1}} u((b-a)y) & \text{for } x_{s,G}^{i-1} < x < (b-a)y, \\ \bar{u}_{s,G}(x) & \text{for } x \leq x_{s,G}^{i-1}. \end{cases}$$

Then $\eta_{s,G}(0, \cdot) = \text{id}$, and $\eta_{s,G}(1, J_{s,G\tau_{s,G}}) = \Sigma_{s,G}$. Thus we prove that $J_{s,G\tau_{s,G}}$ is deformed into $\Sigma_{s,G}$ in the loop space $\Lambda_{s,G}D$. $\square$

Proof of Theorem 1. We claim that for each $q > nN_{s,G}$, where $N_{s,G} = N_{s,G\tau_{s,G}^0}$ is defined in Lemma 17, let us set

$$\tau_{s,G} = \inf_{z \in \beta} \max_{u \in [z]} J_{s,G}(u),$$

where $\beta \in H_q(\Lambda_{s,G}D)$ is nontrivial. We claim that $\tau_{s,G} > \tau_{s,G}^0$ and $\tau_{s,G}$ is a critical value of $J_{s,G}$. In fact, suppose that $\tau_{s,G} \leq \tau_{s,G}^0$ and $\tau_{s,G}$ is not a critical value of $J_{s,G}$. Then $[u] \in J_{s,G\tau_{s,G}^0+1}$ for some $[u] \in \beta$. By Lemma 17, there exists a deformation $\eta_{s,G} : [0, 1] \times J_{s,G\tau_{s,G}^0+1} \rightarrow \Lambda_{s,G}D$ such that $\eta_{s,G}(1, J_{s,G\tau_{s,G}^0+1}) \subset \Sigma_{s,G\tau_{s,G}^0+1}$ with $\dim \Sigma_{s,G\tau_{s,G}^0+1} \leq nN_{s,G\tau_{s,G}^0}$. This implies that $\eta_{s,G}(1, [u]) \subset \Sigma_{s,G\tau_{s,G}^0+1}$. However, $\eta_{s,G}(1, [u]) \in \beta$ and $\beta \in H_q(\Lambda_{s,G}D)$ with $q > nN_{s,G\tau_{s,G}^0}$, which is absurd. $\square$

Proof of Theorem 2. We suppose that $J_{s,G}(u)$ has only finitely many critical points $u_1, u_2, \dots, u_{m_{s,G}}$ such that we can obtain $J_{s,G}(u_k) > \tau_{s,G}^0$, $1 \leq k \leq m_{s,G}$. Let us set

$$K = \{u_1, u_2, \dots, u_{m_{s,G}}\}.$$

We note that $\dim \ker(D^2 J_{s,G}(u_k)) \leq 2n$, for all k . Letting

$$b_{s,G}^* > \max\{nN_{s,G\tau_{s,G}^0}, \text{ind}(J_{s,G}, u_k) + \dim \ker(D^2 J_{s,G}(u_k)) \mid 1 \leq k \leq m_{s,G}\},$$

and

$$d_{s,G} > \max\{\tau_{s,G}^0, J_{s,G}(u_k) \mid 1 \leq k \leq m_{s,G}\},$$

we have

$$C_b(J_{s,G}, u_k) = 0 \quad \forall b_{s,G} \geq b_{s,G}^*, k = 1, 2, \dots, m_{s,G}$$

and

$$H_*(\Lambda_{s,G}D, J_{s,G\tau_{s,G}^0}) = H_*(J_{s,Gd_{s,G}}, J_{s,G\tau_{s,G}^0}).$$

Hence

$$H_b(\Lambda_{s,G}D, J_{s,G\tau_{s,G}^0}) = 0 \quad \forall b_{s,G} > b_{s,G}^*.$$

Since

$$i_* : H_b(\Lambda_{s,G}D) \longrightarrow H_b(\Lambda_{s,G}D, J_{s,G\tau_{s,G}^0}) \text{ is injective for } b_{s,G} \geq b_{s,G}^*$$

we have

$$H_b(\Lambda_{s,G}D) = 0 \text{ for } b_{s,G} \geq b_{s,G}^*,$$

which is absurd. $\square$

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