

Article

Two Isospectral-Nonisospectral Super-Integrable Hierarchies and Related Invariant Solutions

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Abstract: In this article, we adopt two kinds of loop algebras corresponding to the Lie algebra $B(0, 1)$ to introduce two line spectral problems with different numbers of even and odd superfunctions. Through generalizing the time evolution λ_t to a polynomial of λ , two isospectral-nonispectral super integrable hierarchies are derived in terms of Tu scheme and zero-curvature equation. Among them, the first super integrable hierarchy is further reduced to generalized Fokker–Plank equation and special bond pricing equation, as well as an explicit super integrable system under the choice of specific parameters. More specifically, a super integrable coupled equation is derived and the corresponding integrable properties are discussed, including the Lie point symmetries and one-parameter Lie symmetry groups as well as group-invariant solutions associated with characteristic equation.

Keywords: isospectral-nonispectral super-integrable hierarchy; invariant solution; zero curvature equation; Lie algebra



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1. Introduction

In the development of the past few decades, the nonlinear evolution equations have attracted increasing attentions from many researchers with their significant roles in describing the nonlinear dynamic behaviors in various fields. While a group of scholars devoted themselves to the study of solving equation, there were also a group of scholars who devoted themselves to the derivation of integrable hierarchies of equations. Through further reduction, a large number of integrable equations can be obtained, so that some related integrable properties could be discussed. Thus, the fact indicates that the derivation of integrable hierarchies of evolution equations is extremely essential. In recent decades, a variety of powerful methods for deriving integrable hierarchies of equations have also emerged. For instance, one of the most classical methods for generating integrable hierarchies of equations and Hamiltonian structures was proposed by Tu Guizhang in 1989 [1]. Later, the method was called the Tu scheme by Ma Wenxiu [2]. Since then, with the aid of Tu scheme, a great many integrable systems and the corresponding Hamiltonian structures as well as other properties were derived [3–13]. However, we found that the Tu scheme is usually utilized to generate isospectral integrable hierarchies of equations by choosing proper loop algebras. How to use it to generate nonisospectral integrable hierarchies of equations will be a problem worth considering. Until recently, Zhang et al. proposed an approach for generating nonisospectral integrable hierarchies under the assumption that λ_t is a polynomial of λ [14], which is a more general case compared to that presented in [15]. There has been a lot of research work on the application of the method, such as [16,17]. However, it is worth noting that the method has not been employed to generate nonisospectral super integrable hierarchy. For realizing the purpose, in this paper, two line spectral problems are introduced via two kinds of super loop algebras of the super Lie algebra $B(0, 1)$ [18]. The subalgebra $B(0, 1)$ of the super Lie algebra $sl(m/n)$ is defined as follows

$$sl(m/n) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad str X = Tr A - Tr D = 0 \right\},$$

where A is an $m \times m$ matrix, B is an $m \times n$ matrix, C is an $n \times m$ matrix, D is an $n \times n$ matrix. The Lie bracket of $sl(m/n)$ is given by

$$s[X, Y] = XY - (-1)^{P(X)P(Y)} YX, \quad \forall X, Y \in sl(m/n), \quad (1)$$

where the degraion of the element X is defined as

$$P(X) = \begin{cases} 0, & X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad str X = 0, \\ 1, & X = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \end{cases}$$

for which, the matrix $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is called an even element of the super Lie algebra $sl(m/n)$,

also known the bosonic. On the contrary, the matrix $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ is called the odd element or known as the fermionic. The super loop algebra $\tilde{sl}(m/n)$ corresponding to the Lie algebra $sl(m/n)$ reads as

$$s[X(m), Y(n)] \equiv s[X, Y] \otimes \lambda^{m+n}. \quad (2)$$

A natural degree is defined as

$$deg(X \otimes \lambda^n) = n, \quad \forall X \in sl(m/n).$$

By employing the first loop algebra, an isospectral-nonispectral super integrable hierarchy containing two even superfunctions and two odd superfunctions is derived under the case of $\lambda_t = \sum_{j \geq 0} k_j(t) \lambda^{-j}$. Through further reduction, we obtain the generalized Fokker-

Plank equation and special bond pricing equation, as well as the explicit super integrable system, which is further reduced to three super integrable coupled equations. The three super integrable coupled equations have been presented in Ref. [18]. For simplicity, we write only one of them and discuss its group-invariant solutions in the form of power series. By utilizing the second loop algebra, another line spectral problem is introduced, for which the isospectral-nonispectral super integrable hierarchy containing five even superfunctions and four odd superfunctions is obtained in the frame of zero-curvature equation and Tu scheme. For the second situation, the time evolution is designed as $\lambda_t = \sum_{j \geq 0} k_j(t) \lambda^{-2j+1}$, because the degree of the loop algebra is of the form $2m + i, m \in \mathcal{Z}, i = 0, 1; j = 0, 1, 2, 3, 4$, which will be found below.

2. Two Isospectral-Nonisospectral Super-Integrable Hierarchies

In fact, there are many powerful methods for generating isospectral integrable hierarchies, while there are few methods to generate nonisospectral integrable hierarchies. As far as we know, Qiao Zhijun adopted the generalized Lax representations to generate nonisospectral integrable hierarchies and obtained some isospectral and nonisospectral integrable hierarchies with the help of the Gateaux derivatives and Lenard series [19,20]. Ma proposed a technique for generating nonisospectral integrable equations by making use of Lax operators and zero curvature equations and further discussed corresponding algebraic structures [21–23]. There are other approaches for deducing nonisospectral integrable hierarchies, but they are still very few. Hence, the method for generating nonisospectral super integrable hierarchies introduced in our paper will be a very significant work.

The super algebra $B(0, 1)$ has different bases. In this paper, we only consider the following basis

$$B(0, 1) = span\{E_0, E_1, E_2, E_3, E_4\}, \quad (3)$$

where

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Moreover, E_0, E_1 and E_2 are even elements, E_3 and E_4 are odd elements. In terms of the definition (1) of the super Lie algebra $sl(m/n)$, direct computation yields the following commutative and anti-commutative relations

$$s[E_0, E_1] = 2E_1, \quad s[E_0, E_2] = -2E_2, \quad s[E_1, E_2] = E_0, \quad s[E_0, E_3] = E_3,$$

$$s[E_0, E_4] = -E_4, \quad s[E_1, E_3] = 0, \quad s[E_1, E_4] = E_3, \quad s[E_2, E_3] = E_4,$$

$$s[E_2, E_4] = 0, \quad s[E_3, E_4] = -E_0, \quad s[E_3, E_3] = 2E_1, \quad s[E_4, E_4] = -2E_2.$$

At the beginning, we introduce the first kind of simplest super loop algebra

$$\tilde{B}(0, 1) = span\{E_0(n), E_1(n), E_2(n), E_3(n), E_4(n)\}, \tag{4}$$

of which

$$E_i(n) = E_i \lambda^n, \quad i = 0, 1, 2, 3, 4, \quad deg E_i(n) = n$$

equipped with the following operation relations

$$s[E_i(m), E_j(n)] = s[E_i, E_j] \lambda^{m+n}, \quad 0 \leq i, j \leq 4; \quad m, n \in \mathbb{Z}.$$

In what follows, we shall employ the super loop algebra (4) to introduce the following spectral problem

$$\begin{cases} \varphi_x = U\varphi, \\ U = E_0(1) + qE_1(0) + rE_2(0) + \alpha E_3(0) + \beta E_4(0). \end{cases} \tag{5}$$

where q, r are even superfunctions, and α, β are odd superfunctions. Assume

$$\varphi_t = V_1\varphi + V_2\varphi =: V\varphi, \tag{6}$$

where

$$V_1 = \sum_{i \geq 0} (V_{0,i}E_0(-i) + V_{1,i}E_1(-i) + V_{2,i}E_2(-i) + V_{3,i}E_3(-i) + V_{4,i}E_4(-i)),$$

$$V_2 = \sum_{j \geq 0} (\tilde{V}_{0,j}E_0(-j) + \tilde{V}_{1,j}E_1(-j) + \tilde{V}_{2,j}E_2(-j) + \tilde{V}_{3,j}E_3(-j) + \tilde{V}_{4,j}E_4(-j)),$$

$$\lambda_t = \frac{\partial \lambda}{\partial t} = \sum_{j \geq 0} k_j(t) \lambda^{-j},$$

$V_{0,i}, V_{1,i}, V_{2,i}, \tilde{V}_{0,j}, \tilde{V}_{1,j}, \tilde{V}_{2,j}$ are bosonic, and $V_{3,i}, V_{4,i}, \tilde{V}_{3,j}, \tilde{V}_{4,j}$ are fermionic.

Solving the stationary zero curvature equation

$$V_{1,x} = [U, V_1] \tag{7}$$

yields

$$\begin{cases} (V_{0,i})_x = qV_{2,i} - rV_{1,i} - \alpha V_{4,i} + \beta V_{3,i} \\ 2V_{1,i+1} = (V_{1,i})_x + 2qV_{0,i} - 2\alpha V_{3,i} \\ 2V_{2,i+1} = -(V_{2,i})_x + 2rV_{0,i} - 2\beta V_{4,i} \\ V_{3,i+1} = (V_{3,i})_x - qV_{4,i} + \alpha V_{0,i} + \beta V_{1,i} \\ V_{4,i+1} = -(V_{4,i})_x + rV_{3,i} - \alpha V_{2,i} + \beta V_{0,i}. \end{cases} \tag{8}$$

Denote

$$V_{1,+}^{(n)} = \sum_{i=0}^n (V_{0,i}E_0(n-i) + V_{1,i}E_1(n-i) + V_{2,i}E_2(n-i) + V_{3,i}E_3(n-i) + V_{4,i}E_4(n-i)) \tag{9}$$

and let $V_1^{(n)} = V_{1,+}^{(n)}$, then the zero-curvature equation

$$U_t - V_{1,x}^{(n)} + [U, V_1^{(n)}] = 0 \tag{10}$$

admits a super integrable hierarchy (17) presented in Ref. [18].

Under the time evolution $\lambda_t \neq 0$, the resulting zero curvature equation reads

$$V_{2,x} = [U, V_2] + \frac{\partial U}{\partial \lambda} \lambda_t, \tag{11}$$

which leads to

$$\begin{cases} (\bar{V}_{0,j})_x = q\bar{V}_{2,j} - r\bar{V}_{1,j} - \alpha\bar{V}_{4,j} + \beta\bar{V}_{3,j} + k_j(t) \\ 2\bar{V}_{1,j+1} = (\bar{V}_{1,j})_x + 2q\bar{V}_{0,j} - 2\alpha\bar{V}_{3,j} \\ 2\bar{V}_{2,j+1} = -(\bar{V}_{2,j})_x + 2r\bar{V}_{0,j} - 2\beta\bar{V}_{4,j} \\ \bar{V}_{3,j+1} = (\bar{V}_{3,j})_x - q\bar{V}_{4,j} + \alpha\bar{V}_{0,j} + \beta\bar{V}_{1,j} \\ \bar{V}_{4,j+1} = -(\bar{V}_{4,j})_x + r\bar{V}_{3,j} - \alpha\bar{V}_{2,j} + \beta\bar{V}_{0,j}. \end{cases} \tag{12}$$

Denoting

$$V_{2,+}^{(m)} = \sum_{j=0}^m (\bar{V}_{0,j}E_0(m-j) + \bar{V}_{1,j}E_1(m-j) + \bar{V}_{2,j}E_2(m-j) + \bar{V}_{3,j}E_3(m-j) + \bar{V}_{4,j}E_4(m-j)) = \lambda^m V_2 - V_{2,-}^{(m)}, \tag{13}$$

$$\lambda_{t,+}^{(m)} = \sum_{j=0}^m k_j(t)\lambda^{m-j} = \lambda^m \lambda_t - \lambda_{t,-}^{(m)} = \lambda^m \lambda_t - \sum_{j=m+1}^{\infty} k_j(t)\lambda^{m-j}.$$

Based on the purpose for deducing the nonisospectral super integrable hierarchy, we first decompose (11) into the following form

$$-(V_{2,+}^{(m)})_x + [U, V_{2,+}^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{(m)} = (V_{2,-}^{(m)})_x - [U, V_{2,-}^{(m)}] - \frac{\partial U}{\partial \lambda} \lambda_{t,-}^{(m)}. \tag{14}$$

The gradations of the left-hand side in (14) are more than 0, while those of the right-hand side less than 0. Thus, we only take the terms with the gradations being 0 in both sides of (14) and let $V_2^{(m)} = V_{2,+}^{(m)}$, one gets

$$-(V_2^{(m)})_x + [U, V_2^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{(m)} = -2\bar{V}_{1,m+1}E_1(0) + 2\bar{V}_{2,m+1}E_2(0) - \bar{V}_{3,m+1}E_3(0) + \bar{V}_{4,m+1}E_4(0).$$

Hence, the nonisospectral zero curvature equation

$$U_t - V_{1,x}^{(n)} - V_{2,x}^{(m)} + [U, V_1^{(n)} + V_2^{(m)}] = 0 \tag{15}$$

gives an isospectral–nonisospectral super integrable hierarchy

$$u_{t_{n,m}} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_{n,m}} = \begin{pmatrix} 2V_{1,n+1} + 2\bar{V}_{1,m+1} \\ -2V_{2,n+1} - 2\bar{V}_{2,m+1} \\ V_{3,n+1} + \bar{V}_{3,m+1} \\ -V_{4,n+1} - \bar{V}_{4,m+1} \end{pmatrix}. \tag{16}$$

In what follows, we consider some reductions for (16). Before that, we take initial values $V_{0,0} = \sigma$ (arbitrary real number), $V_{1,0} = V_{2,0} = V_{3,0} = V_{4,0} = 0$, $\bar{V}_{0,0} = k_0(t)x$, $\bar{V}_{1,0} = \bar{V}_{2,0} = \bar{V}_{3,0} = \bar{V}_{4,0} = 0$. In terms of (8) and (12), we further obtain

$$\begin{aligned} V_{1,1} &= \sigma q, V_{2,1} = \sigma r, V_{3,1} = \sigma \alpha, V_{4,1} = \sigma \beta, V_{0,1} = 0, V_{1,2} = \frac{\sigma}{2} q_x, \\ V_{2,2} &= -\frac{\sigma}{2} r_x, V_{3,2} = \sigma \alpha_x, V_{4,2} = -\sigma \beta_x, V_{0,2} = -\frac{\sigma}{2} qr + \sigma \alpha \beta, \dots, \\ \bar{V}_{1,1} &= k_0(t)xq, \bar{V}_{2,1} = k_0(t)xr, \bar{V}_{3,1} = k_0(t)x\alpha, \bar{V}_{4,1} = k_0(t)x\beta, \bar{V}_{0,1} = k_1(t)x, \bar{V}_{1,2} = k_1(t)xq + \frac{1}{2}k_0(t)(xq)_x, \\ \bar{V}_{2,2} &= k_1(t)xr - \frac{1}{2}k_0(t)(xr)_x, \bar{V}_{3,2} = k_1(t)x\alpha + k_0(t)(x\alpha)_x, \bar{V}_{4,2} = k_1(t)x\beta - k_0(t)(x\beta)_x, \\ \bar{V}_{0,2} &= -\frac{1}{2}k_0(t)xqr - \frac{1}{2}k_0(t) \int qrdx + k_0(t)x\alpha\beta + k_0(t) \int \alpha\beta dx + k_2(t)x, \dots \end{aligned} \tag{17}$$

When $n = 2, m = 1$, (16) reduces to

$$u_{t_{2,1}} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_{2,1}} = \begin{pmatrix} \frac{\sigma}{2} q_{xx} - \sigma q^2 r + 2\sigma q \alpha \beta - 2\sigma \alpha \alpha_x + 2k_1(t)xq + k_0(t)(xq)_x \\ -\frac{\sigma}{2} r_{xx} + \sigma r^2 q - 2\sigma r \alpha \beta - 2\sigma \beta \beta_x - 2k_1(t)xr + k_0(t)(xr)_x \\ \sigma \alpha_{xx} + \sigma q \beta_x - \frac{\sigma}{2} \alpha qr + \frac{\sigma}{2} \beta q_x - k_1(t)x\alpha\beta + k_1(t)x\alpha + k_0(t)(x\alpha)_x \\ -\sigma \beta_{xx} - \sigma r \alpha_x - \frac{\sigma}{2} \alpha r_x + \frac{\sigma}{2} \beta qr - k_1(t)x\beta + k_0(t)(x\beta)_x \end{pmatrix}. \tag{18}$$

If taking $r = \alpha = \beta = 0$, (18) becomes

$$q_t = \frac{\sigma}{2} q_{xx} + 2k_1(t)xq + k_0(t)(xq)_x, \tag{19}$$

which is the generalized Fokker–Plank equation.

Letting $\sigma = 0$, (19) further reduces

$$q_t = 2k_1(t)xq + k_0(t)(xq)_x, \tag{20}$$

we call it special bond pricing equation.

Remark 1. The generalized Fokker–Plank Equation (19) and special bond pricing Equation (20) are similar to that presented in [16].

When $n = 3, m = 0$, (16) reduces to

$$u_{t_{3,0}} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_{3,0}} = \begin{pmatrix} \frac{\sigma}{4} q_{xxx} - \frac{\sigma}{2} (q^2 r)_x + \sigma (q \alpha \beta)_x - \sigma (\alpha \alpha_x)_x + \frac{\sigma}{2} q^2 r_x - \frac{\sigma}{2} q q_x r - 4\sigma q \alpha \beta_x \\ + 2\sigma q \alpha_x \beta - 2\sigma \alpha \alpha_{xx} - \sigma \alpha \beta q_x + 2k_0(t)xq, \\ \frac{\sigma}{4} r_{xxx} - \frac{\sigma}{2} (r^2 q)_x + \sigma (r \alpha \beta)_x + \sigma (\beta \beta_x)_x - \frac{\sigma}{2} q r r_x + \frac{\sigma}{2} q_x r^2 + 2\sigma \alpha \beta_x \\ - 4\sigma r \alpha_x \beta + 2\sigma \beta \beta_{xx} + \sigma \beta \alpha r_x - 2k_0(t)xr, \\ \sigma \alpha_{xxx} + \sigma (q \beta_x)_x - \frac{\sigma}{2} (\alpha qr)_x + \frac{\sigma}{2} (\beta q_x)_x - \sigma q \beta_{xx} - \sigma q r \alpha_x - \frac{\sigma}{4} \alpha q r_x + \frac{\sigma}{4} \beta q_{xx} + k_0(t)x\alpha, \\ \sigma \beta_{xxx} + \sigma (r \alpha_x)_x + \frac{\sigma}{2} (\alpha r_x)_x - \frac{\sigma}{2} (\beta qr)_x - \sigma r \alpha_{xx} - \sigma q r \beta_x + \frac{\sigma}{4} \alpha r_{xx} - \frac{\sigma}{4} \beta q r_x + \sigma \beta \beta_x \\ + \sigma \beta \alpha \alpha_x - k_0(t)x\beta. \end{pmatrix}. \tag{21}$$

Specially, set $q = 1, \alpha = 0, \sigma = -4, k_0(t) = 0$, (21) reduces to

$$\begin{cases} r_t = -r_{xxx} + 6rr_x - 12\beta\beta_{xx} - 4\beta_x^2, \\ \beta_t = -4\beta_{xxx} + 3\beta r_x + 6r\beta_x, \end{cases} \quad (22)$$

which has been presented in Ref. [18]. In the following, we use the method of Lie-symmetry analysis of differential equations, see, e.g., [24–27], to find Lie point symmetries and one-parameter transformation groups as well as power series solutions for (22). We first assume a vector field has the following infinitesimal generator

$$V = \zeta_1(x, t, u, v) \frac{\partial}{\partial t} + \zeta_2(x, t, u, v) \frac{\partial}{\partial x} + \eta(x, t, u, v) \frac{\partial}{\partial u} + \zeta(x, t, u, v) \frac{\partial}{\partial v}.$$

With the help of Maple software, we obtain the following infinitesimal generator

$$V_1 = \frac{1}{3}x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{2}{3}u \frac{\partial}{\partial u} - \frac{1}{2}v \frac{\partial}{\partial v}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = t \frac{\partial}{\partial x} - \frac{1}{6} \frac{\partial}{\partial u}, \quad V_4 = \frac{\partial}{\partial x}.$$

By solving the following initial problems

$$\begin{aligned} \frac{d\tilde{t}(\varepsilon)}{d\varepsilon} &= \zeta_1(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon), \tilde{u}(\varepsilon), \tilde{v}(\varepsilon)), \tilde{t}(0) = t, \\ \frac{d\tilde{x}(\varepsilon)}{d\varepsilon} &= \zeta_2(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon), \tilde{u}(\varepsilon), \tilde{v}(\varepsilon)), \tilde{x}(0) = x, \\ \frac{d\tilde{u}(\varepsilon)}{d\varepsilon} &= \eta(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon), \tilde{u}(\varepsilon), \tilde{v}(\varepsilon)), \tilde{u}(0) = u, \\ \frac{d\tilde{v}(\varepsilon)}{d\varepsilon} &= \zeta(\tilde{x}(\varepsilon), \tilde{t}(\varepsilon), \tilde{u}(\varepsilon), \tilde{v}(\varepsilon)), \tilde{v}(0) = v, \end{aligned}$$

the one-parameter Lie symmetry group generated by the infinitesimal generator $V_i (i = 1, 2, \dots, 5)$ can be expressed as below, respectively.

$$\begin{aligned} g_1(x, t, u, v) &= e^{V_1}(x, t, u, v) = (e^{\frac{1}{3}\varepsilon}x, e^{\varepsilon}t, e^{-\frac{2}{3}\varepsilon}u, e^{-\frac{1}{2}\varepsilon}v), \\ g_2(x, t, u, v) &= e^{V_2}(x, t, u, v) = (x, \varepsilon + t, u, v), \\ g_3(x, t, u, v) &= e^{V_3}(x, t, u, v) = (t\varepsilon + x, t, -\frac{1}{6}\varepsilon + u, v), \\ g_4(x, t, u, v) &= e^{V_4}(x, t, u, v) = (x + \varepsilon, t, u, v). \end{aligned}$$

Hence, if $u = f(x, t), v = g(x, t)$ is a set of solutions to the integrable system (22), then we can get its new solutions:

$$\begin{aligned} g_1(\varepsilon) \begin{pmatrix} f(x, t) \\ g(x, t) \end{pmatrix} &= \begin{pmatrix} e^{\frac{2}{3}\varepsilon}f(e^{\frac{1}{3}\varepsilon}x, e^{\varepsilon}t) \\ e^{\frac{1}{2}\varepsilon}g(e^{\frac{1}{3}\varepsilon}x, e^{\varepsilon}t) \end{pmatrix}, \\ g_2(\varepsilon) \begin{pmatrix} f(x, t) \\ g(x, t) \end{pmatrix} &= \begin{pmatrix} f(x, t - \varepsilon) \\ g(x, t - \varepsilon) \end{pmatrix}, \\ g_3(\varepsilon) \begin{pmatrix} f(x, t) \\ g(x, t) \end{pmatrix} &= \begin{pmatrix} \frac{1}{6}\varepsilon + f(x - t\varepsilon, t) \\ g(x - t\varepsilon, t) \end{pmatrix}, \\ g_4(\varepsilon) \begin{pmatrix} f(x, t) \\ g(x, t) \end{pmatrix} &= \begin{pmatrix} f(x - \varepsilon, t) \\ g(x - \varepsilon, t) \end{pmatrix}. \end{aligned}$$

In the following, we seek invariant solutions for integrable system (22) with the aid of characteristic equations. For the vector field V_1 , the corresponding characteristic equation presents that

$$\frac{dx}{\frac{1}{3}x} = \frac{dt}{t} = \frac{du}{-\frac{2}{3}u} = \frac{dv}{-\frac{1}{2}v}, \quad (23)$$

which has the group-invariant solution

$$u = x^{-2}f(\xi), \quad v = x^{-\frac{3}{2}}g(\xi), \quad \xi = \frac{x^3}{t}.$$

Inserting the above solution into (22) gives rise to

$$\begin{aligned} -\xi^2 f'(\xi) - 24f(\xi) + 24\xi f'(\xi) + 27\xi^3 f^{(3)}(\xi) + 12f^2(\xi) - 18\xi f(\xi)f'(\xi) + 54g^2(\xi) - 72\xi g(\xi)g'(\xi) + 108\xi^2 g(\xi)g''(\xi) &= 0, \\ -\xi^2 g'(\xi) - \frac{105}{2}g(\xi) + 39\xi g'(\xi) + 126\xi^2 g''(\xi) + 108\xi^3 g'''(\xi) + 15f(\xi)g(\xi) - 9\xi g(\xi)f'(\xi) - 18\xi f(\xi)g'(\xi) &= 0. \end{aligned} \quad (24)$$

For the aim to construct power series solution, we assume that $f(\xi)$ and $g(\xi)$ have the following form

$$f(\xi) = a_0 + a_1\xi + a_2\xi^2 + \dots, \quad g(\xi) = b_0 + b_1\xi + b_2\xi^2 + \dots \quad (25)$$

Substituting (25) into (24) and equating the coefficients of ξ^n ($n \geq 0$) to zero yields

$$\begin{aligned} n = 0: \quad -4a_0 + 2a_0^2 + 9b_0^2 &= 0, \quad -\frac{105}{2}b_0 + 15a_0b_0 = 0. \\ n = 1: \quad a_0a_1 + 6b_0b_1 &= 0, \quad -\frac{27}{2}b_1 - 3a_0b_1 - 6a_1b_0 = 0. \\ n = 2: \quad -a_1 + 24a_2 - 12a_0a_2 - 6a_1^2 + 18b_0b_2 - 18b_1^2 &= 0, \quad -b_1 + \frac{555}{2}b_2 - 21a_0b_2 - 12a_1b_1 - 3a_2b_0 = 0. \\ n \geq 3: \quad -(n-1)a_{n-1} - 24a_n + 24na_n + 27n(n-1)(n-2)a_n + 12 \sum_{k=0}^n a_k a_{n-k} - 18 \sum_{k=0}^n (n-k)a_k a_{n-k} + \sum_{k=0}^n b_k b_{n-k} \\ &- 72 \sum_{k=0}^n (n-k)b_k b_{n-k} + 108 \sum_{k=0}^n (n-k)(n-k-1)b_k b_{n-k} = 0. \\ &-(n-1)b_{n-1} - \frac{105}{2}b_n + 39nb_n + 126n(n-1)b_n + 108n(n-1)(n-2)b_n + 15 \sum_{k=0}^n a_k b_{n-k} - 9 \sum_{k=0}^n (n-k)b_k a_{n-k} \\ &- 18 \sum_{k=0}^n a_k b_{n-k} = 0. \end{aligned}$$

Direct calculation results in the following two situations:

Case I: $b_0 = 0, a_0 = 2, a_1 = 0, b_1 = 0, a_2 = 1, b_2 = 0, \dots$

Case II: $b_0 = \pm\sqrt{\frac{21}{2}}i, a_0 = \frac{7}{2}, a_1 = 0, b_1 = 0, a_2 = \mp\sqrt{\frac{21}{2}} \cdot \frac{21}{36}i, b_2 = -\frac{21}{136}, \dots$

In the same way, a_i and b_j ($i > 2, j > 2$) can be further calculated according to the above recurrence relation. We omit it here due to computational complexity. Similarly, we can also introduce the following group-invariant solution in terms of the characteristic Equation (23):

$$u = t^{-\frac{2}{3}}f(\xi), \quad v = t^{-\frac{1}{2}}g(\xi), \quad \xi = \frac{x^3}{t}.$$

More abundant solutions can be worked out by restricting $f(\xi)$ and $g(\xi)$ satisfy different target equations, which will be investigated in the future work.

For V_3 , the characteristic equation can be expressed as

$$\frac{dx}{t} = \frac{dt}{0} = \frac{du}{-\frac{1}{6}} = \frac{dv}{0},$$

which leads to the following group-invariant solution

$$u = f(\xi) - \frac{1}{6}\frac{x}{t}, \quad v = g(\xi), \quad \xi = t. \quad (26)$$

By inserting (26) into (22), one infers that

$$u = -\frac{1}{6} \frac{x}{t} + \frac{c}{t}, \quad v = ct^{-\frac{1}{2}},$$

where c is an arbitrary integral constant.

The second super loop algebra is that

$$\tilde{B}(0, 1) = \text{span}\{E_j(i, m), j = 0, 1, 2, 3, 4; i = 0, 1\}, \tag{27}$$

where

$$E_j(i, m) = E_j \lambda^{2m+i}, \quad \text{deg} E_j(i, m) = 2m + i, \quad m \in \mathbb{Z}, \quad i = 0, 1; \quad j = 0, 1, 2, 3, 4.$$

The corresponding super operation relations are presented in Appendix A.

In what follows, we shall adopt the super loop algebra to deduce another isospectral-nonisospectral super integrable hierarchy.

Consider the following spectral problem

$$\begin{cases} \psi_x = U\psi, \\ U = E_0(1, 0) + u_1 E_1(0, 0) + u_2 E_2(0, 0) + \alpha_1 E_3(0, 0) + \beta_1 E_4(0, 0) + u_3 E_1(1, -1) \\ \quad + u_4 E_2(1, -1) + \alpha_2 E_3(1, -1) + \beta_2 E_4(1, -1) + u_5 E_0(1, -1), \end{cases} \tag{28}$$

where E_0, E_1, E_2 are even, E_3, E_4 are odd, u_1, u_2, u_3, u_4, u_5 are even superfunctions and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are the odd ones. Suppose

$$\psi_t = V_1 \psi + V_2 \psi =: V \psi, \tag{29}$$

where

$$V_1 = \sum_{\mu \geq 0} \left(\sum_{j=0}^4 \sum_{i=0}^1 V_j(i, \mu) E_j(i, -\mu) \right),$$

$$V_2 = \sum_{\nu \geq 0} \left(\sum_{j=0}^4 \sum_{i=0}^1 \tilde{V}_j(i, \nu) E_j(i, -\nu) \right), \quad \lambda_t = \frac{\partial \lambda}{\partial t} = \sum_{\nu \geq 0} k_\nu(t) \lambda^{-2\nu+1}.$$

The stationary zero curvature Equation (7) produces the corresponding recursion relations which are shown in Appendix B. Analogously, a solution for the nonisospectral zero curvature Equation (11) is presented in Appendix C.

Take the initial values $V_0(0, 0) = \sigma(\text{even}), V_1(1, 0) = V_0(1, 0) = V_2(1, 0) = V_2(0, 0) = V_3(1, 0) = V_3(0, 0) = V_4(1, 0) = V_4(0, 0) = V_1(0, 0) = 0, \tilde{V}_0(0, 0) = \sigma(\text{even}), \tilde{V}_1(1, 0) = \tilde{V}_0(1, 0) = \tilde{V}_2(1, 0) = \tilde{V}_2(0, 0) = \tilde{V}_3(1, 0) = \tilde{V}_3(0, 0) = \tilde{V}_4(1, 0) = \tilde{V}_4(0, 0) = \tilde{V}_1(0, 0) = 0,$ and then one has

$$\begin{aligned} V_1(1, 1) &= \sigma u_1, V_2(1, 1) = \sigma u_2, V_3(1, 1) = \sigma \alpha_1, V_4(1, 1) = \sigma \beta_1, V_0(1, 1) = 0, V_1(0, 1) = \frac{\sigma}{2} u_{1,x} + \sigma u_3, \\ V_2(0, 1) &= -\frac{\sigma}{2} u_{2,x} + \sigma u_4, V_3(0, 1) = \sigma \alpha_{1,x} + \sigma \alpha_2, V_4(0, 1) = -\sigma \beta_{1,x} + \sigma \beta_2, V_0(0, 1) = -\frac{\sigma}{2} u_1 u_2 + \sigma \alpha_1 \beta_1, \\ \tilde{V}_1(1, 1) &= \sigma u_1, \tilde{V}_2(1, 1) = \sigma u_2, \tilde{V}_3(1, 1) = \sigma \alpha_1, \tilde{V}_4(1, 1) = \sigma \beta_1, \tilde{V}_0(1, 1) = k_1(t)x - k_0(t) \int u_5 dx, \\ \tilde{V}_1(0, 1) &= \frac{\sigma}{2} u_{1,x} + \sigma u_3 + k_1 x u_1 - k_0 u_1 \int u_5 dx + \frac{1}{2} u_3 k_0, \tilde{V}_2(0, 1) = -\frac{\sigma}{2} u_{2,x} + \sigma u_4 + k_1 x u_2 - k_0 u_2 \int u_5 dx - \frac{1}{2} k_0 u_4, \\ \tilde{V}_3(0, 1) &= \sigma \alpha_{1,x} + \sigma \alpha_2 + k_1 x \alpha_1 - k_0 \alpha_1 \int u_5 dx + k_0 \alpha_2, \tilde{V}_4(0, 1) = -\sigma \beta_{1,x} + \sigma \beta_2 + k_1 x \beta_1 - k_0 \beta_1 \int u_5 dx - k_0 \beta_2, \\ \tilde{V}_0(0, 1) &= -\frac{\sigma}{2} u_1 u_2 + \sigma \alpha_1 \beta_1 + \int \left(-\frac{k_0}{2} u_1 u_4 - \frac{k_0}{2} u_2 u_3 + k_0 \alpha_2 \beta_1 + k_0 \alpha_1 \beta_2 \right) dx, \dots \end{aligned} \tag{30}$$

Denote $V_{1,+}^{(n)} = \sum_{\mu=0}^n (\sum_{j=0}^4 \sum_{i=0}^1 V_j(i, \mu) E_j(i, n - \mu)) = \lambda^{2n} V_1 - V_{1,-}^{(n)}$ and let $V_1^{(n)} = V_{1,+}^{(n)}$, then the isospectral zero-curvature equation

$$U_t - V_{1,x}^{(n)} + [U, V_1^{(n)}] = 0 \tag{31}$$

leads to an isospectral super integrable hierarchy (39) presented in Ref. [18].

Note that

$$\begin{aligned} V_{2,+}^{(m)} &= \sum_{\nu=0}^m (\sum_{j=0}^4 \sum_{i=0}^1 \bar{V}_j(i, \nu) E_j(i, m - \nu)) = \lambda^{2m} V_2 - V_{2,-}^{(m)}, \\ \lambda_{t,+}^{(m)} &= \sum_{\nu=0}^m k_\nu(t) \lambda^{2m-2\nu+1} = \lambda^{2m} \lambda_t - \lambda_{t,-}^{(m)} = \lambda^{2m} \lambda_t - \sum_{\nu=m+1}^{\infty} k_\nu(t) \lambda^{2m-2\nu+1}, \end{aligned} \tag{32}$$

then the nonisospectral zero-curvature equation

$$V_{2,x} = [U, V_2] + \frac{\partial U}{\partial \lambda} \lambda_t \tag{33}$$

can be broken down into

$$-(V_{2,+}^{(m)})_x + [U, V_{2,+}^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{(m)} = (V_{2,-}^{(m)})_x - [U, V_{2,-}^{(m)}] - \frac{\partial U}{\partial \lambda} \lambda_{t,-}^{(m)}. \tag{34}$$

We find that the gradation of the left-hand side of (34) is more than -1 , while that of the right-hand side is less than 0 . Taking the terms with the gradations being 0 and -1 in both sides of (34) and let $V_2^{(m)} = V_{2,+}^{(m)}$, one has

$$\begin{aligned} -(V_{2,+}^{(m)})_x + [U, V_{2,+}^{(m)}] + \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{(m)} &= -2\bar{V}_1(1, m + 1)E_1(0, 0) + 2\bar{V}_2(1, m + 1)E_2(0, 0) - \bar{V}_3(1, m + 1)E_3(0, 0) \\ &+ \bar{V}_4(1, m + 1)E_4(0, 0) + [\bar{V}_{1,x}(1, m + 1) - 2\bar{V}_1(0, m + 1) + 2u_1\bar{V}_0(1, m + 1) - 2\alpha_1\bar{V}_3(1, m + 1)]E_1(1, -1) \\ &+ [\bar{V}_{2,x}(1, m + 1) + 2\bar{V}_2(0, m + 1) - 2u_2\bar{V}_0(1, m + 1) + 2\beta_1\bar{V}_4(1, m + 1)]E_2(1, -1) \\ &+ [\bar{V}_{3,x}(1, m + 1) - \bar{V}_3(0, m + 1) - u_1\bar{V}_4(1, m + 1) + \beta_1\bar{V}_1(1, m + 1) + \alpha_1\bar{V}_0(1, m + 1)]E_3(1, -1) \\ &+ [\bar{V}_{4,x}(1, m + 1) + \bar{V}_4(0, m + 1) - u_2\bar{V}_3(1, m + 1) - \beta_1\bar{V}_0(1, m + 1) + \alpha_1\bar{V}_2(1, m + 1)]E_4(1, -1) \\ &+ [\bar{V}_{0,x}(1, m + 1) - u_1\bar{V}_2(0, m + 1) + u_2\bar{V}_1(1, m + 1) + \alpha_1\bar{V}_4(1, m + 1) + \beta_1\bar{V}_3(1, m + 1)]E_0(1, -1) - k_{m+1}(t)E_0(1, -1). \end{aligned}$$

Hence, the nonisospectral zero curvature equation

$$U_t - V_{1,x}^{(n)} - V_{2,x}^{(m)} + [U, V_1^{(n)} + V_2^{(m)}] = 0 \tag{35}$$

yields another isospectral-nonisospectral super integrable hierarchy

$$\begin{aligned} u_{t,n,m} &= (u_1, u_2, u_3, u_4, u_5, \alpha_1, \beta_1, \alpha_2, \beta_2)_{t,n,m}^T \\ &= \begin{pmatrix} 2V_1(1, n + 1) \\ -2V_2(1, n + 1) \\ -V_{1,x}(1, n + 1) + 2V_1(0, n + 1) - 2u_1V_0(1, n + 1) + 2\alpha_1V_3(1, n + 1) \\ -V_{2,x}(1, n + 1) - 2V_2(0, n + 1) + 2u_2V_0(1, n + 1) - 2\beta_1V_4(1, n + 1) \\ -V_{0,x}(1, n + 1) + u_1V_2(1, n + 1) - u_2V_1(1, n + 1) - \alpha_1V_4(1, n + 1) - \beta_1V_3(1, n + 1) \\ V_3(1, n + 1) \\ -V_4(1, n + 1) \\ -V_{3,x}(1, n + 1) + V_3(0, n + 1) + u_1V_4(1, n + 1) - \beta_1V_1(1, n + 1) - \alpha_1V_0(1, n + 1) \\ -V_{4,x}(1, n + 1) - V_4(0, n + 1) + u_2V_3(1, n + 1) - \alpha_1V_2(1, n + 1) + \beta_1V_0(1, n + 1) \end{pmatrix} \end{aligned} \tag{36}$$

$$+ \begin{pmatrix} 2\bar{V}_1(1, m+1) \\ -2\bar{V}_2(1, m+1) \\ -\bar{V}_{1,x}(1, m+1) + 2\bar{V}_1(0, m+1) - 2u_1\bar{V}_0(1, m+1) + 2\alpha_1\bar{V}_3(1, m+1) \\ -\bar{V}_{2,x}(1, m+1) - 2\bar{V}_2(0, m+1) + 2u_2\bar{V}_0(1, m+1) - 2\beta_1\bar{V}_4(1, m+1) \\ -\bar{V}_{0,x}(1, m+1) + u_1\bar{V}_2(1, m+1) - u_2\bar{V}_1(1, m+1) - \alpha_1\bar{V}_4(1, m+1) - \beta_1\bar{V}_3(1, m+1) + k_{m+1}(t) \\ \bar{V}_3(1, m+1) \\ -\bar{V}_4(1, m+1) \\ -\bar{V}_{3,x}(1, m+1) + \bar{V}_3(0, m+1) + u_1\bar{V}_4(1, m+1) - \beta_1\bar{V}_1(1, m+1) - \alpha_1\bar{V}_0(1, m+1) \\ -\bar{V}_{4,x}(1, m+1) - \bar{V}_4(0, m+1) + u_2\bar{V}_3(1, m+1) - \alpha_1\bar{V}_2(1, m+1) + \beta_1\bar{V}_0(1, m+1) \end{pmatrix}.$$

By taking specific value for n and m , the more abundant isospectral-nonispectral super integrable hierarchy will be deduced.

Remarks: In the subsection, we considered two cases where the degrees of the loop algebra are $m+i$ and $2m+i$ ($i=0,1; m \in \mathcal{Z}$). It followed the Tu scheme that two types of super-integrable hierarchies were generated, respectively. Of course, we could consider the degree of the loop algebra reads $Nm+i$, $m \in \mathcal{Z}$, N is positive nature number. By following the approach presented in the paper, we could obtain other rich super-integrable hierarchies which possess some physical senses.

3. Comparison and Discussion

We have known that the Tu scheme is regarded as one of the most effective methods to generate isospectral integrable hierarchies of evolution equations. However, no researchers are devoted to discussing how to apply it to generate nonispectral integrable hierarchies, along with the time-evolution of the spectral parameter as the polynomial form in the spectral parameter λ . In Ref. [14], we proposed a method for generating isospectral and nonispectral integrable hierarchies by improving the Tu scheme. Inspired by the work, in the paper, we further extend this method to obtain isospectral-nonispectral super-integrable hierarchies. It is obvious to note that our results cover the results in Ref. [18] because the super-integrable hierarchies presented in the paper can be reduced to that in Ref. [18] when $\lambda_i = 0$. In a word, this paper presents the method for producing super-integrable equations by making use of the Tu scheme for the first time. By applying the method, we derived the nonispectral super-integrable bond pricing equation and the generalized Fokker–Plank equation, which have important applications in the finance field.

4. Conclusions

In this article, two nonispectral problems with different numbers of even and odd superfunctions are established, respectively, by constructing two kinds of loop algebras of the super Lie algebra $B(0,1)$. It follows that two isospectral-nonispectral super integrable hierarchies are derived and several important equations are produced through further reduction, including the generalized Fokker–Plank equation and special bond pricing equation, as well as the an explicit super integrable system. In particular, we investigate the Lie point symmetries and one-parameter Lie symmetry groups, as well as group-invariant solutions for the integrable coupled Equation (22), which is reduced from the super integrable system (21). There are other properties for super integrable equation are worth further exploring, such as the Darboux transformation, Bäcklund transformation, and some others. Besides, we only formally wrote out the first terms of f and g in (2.23). Concerning the convergence of the series, we did not further consider it.

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Appendix A

$$\begin{aligned}
 s[E_0(i, m), E_1(j, n)] &= \begin{cases} 2E_1(i + j, m + n), i + j < 2, \\ 2E_1(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_0(i, m), E_2(j, n)] &= \begin{cases} -2E_2(i + j, m + n), i + j < 2, \\ -2E_2(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_1(i, m), E_2(j, n)] &= \begin{cases} E_0(i + j, m + n), i + j < 2, \\ E_0(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_0(i, m), E_3(j, n)] &= \begin{cases} E_3(i + j, m + n), i + j < 2, \\ E_3(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_0(i, m), E_4(j, n)] &= \begin{cases} -E_4(i + j, m + n), i + j < 2, \\ -E_4(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_1(i, m), E_3(j, n)] &= s[E_2(i, m), E_4(j, n)] = 0, \\
 s[E_1(i, m), E_4(j, n)] &= \begin{cases} E_3(i + j, m + n), i + j < 2, \\ E_3(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_2(i, m), E_3(j, n)] &= \begin{cases} E_4(i + j, m + n), i + j < 2, \\ E_4(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_3(i, m), E_4(j, n)] &= \begin{cases} -E_0(i + j, m + n), i + j < 2, \\ -E_0(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_0(i, m), E_3(j, n)] &= \begin{cases} 2E_1(i + j, m + n), i + j < 2, \\ 2E_1(i + j - 2, m + n + 1), i + j \geq 2, \end{cases} \\
 s[E_4(i, m), E_4(j, n)] &= \begin{cases} -2E_2(i + j, m + n), i + j < 2, \\ -2E_2(i + j - 2, m + n + 1), i + j \geq 2. \end{cases}
 \end{aligned}$$

Appendix B

$$\begin{aligned}
 V_{0,x}(0, \mu) &= u_1 V_2(0, \mu) - u_2 V_1(0, \mu) + \beta_1 V_3(0, \mu) + \beta_2 V_3(1, \mu) + u_3 V_2(1, \mu) - u_4 V_1(1, \mu) - \alpha_1 V_4(0, \mu) - \alpha_2 V_4(1, \mu), \\
 V_{0,x}(1, \mu + 1) &= u_1 V_2(1, \mu + 1) - u_2 V_1(1, \mu + 1) + \beta_1 V_3(1, \mu + 1) - \alpha_1 V_4(1, \mu + 1) - \alpha_2 V_4(0, \mu) + \beta_2 V_3(0, \mu) \\
 &\quad + u_3 V_2(0, \mu) - u_4 V_1(0, \mu), \\
 V_{1,x}(0, \mu) &= 2V_1(1, \mu + 1) - 2u_1 V_0(0, \mu) + 2\alpha_1 V_3(0, \mu) - 2u_3 V_0(1, \mu) + 2\alpha_2 V_3(1, \mu) + 2u_5 V_1(1, \mu), \\
 V_{1,x}(1, \mu + 1) &= 2V_1(1, \mu + 1) - 2u_1 V_0(1, \mu + 1) + 2\alpha_1 V_3(1, \mu + 1) - 2u_3 V_0(0, \mu) + 2\alpha_2 V_3(0, \mu) + 2u_5 V_1(1, \mu), \\
 V_{2,x}(0, \mu) &= -2V_2(1, \mu + 1) + 2u_2 V_0(0, \mu) - 2\beta_1 V_4(0, \mu) + 2u_4 V_0(1, \mu) - 2\beta_2 V_4(1, \mu) - 2u_5 V_2(1, \mu), \\
 V_{2,x}(1, \mu + 1) &= -2V_2(0, \mu + 1) + 2u_2 V_0(1, \mu + 1) - 2\beta_1 V_4(1, \mu + 1) - 2\beta_2 V_4(0, \mu) - 2u_5 V_2(0, \mu) + 2u_4 V_0(0, \mu), \\
 V_{3,x}(0, \mu) &= V_3(1, \mu + 1) + u_1 V_4(0, \mu) - \alpha_1 V_0(0, \mu) - \beta_1 V_1(0, \mu) + u_3 V_4(1, \mu) - \alpha_2 V_0(1, \mu) - \beta_2 V_1(1, \mu) + u_5 V_3(1, \mu), \\
 V_{3,x}(1, \mu + 1) &= V_3(0, \mu + 1) + u_1 V_4(1, \mu + 1) - \alpha_1 V_0(1, \mu + 1) - \beta_1 V_1(1, \mu + 1) + u_3 V_4(0, \mu) - \alpha_2 V_0(0, \mu) \\
 &\quad - \beta_2 V_1(0, \mu) + u_5 V_3(0, \mu), \\
 V_{4,x}(0, \mu) &= -V_4(1, \mu + 1) + u_2 V_3(0, \mu) - \alpha_1 V_2(0, \mu) + \beta_1 V_0(0, \mu) + u_4 V_3(1, \mu) - \alpha_2 V_2(1, \mu) \\
 &\quad + \beta_2 V_0(1, \mu) - u_5 V_4(1, \mu), \\
 V_{4,x}(1, \mu + 1) &= -V_4(0, \mu + 1) + u_2 V_3(1, \mu + 1) - \alpha_1 V_2(1, \mu + 1) + u_4 V_3(0, \mu) - u_5 V_4(0, \mu) + \beta_2 V_0(0, \mu) \\
 &\quad - \alpha_2 V_2(0, \mu) + \beta_1 V_0(1, \mu + 1).
 \end{aligned}$$

Appendix C

$$\begin{aligned}
\bar{V}_{0,x}(0, \nu) &= u_1 \bar{V}_2(0, \nu) - u_2 \bar{V}_1(0, \nu) + \beta_1 \bar{V}_3(0, \nu) + \beta_2 \bar{V}_3(1, \nu) + u_3 \bar{V}_2(1, \nu) - u_4 \bar{V}_1(1, \nu) - \alpha_1 \bar{V}_4(0, \nu) - \alpha_2 \bar{V}_4(1, \nu), \\
\bar{V}_{0,x}(1, \nu + 1) &= u_1 \bar{V}_2(1, \nu + 1) - u_2 \bar{V}_1(1, \nu + 1) + \beta_1 \bar{V}_3(1, \nu + 1) - \alpha_1 \bar{V}_4(1, \nu + 1) - \alpha_2 \bar{V}_4(0, \nu) + \beta_2 \bar{V}_3(0, \nu) \\
&\quad + u_3 \bar{V}_2(0, \nu) - u_4 \bar{V}_1(0, \nu) + k_{\nu+1}(t) - u_5 k_\nu(t), \\
\bar{V}_{1,x}(0, \nu) &= 2\bar{V}_1(1, \nu + 1) - 2u_1 \bar{V}_0(0, \nu) + 2\alpha_1 \bar{V}_3(0, \nu) - 2u_3 \bar{V}_0(1, \nu) + 2\alpha_2 \bar{V}_3(1, \nu) + 2u_5 \bar{V}_1(1, \nu), \\
\bar{V}_{1,x}(1, \nu + 1) &= 2\bar{V}_1(1, \nu + 1) - 2u_1 \bar{V}_0(1, \nu + 1) + 2\alpha_1 \bar{V}_3(1, \nu + 1) - 2u_3 \bar{V}_0(0, \nu) + 2\alpha_2 \bar{V}_3(0, \nu) + 2u_5 \bar{V}_1(0, \nu) - u_3 k_\nu(t), \\
\bar{V}_{2,x}(0, \nu) &= -2\bar{V}_2(1, \nu + 1) + 2u_2 \bar{V}_0(0, \nu) - 2\beta_1 \bar{V}_4(0, \nu) + 2u_4 \bar{V}_0(1, \nu) - 2\beta_2 \bar{V}_4(1, \nu) - 2u_5 \bar{V}_2(1, \nu), \\
\bar{V}_{2,x}(1, \nu + 1) &= -2\bar{V}_2(0, \nu + 1) + 2u_2 \bar{V}_0(1, \nu + 1) - 2\beta_1 \bar{V}_4(1, \nu + 1) - 2\beta_2 \bar{V}_4(0, \nu) - 2u_5 \bar{V}_2(0, \nu) + 2u_4 \bar{V}_0(0, \nu) - u_4 k_\nu(t), \\
\bar{V}_{3,x}(0, \nu) &= \bar{V}_3(1, \nu + 1) + u_1 \bar{V}_4(0, \nu) - \alpha_1 \bar{V}_0(0, \nu) - \beta_1 \bar{V}_1(0, \nu) + u_3 \bar{V}_4(1, \nu) - \alpha_2 \bar{V}_0(1, \nu) - \beta_2 \bar{V}_1(1, \nu) + u_5 \bar{V}_3(1, \nu), \\
\bar{V}_{3,x}(1, \nu + 1) &= \bar{V}_3(0, \nu + 1) + u_1 \bar{V}_4(1, \nu + 1) - \alpha_1 \bar{V}_0(1, \nu + 1) - \beta_1 \bar{V}_1(1, \nu + 1) + u_3 \bar{V}_4(0, \nu) - \alpha_2 \bar{V}_0(0, \nu) \\
&\quad - \beta_2 \bar{V}_1(0, \mu) + u_5 \bar{V}_3(0, \mu) - \alpha_2 k_\nu(t), \\
\bar{V}_{4,x}(0, \nu) &= -\bar{V}_4(1, \nu + 1) + u_2 \bar{V}_3(0, \nu) - \alpha_1 \bar{V}_2(0, \nu) + \beta_1 \bar{V}_0(0, \nu) + u_4 \bar{V}_3(1, \nu) - \alpha_2 \bar{V}_2(1, \nu) + \beta_2 \bar{V}_0(1, \mu) - u_5 \bar{V}_4(1, \mu), \\
\bar{V}_{4,x}(1, \nu + 1) &= -\bar{V}_4(0, \nu + 1) + u_2 \bar{V}_3(1, \nu + 1) - \alpha_1 \bar{V}_2(1, \nu + 1) + u_4 \bar{V}_3(0, \nu) - u_5 \bar{V}_4(0, \nu) + \beta_2 \bar{V}_0(0, \nu) - \alpha_2 \bar{V}_2(0, \nu) \\
&\quad + \beta_1 \bar{V}_0(1, \nu + 1) - \beta_2 k_\nu(t).
\end{aligned}$$

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