



# Article Stone Duality for Kolmogorov Locally Small Spaces

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Abstract: In this paper, we prove new versions of Stone Duality. The main version is the following: the category of Kolmogorov locally small spaces and bounded continuous mappings is equivalent to the category of spectral spaces with decent lumps and with bornologies in the lattices of (quasi-) compact open sets as objects and spectral mappings respecting those decent lumps and satisfying a boundedness condition as morphisms. Furthermore, it is dually equivalent to the category of bounded lattices with bornologies and with decent lumps of prime filters as objects and homomorphisms of bounded lattices respecting those decent lumps and satisfying a domination condition as morphisms. This helps to understand Kolmogorov locally small spaces and morphisms between them. We comment also on spectralifications of topological spaces.

**Keywords:** Stone Duality; spectral space; distributive lattice; locally small space; equivalence of categories; spectralification

#### 1. Introduction

Stone Duality is one of the most important dualities in mathematics, and equivalences or dualities between categories are a form of symmetry on the category theory level. Stone Duality is very widely known for Boolean algebras, and a little less known for bounded distributive lattices. In fact, M. H. Stone's two fundamental papers [1,2] described duality (at least on the object level) between generalized Boolean algebras (or Boolean rings) and Hausdorff locally compact Boolean spaces, where usual Boolean algebras (or unital Boolean rings) correspond to Hausdorff compact Boolean spaces. He achieved a beautiful theory of ideals in Boolean rings and a beautiful theory of representations of Boolean rings in powersets. The case of distributive lattices was considered by M. H. Stone in [3]. Many versions of this duality exist (see, for example, [4] or [5] for further literature), including versions of Priestley Duality proved by H. Priestley in [6] with many consequences developed in [7]. Stone Duality for bounded distributive lattices in the category theory language, while considered already in a much broader context in monographs by G. Grätzer [8] and P. T. Johnstone [9], has been presented in detail in a recent monograph by M. Dickmann, N. Schwartz and M. Tressl [10] on spectral spaces.

Algebraic and analytic geometry and model theory use Stone Duality for bounded distributive lattices. In real algebraic and analytic geometry the spectral topology (also called the Harrison topology) on the real spectrum is most important (see [11–13]). In complex algebraic and analytic geometry, the spectral topology on the Zariski spectrum is similarly important (see [14] or [15], Chapter II). On the other hand, model theory uses the constructible topology (also called the patch topology) more often ([16,17]), sometimes allowing retopologization to the spectral topology (as in the case of the o-minimal spectrum, see [18]).

The purpose of this study is to extend the method of taking the real spectrum or its analogues to the case of infinite gluings of the small spaces considered in real algebraic or analytic geometry or in model theory (where small spaces are quite often unnamed, see [12], Definition 7.1.14 or [11], p. 12) and to make another step in building general topology for locally small spaces, which can be considered as topological spaces with



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**Copyright:** © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). additional structure. New versions of Stone Duality are proved: for small spaces, for locally small spaces with usual morphisms (bounded continuous mappings) and for locally small spaces with bounded strongly continuous mappings as morphisms. In each of the cases, the Kolmogorov separation axiom ( $T_0$ ) is assumed.

Locally small spaces may be understood to be a special kind ([19]) of generalized topological spaces in the sense of Delfs and Knebusch ([20]) introduced in 1985, which in turn may be seen as a special form of categories with Grothendieck topologies (see [20], p. 2, [21]) or sets with G-topologies of [22]. Locally small spaces were implicitly used in o-minimal homotopy theory ([20,23]), including a context of locally definable manifolds (see [23,24], for example). The possibility of gluing together infinitely many pieces is essential in these issues. A simpler language for locally small spaces was introduced and used in [19,25]; compare also [26]. We continue developing the theory of locally small spaces in this simple language, analogical to the language of Lugojan's generalized topology ([27]) or Császár's generalized topology ([28]), where a family of subsets of the underlying set is satisfying some, but not all, conditions for a topology.

The main result of the paper reads as follows: the category of Kolmogorov locally small spaces and bounded continuous mappings is equivalent to the category of spectral spaces with distinguished decent lumps and with bornologies in the lattices of (quasi) compact open sets as objects and spectral mappings respecting those decent lumps and satisfying a boundedness condition as morphisms and is dually equivalent to the category of bounded distributive lattices with bornologies and with decent lumps of prime filters as objects and homomorphisms of bounded lattices satisfying a domination condition and respecting those decent lumps as morphisms. Bornologies on sets were used in [19,21,25] and bornologies in bounded lattices are defined in this paper. As a consequence, spectralifications of a Kolmogorov topological space may be constructed by choosing lattice bases of the topology.

Small spaces are a special case of locally small spaces, with some compactness flavour. While we meet small spaces as these underlying definable spaces over structures with topologies, we meet locally small spaces as those underlying analogical locally definable spaces ([19,25]). We show that a Kolmogorov small space is essentially a patch dense subset of a spectral space. More precisely: the category of Kolmogorov small spaces and continuous mappings is equivalent to the category of spectral spaces with distinguished patch dense subsets and spectral mappings respecting those patch dense subsets and is dually equivalent to the category of bounded distributive lattices with distinguished patch dense sets of prime filters and homomorphisms of bounded lattices respecting those patch dense sets.

We have another version of Stone Duality: for Kolmogorov locally small spaces with bounded strongly continuous mappings. This category is equivalent to the category of up-spectral spaces with distinguished patch dense subsets as objects and strongly spectral mappings respecting those patch dense subsets as morphisms and is dually equivalent to the category of distributive lattices with zeros and distinguished patch dense sets of prime filters as objects and lattice homomorphisms respecting zeros and those patch dense sets and satisfying a condition of domination as morphisms.

These new versions of Stone Duality give more understanding of objects and morphisms of the categories we introduce. In particular, a Kolmogorov locally small space is essentially a patch dense subset of a spectral (or up-spectral) space. Bounded continuous mappings are restrictions of spectral mappings.

The paper is organized in the following way: Section 2 introduces categories  $SS_0$  and  $LSS_0$ , Section 3 deals with SpecD and SpecBD, Section 4 introduces LatD and LatBD. Section 5 gives the main theorem for  $LSS_0$  and a version for  $SS_0$ . Section 6 introduces the categories uSpec and  $uSpec^s$ . Section 7 deals with ZLat and establishes a dual equivalency between  $uSpec^s$  and ZLat. Section 8 provides Stone Duality for  $LSS_0^s$ . Section 9 deals with spectralifications of Kolmogorov spaces. Regarding the set-theoretic axiomatics for this paper, we follow Saunders Mac Lane's version of Zermelo–Fraenkel axioms with the axiom of choice plus the existence of a set which is a universe ([29], p. 23).

We shall freely use the notation for family intersection and family difference, compatible with [19,21,25,26]:

 $\mathcal{U} \cap_1 \mathcal{V} = \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \}, \quad \mathcal{U} \setminus_1 \mathcal{V} = \{ U \setminus V \mid U \in \mathcal{U}, V \in \mathcal{V} \}.$ 

## 2. The Categories SS<sub>0</sub> and LSS<sub>0</sub>

This section gives the basic concepts connected with small and locally small spaces in the simplified language introduced in [19] and distinguishes the Kolmogorov spaces.

**Definition 1** ([19], Definition 2.1). A locally small space is a pair  $(X, \mathcal{L}_X)$ , where X is any set and  $\mathcal{L}_X \subseteq \mathcal{P}(X)$  satisfies the following conditions:

(LS1)  $\emptyset \in \mathcal{L}_X$ ,

(LS2) if  $A, B \in \mathcal{L}_X$ , then  $A \cap B, A \cup B \in \mathcal{L}_X$ ,

(LS3)  $\forall x \in X \exists A_x \in \mathcal{L}_X x \in A_x \text{ (i.e., } \bigcup \mathcal{L}_X = X).$ 

*Elements of*  $\mathcal{L}_X$  *are called* small open *subsets (or* smops) *of X*.

**Definition 2** ([19], Definition 2.21). *A* small space *is such a locally small space*  $(X, \mathcal{L}_X)$  *that*  $X \in \mathcal{L}_X$ .

**Definition 3.** A locally small space  $(X, \mathcal{L}_X)$  will be called  $T_0$  (or Kolmogorov) if the family  $\mathcal{L}_X$  separates points ([10], Remainder 1.1.4), which means that for  $x, y \in X$  the following condition is satisfied:

*if*  $x \in A \iff y \in A$  *for each*  $A \in \mathcal{L}_X$ *, then* x = y*.* 

**Definition 4** ([19], Definition 2.9). If  $(X, \mathcal{L}_X)$  is a locally small space, then the topology  $\mathcal{L}_X^{wo} = \tau(\mathcal{L}_X)$ , generated by  $\mathcal{L}_X$  in  $\mathcal{P}(X)$ , is called the family of weakly open sets in  $(X, \mathcal{L}_X)$ .

**Fact 1.** For a small space  $(X, \mathcal{L}_X)$ , the following conditions are equivalent:

- (1)  $(X, \mathcal{L}_X)$  is  $T_0$ ,
- (2) the topological space  $(X, \mathcal{L}_X^{wo})$  is  $T_0$ .

**Example 1.** (1) The small spaces  $\mathbb{R}_{om} = (\mathbb{R}, \mathcal{L}_{om}), \mathbb{R}_{rom} = (\mathbb{R}, \mathcal{L}_{rom}), \mathbb{R}_{slom} = (\mathbb{R}, \mathcal{L}_{slom}), \mathbb{R}_{st} = (\mathbb{R}, \tau_{nat})$  from ([19], Example 2.14), compare ([26], Definition 1.2), have the natural topology  $\tau_{nat}$  on  $\mathbb{R}$  as the topology of weakly open sets, so they are Kolmogorov small spaces. In the above, we have:

- (*i*)  $\mathcal{L}_{om} =$ the family of all finite unions of open intervals,
- (ii)  $\mathcal{L}_{rom} = the family of all finite unions of open intervals with rational numbers or infinities as endpoints,$
- (iii)  $\mathcal{L}_{slom} =$  the family of all locally finite (in the traditional sense) unions of bounded open intervals.

(2) The space  $(\mathbb{R}, \mathcal{L}_{iom})$ , where  $\mathcal{L}_{iom}$  is the family of all finite unions of open intervals with integers or infinities as ends, is not Kolmogorov.

**Definition 5** ([19]). For a locally small space  $(X, \mathcal{L}_X)$ , we define the family of open sets as

$$\mathcal{L}_X^o = \{ M \subseteq X \mid M \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X \}.$$

**Remark 1.** The family  $\mathcal{L}_X^o$  is a bounded sublattice of  $\mathcal{P}(X)$  containing  $\mathcal{L}_X$ . The open sets are those subsets of X that are "compatible with" smops.

**Example 2.** Consider the following families of subsets of the set  $\mathbb{R}$  of real numbers:

(*i*)  $\mathcal{L}_{lom} = the family of all finite unions of bounded open intervals,$ 

- (ii)  $\mathcal{L}_{lom}^o = \mathcal{L}_{slom}$  = the family of all locally finite unions of bounded open intervals.
- (iii)  $\mathcal{L}_{lrom} =$ the family of all finite unions of bounded open intervals with rational endpoints,
- (iv)  $\mathcal{L}^o_{lrom} = the family of all locally finite unions of open intervals with rational endpoints.$

Then  $\mathbb{R}_{lom} = (\mathbb{R}, \mathcal{L}_{lom})$  and  $\mathbb{R}_{lrom} = (\mathbb{R}, \mathcal{L}_{lrom})$  are Kolmogorov locally small spaces (compare ([19], Example 2.14) and ([26], Definition 1.2)) that are not small. On the other hand,  $(\mathbb{R}, \mathcal{L}_{lom}^o)$  and  $(\mathbb{R}, \mathcal{L}_{lrom}^o)$  are small.

**Definition 6.** Assume  $(X, \mathcal{L}_X)$  and  $(Y, \mathcal{L}_Y)$  are locally small spaces. Then a mapping  $f : X \to Y$  is:

- (a) bounded ([19], Definition 2.40) if  $\mathcal{L}_X$  refines  $f^{-1}(\mathcal{L}_Y)$ , which means that each  $A \in \mathcal{L}_X$ admits  $B \in \mathcal{L}_Y$  such that  $A \subseteq f^{-1}(B)$ ,
- (b) continuous ([19], Definition 2.40) if  $f^{-1}(\mathcal{L}_Y) \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X$  (i.e.,  $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X^o$ ),
- (c) strongly continuous if  $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X$ .

**Definition 7.** We consider the following categories:

- (a) the category LSS of locally small spaces and their bounded continuous mappings ([19], Remark 2.46),
- (b) the full subcategory  $LSS_0$  in LSS of  $T_0$  locally small spaces,
- (c) the full subcategory **SS** in **LSS** of small spaces ([19], Remark 2.48),
- (d) the full subcategory  $SS_0$  in LSS of  $T_0$  small spaces.

# 3. The Categories SpecD and SpecBD

This section restates the classical Stone Duality and introduces two categories of spectral spaces with additional data needed in the main statements on the equivalency of categories.

**Definition 8.** For any topological space  $X = (X, \tau_X)$ , we consider the following families of subsets:

- (a) the family CO(X) of (not necessarily Hausdorff) compact open subsets of X,
- (b) the family ICO(X) of intersection compact open subsets of X. (An open subset Y of X is intersection compact open if for every compact open set V their intersection  $V \cap Y$  is compact, see [14,30].)

**Definition 9.** A spectral space is a topological space  $X = (X, \tau_X)$  satisfying the following conditions (compare ([10], Definition 1.1.5)):

- (S1)  $X \in CO(X)$ ,
- (S2) CO(X) is a basis of  $\tau_X$ ,
- (S3)  $CO(X) \cap_1 CO(X) \subseteq CO(X)$ ,
- (S4)  $(X, \tau_X)$  is  $T_0$ ,
- (S5)  $(X, \tau_X)$  is sober (this means for us: each non-empty irreducible closed set is the closure of a one-point set).

Hochster ([14]) proved that every spectral space is homeomorphic to the Zariski spectrum of some commutative unital ring.

**Definition 10.** A mapping  $g : X \to Y$  between spectral spaces is spectral if the preimage of any compact open subset of Y is a compact open subset of X, shortly:  $g^{-1}(CO(Y)) \subseteq CO(X)$ , see ([10], Definition 1.2.2). We have the category **Spec** of spectral spaces and spectral mappings.

**Remark 2** (The classical Stone Duality). *The category* **Lat** *of bounded distributive lattices with homomorphisms of bounded lattices is dually equivalent to the category* **Spec***. While* ([10], *Chapter 3) uses contravariant functors and homomorphisms into a two-element lattice, we restate Stone Duality using covariant functors and prime filters. Namely, we have:* 

1. The functor  $Sp : Lat^{op} \rightarrow Spec$  is given by:

- (a)  $Sp(L) = (\mathfrak{PF}(L), \tau(\widetilde{L}))$  for  $L = (L, \lor, \land, 0, 1)$  a bounded distributive lattice, where  $\mathfrak{PF}(L)$  is the set of all prime filters in L with topology  $\tau(\widetilde{L})$  on  $\mathfrak{PF}(L)$  generated by the family  $\widetilde{L}$ , where  $\widetilde{L} = \{\widetilde{a} \mid a \in L\} \subseteq \mathfrak{P}(\mathfrak{PF}(L))$  and  $\widetilde{a} = \{F \in \mathfrak{PF}(L) \mid a \in F\}$ ,
- (b)  $Sp(h^{op}) = h^{\bullet}$  for a homomorphism of bounded lattices  $h : L \to M$  where, for  $G \in \mathfrak{PF}(M)$ , we have

$$h^{\bullet}(G) = \{a \in L \mid h(a) \in G\} \in \mathfrak{PF}(L).$$

- 2. The functor  $Co: \mathbf{Spec} \to \mathbf{Lat}^{op}$  is given by:
  - (a) Co(X) = CO(X) with obvious lattice operations on CO(X),
  - (b)  $Co(g) = (\mathcal{L}g)^{op}$ , where  $\mathcal{L}g : CO(Y) \to CO(X)$  is defined by  $(\mathcal{L}g)(W) = g^{-1}(W)$  for a spectral  $g : X \to Y$  and  $W \in CO(Y)$ .

Then the compositions SpCo, CoSp are naturally isomorphic to the identity functors  $Id_{Spec}$ ,  $Id_{Lat^{op}}$ , respectively. Consequences of the classical Stone Duality ([10], 3.2.5) include:

- (*i*) the fact that each bounded distributive lattice  $L = (L, \lor, \land, 0, 1)$  is isomorphic to the lattice  $(\widetilde{L}, \cup, \cap, \emptyset, \mathfrak{PF}(L))$  of subsets of  $\mathfrak{PF}(L)$  and
- (ii) the equality  $\tilde{L} = CO(\mathfrak{PF}(L))$ .

**Definition 11.** An object of **SpecD** is a pair  $((X, \tau_X), X_d)$  where  $(X, \tau_X)$  is a spectral space and  $X_d$  is a subset of X satisfying:

$$\forall U, V \in CO(X) \quad U \neq V \implies U \cap X_d \neq V \cap X_d.$$

Then  $X_d$  is called a decent subset of X.

A morphism of **SpecD** between  $((X, \tau_X), X_d)$  and  $((Y, \tau_Y), Y_d)$  is a spectral mapping  $g : X \to Y$  between spectral spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$  that respects the decent subset, that is,  $g(X_d) \subseteq Y_d$ .

**Fact 2.** If  $X_d$  is a decent subset of a spectral space  $(X, \tau_X)$ , then the lattice  $(CO(X), \cup, \cap, \emptyset, X)$  is isomorphic to the lattice  $(CO(X)_d, \cup, \cap, \emptyset, X_d)$ , where

$$CO(X)_d = CO(X) \cap_1 X_d = \{U \cap X_d \mid U \in CO(X)\}.$$

**Remark 3.** If  $((X, \tau_X), X_d)$  is an object of **SpecD**, then, by ([10], 3.2.8), both the spaces  $\mathcal{PF}(CO(X))$  and  $\mathcal{PF}(CO(X)_d)$  with their spectral topologies are homeomorphic to  $(X, \tau_X)$ . A point  $x \in X$  corresponds to

$$\hat{x} = \{ V \in CO(X) \mid x \in V \} \text{ in } PF(CO(X)) \text{ and to}$$

$$\hat{x}^d = \{ U \in CO(X)_d \mid x \in U \} \text{ in } PF(CO(X)_d), \text{ respectively} \}$$

**Definition 12** ([10], Proposition 1.3.13). *Let*  $(X, \tau_X)$  *be a spectral space. Then the* patch topology *(or the* constructible topology) *on* X *is the topology that has the family*  $CO(X) \setminus_1 CO(X)$  *as a basis.* 

**Proposition 1.** For a spectral space  $(X, \tau_X)$  and  $X_d \subseteq X$ , the following conditions are equivalent:

- (1)  $X_d$  is patch dense,
- (2)  $X_d$  is decent.

**Proof.** If the set  $X_d$  is decent in  $(X, \tau_X)$  and U is a non-empty patch open set in  $(X, \tau_X)$ , then we may assume  $U = A \setminus B$  with  $A, B \in CO(X)$ . Since  $U = A \triangle (A \cap B)$  is non-empty, A and  $A \cap B$  are different in CO(X), so  $A \cap X_d$  and  $A \cap B \cap X_d$  are different in  $CO(X)_d$ . This means  $(A \setminus B) \cap X_d$  is non-empty. Hence  $X_d$  is patch dense.

If  $X_d$  is patch dense in  $(X, \tau_X)$  and A, B are different members of CO(X), then  $A \triangle B$  is a non-empty patch open set. Hence  $X_d$  intersects  $A \triangle B$  and  $A \cap X_d$  is different from  $B \cap X_d$ in  $CO(X)_d$ . This means  $X_d$  is decent in  $(X, \tau_X)$ .  $\Box$ 

**Example 3.** The real spectrum of  $\mathbb{R}[X]$ , often denoted by  $\mathbb{R}$  (see 7.1.4 b and 7.2.6 in [12]), can be up to a homeomorphism described in the following way: it contains points  $r^-$ , r,  $r^+$  for each real number r, the infinities  $-\infty$ ,  $+\infty$  and admits the obvious linear order. As a basis of the topology on  $\mathbb{R}$ , we take the family  $\mathbb{B}$  containing: finite intervals  $[r^+, s^-] = (r, s)$  for  $r, s \in \mathbb{R}$ , r < s and infinite intervals  $[-\infty, s^-] = (-\infty, s)$ ,  $[r^+, +\infty] = (r, +\infty)$  for any  $r, s \in \mathbb{R}$ .

Then  $CO(\mathbb{R})$  is the family of finite unions of basic sets and the topological space  $(\mathbb{R}, \tau(\mathcal{B}))$  is spectral. The set  $\mathbb{R}$  of real numbers is decent in this spectral space, so  $((\mathbb{R}, \tau(\mathcal{B})), \mathbb{R})$  is an object of **SpecD**. (The operation  $\tilde{\cdot}$  mentioned in this example is an isomorphism between the Boolean algebra of semialgebraic sets in  $\mathbb{R}$  and the Boolean algebra of constructible sets in  $\mathbb{R}$ , see ([12], Proposition 7.2.3)).

Any semialgebraic mapping  $g : \mathbb{R} \to \mathbb{R}$  (i.e., g has a semialgebraic graph) extends (uniquely) to a maping  $\tilde{g} : \mathbb{R} \to \mathbb{R}$  satisfying the condition  $\tilde{g}^{-1}(\tilde{T}) = g^{-1}(T)$  for any semialgebraic  $T \subseteq \mathbb{R}$ , as in ([12], Proposition 7.2.8), which means that  $\tilde{g} : ((\mathbb{R}, \tau(\mathcal{B})), \mathbb{R}) \to ((\mathbb{R}, \tau(\mathcal{B})), \mathbb{R})$  is a morphism of **SpecD**.

**Definition 13.** A bornology in a bounded lattice  $(L, \lor, \land, 0, 1)$  is an ideal  $B \subseteq L$  such that

$$\bigvee B = 1.$$

**Definition 14.** An object of **SpecBD** is a system  $((X, \tau_X), CO_s(X), X_d)$  where  $(X, \tau_X)$  is a spectral space,  $CO_s(X)$  is a bornology in the bounded lattice CO(X) and  $X_d$  satisfies the following conditions:

- (1)  $X_d \subseteq \bigcup CO_s(X)$ ,
- (2)  $R_d : CO(X) \ni A \mapsto A \cap X_d \in CO(X)_d$  is an isomorphism of lattices,
- (3)  $CO(X)_d = (CO_s(X)_d)^o \subseteq \mathcal{P}(X_d).$

Such  $X_d$  will be called a decent lump of X.

A morphism from  $((X, \tau_X), CO_s(X), X_d)$  to  $((Y, \tau_Y), CO_s(Y), Y_d)$  in **SpecBD** is such a spectral mapping between spectral spaces  $g : (X, \tau_X) \to (Y, \tau_Y)$  that:

(a) satisfies the condition of boundedness

$$\forall A \in CO_s(X) \exists B \in CO_s(Y) \quad g(A) \subseteq B,$$

(b) respects the decent lump:  $g(X_d) \subseteq Y_d$ .

**Example 4.** Each of the spectral spaces  $\mathfrak{PF}(\mathcal{L}_{lom}^{o})$ ,  $\mathfrak{PF}(\mathcal{L}_{lrom}^{o})$  decomposes into two parts: prime filters may or may not intersect  $\mathcal{L}_{lom}$ ,  $\mathcal{L}_{lrom}$ , respectively. Those elements of  $\mathfrak{PF}(\mathcal{L}_{lom}^{o})$  that intersect  $\mathcal{L}_{lom}$  correspond bijectively to the elements of  $\mathfrak{PF}(\mathcal{L}_{lom})$ . The latter set may be topologically identified with an open set in  $\mathfrak{PF}(\mathcal{L}_{lom}^{o})$  or an open set  $\bigcup_{r,s\in\mathbb{R}}[r^+,s^-] = \mathbb{R} \setminus \{-\infty,+\infty\}$  in  $\mathbb{R}$ , using the notation of Example 3. On the other hand,  $\mathfrak{PF}(\mathcal{L}_{lom}^{o})$  has uncountably many prime filters that do not intersect  $\mathcal{L}_{lom}$  (some of them may be constructed using ultrafilters on the set of natural numbers). Similar facts hold true for  $\mathfrak{PF}(\mathcal{L}_{lrom}^{o})$ .

#### 4. The Categories LatD and LatBD

This section introduces two categories of bounded distributive lattices with additional data appearing in the main statements on the equivalency of categories.

**Definition 15.** Objects of LatD are pairs  $(L, \mathbf{D}_L)$  with  $L = (L, \lor, \land, 0, 1)$  a bounded distributive *lattice and*  $\mathbf{D}_L \subseteq \mathfrak{PF}(L)$  *satisfying* 

$$\forall a, b \in L \quad a \neq b \implies \tilde{a}^d \neq \tilde{b}^d \ (where \ \tilde{a}^d = \{F \in \mathbf{D}_L \mid a \in F\} = \tilde{a} \cap \mathbf{D}_L).$$

*Then*  $\mathbf{D}_L$  *is called a* decent set of prime filters *on L*.

Morphisms of LatD are such homomorphisms of bounded lattices  $h : L \to M$  that  $h^{\bullet}(\mathbf{D}_M) \subseteq \mathbf{D}_L$ .

**Fact 3.** If  $\mathbf{D}_L$  is a decent set of prime filters of  $(L, \lor, \land, 0, 1)$ , then the bounded lattice  $(\tilde{L}^d, \cup, \cap, \emptyset, \mathbf{D}_L)$ , where  $\tilde{L}^d = \{\tilde{a}^d \mid a \in L\}$ , is isomorphic to  $(L, \lor, \land, 0, 1)$ . Moreover,  $\tilde{L}^d = CO(\mathfrak{PF}(L)) \cap_1 \mathbf{D}_L$ .

**Definition 16.** An object of **LatBD** is a system  $(L, L_s, \mathbf{D}_L)$  with  $L = (L, \lor, \land, 0, 1)$  a bounded distributive lattice,  $L_s$  a bornology in L and  $\mathbf{D}_L$  satisfying the conditions:

- (1)  $\mathbf{D}_L \subseteq \bigcup \widetilde{L}_s \subseteq \mathcal{PF}(L),$
- (2)  $\forall a, b \in L \quad a \neq b \implies \tilde{a}^d \neq \tilde{b}^d$ , where  $\tilde{a}^d = \{F \in \mathbf{D}_L \mid a \in F\}$ ,
- (3)  $\widetilde{L} \cap_1 \mathbf{D}_L = (\widetilde{L}_s \cap_1 \mathbf{D}_L)^o \subseteq \mathcal{P}(\mathbf{D}_L).$

Such  $D_L$  will be called a decent lump of prime filters on L.

A morphism of **LatBD** from  $(L, L_s, \mathbf{D}_L)$  to  $(M, M_s, \mathbf{D}_M)$  is such a homomorphism of bounded lattices  $h : L \to M$  that:

- (a) satisfies the condition of domination  $\forall a \in M_s \exists b \in L_s \ a \lor h(b) = h(b)$ ,
- (b) respects the decent lump of prime filters:  $h^{\bullet}(\mathbf{D}_M) \subseteq \mathbf{D}_L$ .

#### 5. Stone Duality for LSS<sub>0</sub> and SS<sub>0</sub>

This section presents the main new version of Stone Duality for locally small spaces (Theorem 1) and its restricted version for small spaces (Theorem 2).

**Proposition 2.** Assume  $(X, \mathcal{L}_X)$  is a locally small space. Then

$$\mathcal{L}_X \cong \widetilde{\mathcal{L}_X} \cap_1 \hat{X} \text{ and } \mathcal{L}_X^o \cong \widetilde{\mathcal{L}_X^o} \cap_1 \hat{X} = \widetilde{(\mathcal{L}_X)^o} \cap_1 \hat{X} = \widetilde{(\mathcal{L}_X} \cap_1 \hat{X})^o \subseteq \mathcal{P}(\hat{X}),$$

where  $\widetilde{\mathcal{L}}_X = \{ \widetilde{A} \mid A \in \mathcal{L}_X \}, \widetilde{A} = \{ F \in \mathfrak{PF}(\mathcal{L}_X^o) \mid A \in F \},\$ 

$$\hat{X} = \{\hat{x} \mid x \in X\} \subseteq \mathfrak{PF}(\mathcal{L}_X^o), and \quad \hat{x} = \{W \in \mathcal{L}_X^o \mid x \in W\}.$$

**Proof.** It is clear that  $\widetilde{\mathcal{L}}_X^o \cap_1 \hat{X} \subseteq (\widetilde{\mathcal{L}}_X)^o \cap_1 \hat{X} \subseteq (\widetilde{\mathcal{L}}_X \cap_1 \hat{X})^o$ . Moreover, the Boolean algebras  $\mathcal{P}(X)$  and  $\mathcal{P}(\hat{X})$  are isomorphic, where the sublattice  $\widetilde{\mathcal{L}}_X^o \cap_1 \hat{X}$  corresponds to  $\mathcal{L}_X^o$  and the sublattice  $\widetilde{\mathcal{L}}_X \cap_1 \hat{X}$  corresponds to  $\mathcal{L}_X$ . That is why  $\mathcal{L}_X \cong \widetilde{\mathcal{L}}_X \cap_1 \hat{X}$  and  $\mathcal{L}_X^o \cong \widetilde{\mathcal{L}}_X^o \cap_1 \hat{X} = (\widetilde{\mathcal{L}}_X \cap_1 \hat{X})^o \subseteq \mathcal{P}(\hat{X})$ .  $\Box$ 

**Theorem 1.** The categories LSS<sub>0</sub>, LatBD<sup>op</sup> and SpecBD are equivalent.

## **Proof. Step 1:** Defining functor $\overline{R}$ : **SpecBD** $\rightarrow$ **LSS**<sub>0</sub>.

We define the *restriction functor*  $\bar{R}$  : **SpecBD**  $\rightarrow$  **LSS**<sub>0</sub> by formulas

$$\bar{R}((X,\tau_X),CO_s(X),X_d) = (X_d,CO_s(X)_d), \quad \bar{R}(g) = g_d,$$

where  $g_d : X_d \to Y_d$  is the restriction of  $g : X \to Y$  in the domain and in the codomain to the decent lumps. It is clear that  $CO_s(X)_d$  is a sublattice of  $\mathcal{P}(X_d)$  with zero that covers  $X_d$ . Now CO(X) separates points of X since it is a basis of the topology  $\tau_X$ . Hence both  $CO(X)_d$  and  $CO_s(X)_d$  separate points of  $X_d$ .

For a morphism  $g : X \to Y$  in **SpecBD**, we have

$$g_d^{-1}(CO_s(Y)_d) \subseteq g^{-1}(CO(Y)_d) \cap_1 X_d \subseteq CO(X)_d \subseteq (CO_s(X)_d)^o$$

by (3) of Definition 14, so  $g_d : (X_d, CO_s(X)_d) \to (Y_d, CO_s(Y)_d)$  is continuous. That  $g_d$  is a bounded mapping between locally small spaces follows from g satisfying the condition of boundedness. Since the rest of the conditions are obvious,  $\overline{R}$  is indeed a functor.

# **Step 2:** Defining functor $\overline{S}$ : **LatBD**<sup>*op*</sup> $\rightarrow$ **SpecBD**. We define the *spectrum functor* $\overline{S}$ : **LatBD**<sup>*op*</sup> $\rightarrow$ **SpecBD** by

 $\bar{S}(L, L_s, \mathbf{D}_L) = ((\mathfrak{PF}(L), \tau(\tilde{L})), \tilde{L}_s, \mathbf{D}_L), \quad \bar{S}(h^{op}) = h^{\bullet},$ 

where  $\tau(\tilde{L})$  is as in Remark 2 and  $h^{op}$  in **LatBD**<sup>op</sup> is the morphism *h* in **LatBD** inverted. The lattice  $CO_s(\mathfrak{PF}(L)) = \tilde{L}_s$  is a bornology in  $CO(\mathfrak{PF}(L))$ . Moreover, we have an isomorphism of lattices

$$CO(\mathfrak{PF}(L)) = \tilde{L} \ni \tilde{a} \mapsto \tilde{a}^d \in \tilde{L}^d = CO(\mathfrak{PF}(L)) \cap_1 \mathbf{D}_L$$

and  $\mathbf{D}_L \subseteq \bigcup \widetilde{L}_s$ , by Definition 16. Now (3) of Definition 14 follows from (3) of Definition 16, so  $\mathbf{D}_L$  is a decent lump.

For a morphism  $h : L \to M$  of LatBD we have  $(h^{\bullet})^{-1}(\tilde{b}) = \{G \in \mathfrak{PF}(M) \mid b \in h^{\bullet}G\} = \tilde{h(b)}$  for  $b \in L$ . This means  $(h^{\bullet})^{-1}(\tilde{L}) \subseteq \tilde{M}$ , so  $h^{\bullet} : \mathfrak{PF}(M) \to \mathfrak{PF}(L)$  is spectral, satisfies the condition of boundedness and respects the decent lump:  $h^{\bullet}(\mathbf{D}_M) \subseteq \mathbf{D}_L$ . Since the rest of the conditions are obvious,  $\bar{S}$  is indeed a functor. Step 3: Defining functor  $\bar{A} : \mathbf{LSS}_0 \to \mathbf{LatBD}^{op}$ .

We define the *algebraization functor*  $\overline{A}$  : **LSS**<sub>0</sub>  $\rightarrow$  **LatBD**<sup>op</sup> by

$$A(X,\mathcal{L}_X) = (\mathcal{L}^o_X,\mathcal{L}_X,X), \quad A(f) = (\mathcal{L}^o f)^{op},$$

where  $\mathcal{L}_X^o = (\mathcal{L}_X^o, \cup, \cap, \emptyset, X)$  is a bounded distributive lattice,  $\hat{X} = \hat{X}(\mathcal{L}_X^o) = \{\hat{x} \mid x \in X\} \subseteq \mathcal{PF}(\mathcal{L}_X^o)$  with  $\hat{x} = \{A \in \mathcal{L}_X^o \mid x \in A\}$ , and, for a strictly continuous mapping  $f : (X, \mathcal{L}_X) \to (Y, \mathcal{L}_Y)$ , the mapping  $\mathcal{L}^o f : \mathcal{L}_Y^o \to \mathcal{L}_X^o$  is defined by  $(\mathcal{L}^o f)(W) = f^{-1}(W)$  for  $W \in \mathcal{L}_Y^o$ .

The lattice  $\mathcal{L}_X$  is a bornology in  $\mathcal{L}_X^o$  by the definition of  $\mathcal{L}_X^o$ . Since  $X \subseteq \bigcup \mathcal{L}_X$ , we have  $\hat{X} \subseteq \bigcup \widetilde{\mathcal{L}}_X$ . For  $A \neq B \in \mathcal{L}_X$ , there exists  $x \in A \triangle B$ , so  $\hat{x} \in (\widetilde{A} \triangle \widetilde{B}) \cap \hat{X}$  and  $\widetilde{A} \cap \hat{X} \neq \widetilde{B} \cap \hat{X}$ . By the proof of Proposition 2,  $(\mathcal{L}_X^o, \mathcal{L}_X, \hat{X})$  satisfies (3) of Definition 16 and  $\hat{X}(\mathcal{L}_X^o)$  is a decent lump of prime filters on  $\mathcal{L}_X^o$ .

Moreover,  $\mathcal{L}^o f : \mathcal{L}^o_Y \to \mathcal{L}^o_X$  is a morphism in **LatBD** as a homomorphism of bounded lattices satisfying

$$(\mathcal{L}^{o}f)^{\bullet}(\hat{x}) = \{ W \in \mathcal{L}^{o}_{Y} \mid x \in f^{-1}(W) \} = \widehat{f(x)}, \text{ so } (\mathcal{L}^{o}f)^{\bullet}(\hat{X}) \subseteq \hat{Y},$$

with the domination condition being the boundedness of the strictly continuous mapping f. Since the rest of the conditions are obvious,  $\bar{A}$  is indeed a functor. **Step 4:** The functor  $\bar{R}S\bar{A}$  is naturally isomorphic to  $Id_{LSS_0}$ .

We have  $\overline{RSA}(X, \mathcal{L}_X) = \overline{RS}(\mathcal{L}_X^o, \mathcal{L}_X, \hat{X}) = \overline{R}(\mathfrak{PF}(\mathcal{L}_X^o), \widetilde{\mathcal{L}_X}, \hat{X}) = (\hat{X}, \widetilde{\mathcal{L}_X}^d)$ , where  $\widetilde{\mathcal{L}_X}^d = \widetilde{\mathcal{L}_X} \cap_1 \hat{X}$ , and, for a morphism  $f : (X, \mathcal{L}_X) \to (Y, \mathcal{L}_Y)$  in **LSS**<sub>0</sub>, we have  $\overline{RSA}(f) = ((\mathcal{L}^o f)^{\bullet})_d$ .

Define a natural transformation  $\eta$  from  $\bar{R}\bar{S}\bar{A}$  to  $Id_{LSS_0}$  by

$$\eta_X(\hat{X}, \widetilde{\mathcal{L}_X}^a) \to (X, \mathcal{L}_X)$$
, where  $\eta_X(\hat{x}) = x$ .

Then  $f \circ \eta_X(\hat{x}) = \eta_Y \circ ((\mathcal{L}^o f)^{\bullet})_d(\hat{x}) = f(x)$  for  $\hat{x} \in \hat{X}$  and, by the obvious isomorphism between  $\mathcal{P}(X)$  and  $\mathcal{P}(\hat{X})$  (compare Proposition 2), each  $\eta_X$  is an isomorphism in **LSS**<sub>0</sub>, so  $\eta$  is truly a natural isomorphism.

**Step 5:** The functor  $\overline{SAR}$  is naturally isomorphic to  $Id_{SpecBD}$ .

We have  $\overline{SAR}((X,\tau_X), CO_s(X), X_d) = \overline{SA}(X_d, CO_s(X)_d) = \overline{S}((CO_s(X)_d)^o, CO_s(X)_d),$  $\widehat{X_d}^d) = (\mathfrak{PF}((CO_s(X)_d)^o), \widetilde{CO_s(X)_d}, \widehat{X_d}^d),$ 

with the topology  $\tau(\widetilde{CO(X)}_d)$  on  $\mathcal{PF}((CO_s(X)_d)^o)$  since, by Definition 14, we have  $(CO_s(X)_d)^o = CO(X)_d$ . Here we put  $\widehat{X}_d^d = \{\hat{x}^d \mid x \in X_d\}$  and  $\hat{x}^d = \{V \cap X_d \mid x \in V \in CO(X)\} \in \mathcal{PF}(CO(X)_d)$  for  $x \in X$ . For a morphism  $g : X \to Y$  in **SpecBD**, we have  $\overline{SAR}(g) = (\mathcal{L}^o g_d)^{\bullet}$ .

Define a natural transformation  $\theta$  from  $\bar{S}\bar{A}\bar{R}$  to  $Id_{SpecBD}$  by

$$\theta_X : (\mathfrak{PF}(CO(X)_d), \widetilde{CO_s(X)_d}, \widehat{X_d}^d) \to (X, CO_s(X), X_d) \text{ with } \theta_X(\hat{x}^d) = x$$

Notice that  $(\mathcal{L}^{o}g_{d})^{\bullet}(\hat{x}^{d}) = \{W \cap Y_{d} \mid g(x) \in W \in CO(Y)\} = \widehat{g(x)}^{d}$  for  $x \in X$ . This means  $g \circ \theta_{X} = \theta_{Y} \circ (\mathcal{L}^{o}g_{d})^{\bullet}$  and each  $\theta_{X}$  satisfies  $\theta_{X}(\widehat{X}_{d}^{d}) = X_{d}$  and  $\theta_{X}(\widetilde{A}_{d}) = \theta_{X}(\{\hat{x}^{d} \in \mathfrak{PF}(CO(X)_{d}) \mid x \in A\}) = A$  for  $A \in CO(X)$ , so  $\theta_{X}^{-1}(CO(X)) = \widetilde{CO(X)_{d}}$  and  $\theta_{X}^{-1}(CO_{s}(X)) = \widetilde{CO(X)_{d}}$ . Hence  $\theta$  is truly a natural isomorphism. **Step 6:** The functor  $\overline{ARS}$  is naturally isomorphic to  $Id_{LatBD^{op}}$ .

We get  $\overline{ARS}(L, L_s, \mathbf{D}_L) = \overline{AR}((\mathfrak{PF}(L), \tau(\widetilde{L})), \widetilde{L}_s, \mathbf{D}_L) = \overline{A}(\mathbf{D}_L, \widetilde{L}_s \cap_1 \mathbf{D}_L) = ((\widetilde{L}_s \cap_1 \mathbf{D}_L)^o, \widetilde{L}_s \cap_1 \mathbf{D}_L, \widehat{\mathbf{D}}_L^d)$ . Here  $\widehat{\mathbf{D}}_L^d = \{\widehat{F}^d \mid F \in \mathbf{D}_L\}$ , where  $\widehat{F}^d = \{\widetilde{a}^d \in \widetilde{L}^d \mid F \in \widetilde{a}^d\}$ ,  $\widetilde{a}^d = \{F \in \mathbf{D}_L \mid a \in F\}$ . By Definition 16, we have  $(\widetilde{L}_s \cap_1 \mathbf{D}_L)^o = \widetilde{L} \cap_1 \mathbf{D}_L$ , shortly:  $(\widetilde{L}_s^d)^o = \widetilde{L}^d$ . For a morphism  $h : L \to M$  in LatBD, we have  $\overline{ARS}(h^{op}) = (\mathcal{L}^o(h^{\bullet})_d)^{op}$ .

Define a natural transformation  $\kappa^{op}$  from  $\overline{ARS}$  to  $Id_{LatBD^{op}}$  by putting  $\kappa_L^{op}$  :  $(\widetilde{L}^d, \widetilde{L}_s^d, \widehat{D_L}^d) \to (L, L_s, D_L)$  in LatBD<sup>op</sup> to be the map

$$\kappa_L : (L, L_s, \mathbf{D}_L) \to (\widetilde{L}^d, \widetilde{L}_s^d, \widehat{\mathbf{D}}_L^d)$$
 given by  $\kappa_L(a) = \widetilde{a}^d$ .

We are to check that  $\kappa_L^{op} \circ \overline{ARS}(h^{op}) = h^{op} \circ \kappa_M^{op}$  or  $\kappa_M \circ h = \mathcal{L}^o(h^{\bullet})_d \circ \kappa_L$ . Now  $(\mathcal{L}^o(h^{\bullet})_d \circ \kappa_L)(a) = (h^{\bullet})_d^{-1}(\tilde{a}^d) = \{G \in \mathbf{D}_M \mid h(a) \in G\} = \widetilde{h(a)}^d = (\kappa_M \circ h)(a)$ . Each  $\kappa_L : L \to \widetilde{L}^d$  is an isomorphism of bounded lattices satisfying  $\kappa_L^{\bullet}(\widehat{\mathbf{D}}_L^d) = \mathbf{D}_L$  and  $\kappa_L(L_s) = \widetilde{L}_s^d$ , so  $\kappa^{op}$  is truly a natural isomorphism.  $\Box$ 

**Example 5.** The sine mapping  $\sin : \mathbb{R}_{lom} \to \mathbb{R}_{lom}$  is bounded continuous but not strongly continuous. Consequently,  $\bar{S}\bar{A}(\sin) = (\mathcal{L}^o \sin)^{\bullet} : \mathfrak{PF}(\mathcal{L}^o_{lom}) \to \mathfrak{PF}(\mathcal{L}^o_{lom})$  is a spectral mapping between spectral spaces but  $(\mathcal{L}^o \sin)(\mathcal{L}_{lom})$  is not contained in  $\mathcal{L}_{lom}$ , so  $\bar{A}$  does not provide an endomorphism of the lattice  $\mathcal{L}_{lom}$ .

**Theorem 2.** The categories  $SS_0$ , Lat $D^{op}$  and SpecD are equivalent.

**Proof.** In the proof of Theorem 1, we restrict to the case  $\mathcal{L}_X^o = \mathcal{L}_X$ ,  $L_s = L$  and  $CO_s(X) = CO(X)$ .  $\Box$ 

**Corollary 1.** Any bounded continuous mapping between Kolmogorov locally small spaces is a restriction of a spectral mapping between spectral spaces to some patch dense subsets.

# 6. The Categories uSpec and uSpec<sup>s</sup>

This section collects main facts about up-spectral spaces, gives an equivalency result for the category of up-spectral spaces with spectral mappings (Theorem 5) and distinguishes the category of up-spectral spaces with strongly spectral mappings.

**Definition 17.** For a topological space  $(X, \tau_X)$ , we denote:

 $SO(X) = \{ U \in \tau_X \mid U \text{ has spectral subspace topology} \}.$ 

**Definition 18.** A topological space  $(X, \tau_X)$  is strongly locally spectral *if it satisfies the following conditions:* 

- (1) *it is* locally spectral ([14,30]): SO(X) covers X,
- (2) *it is* semispectral ([14,30]):  $CO(X) \cap_1 CO(X) \subseteq CO(X)$ .

**Proposition 3.** For any strongly locally spectral space  $(X, \tau_X)$ , we have

$$CO(X) = SO(X).$$

**Proof.** Obviously,  $SO(X) \subseteq CO(X)$ . Let  $A \in CO(X)$ . Then A is covered by a finite family  $W_1, ..., W_n$  of spectral open sets. Since a finite union of spectral spaces glued together along compact open subsets is spectral, the set  $W_1 \cup ... \cup W_n$  is spectral and its compact open subset A belongs to SO(X).  $\Box$ 

**Proposition 4.** In a strongly locally spectral space  $(X, \tau_X)$ , we have

$$CO(X)^o = ICO(X) \subseteq \mathcal{P}(X).$$

**Proof.** If  $V \in CO(X)^o$ , then *V* is a union of compact open sets since CO(X) covers *X*. Hence  $V \in \tau_X$  and *V* satisfies the definition of a member of ICO(X).

If  $V \in ICO(X)$  and A is any member of CO(X), then  $V \cap A \in CO(X)$ . This means  $V \in CO(X)^{o}$ .  $\Box$ 

**Proposition 5.** A strongly locally spectral space on a set X may be equivalently defined by:

- (a) the topology  $\tau_X$ ,
- (b) the family of compact open subsets CO(X),
- (c) the family of spectral open subsets SO(X),
- (d) the family of intersection compact open subsets ICO(X).

**Proof.** Elements of CO(X) = SO(X) are the compact elements of  $\tau_X$ . Elements of ICO(X) are the sets compatible with those of CO(X). Elements of  $\tau_X$  are the unions of subfamilies of ICO(X). Each of the considered families induces all the other.  $\Box$ 

**Remark 4.** Hochster proved ([14], Proposition 16) that being strongly locally spectral is equivalent to:

- (a) being the underlying space of some scheme,
- (b) being homeomorphic with an open subspace of a spectral space.

**Remark 5.** It is also known that a topological space is strongly locally spectral if and only if it is:

- (*a*) almost-spectral, see ([31], Theorem 7), which means that it is the prime spectrum of a commutative ring or the prime spectrum of a distributive lattice with zero.
- *(b)* up-spectral, see ([31], Theorem 8), which means that it satisfies all conditions of Definition 9 but (S1).

Because of the above, strongly locally spectral spaces will be called up-spectral from now on.

**Definition 19.** *For L a distributive lattice with zero and*  $a \in L$ *, we set* 

$$D(a) = \{ p \in \mathfrak{PI}(L) \mid a \notin p \}$$

where  $\mathcal{PJ}(L)$  is the set of all prime ideals in L.

Below, we restate in a modern language two theorems of M. H. Stone published in 1938. They give another description of up-spectral spaces.

**Theorem 3** ([3], Theorem 15). Let  $(L, \lor, \land, 0)$  be a distributive lattice with zero. Then the sets

$$D(I) = \{p \in \mathfrak{PI}(L) \mid I \not\subseteq p\}, \text{ where } I \text{ is an ideal in } L,$$

form a  $T_0$  topology on  $\mathcal{PJ}(L)$  with a basis

 $\{D(a) \mid a \in L\} = CO(\mathcal{PI}(L))$ 

closed under finite intersections and satisfying the condition

(\*) for a closed set F and a subfamily  $\mathcal{C} \subseteq CO(\mathcal{PJ}(L))$  centered on F (this means: for any finite family  $C_1, ..., C_n$  of members of  $\mathcal{C}$  the set  $F \cap C_1 \cap ... \cap C_n$  is nonempty), the intersection  $F \cap \bigcap \mathcal{C}$  is nonempty.

**Theorem 4** ([3], Theorem 16). Let  $(X, \tau_X)$  be a topological  $T_0$ -space where CO(X) is a basis of the topology closed under finite intersections and satisfying the condition  $(\star)$  from the previous theorem. Then:

- (1)  $(CO(X), \cup, \cap, \emptyset)$  is a distributive lattice with zero,
- (2)  $\Psi : \mathfrak{I}(CO(X)) \ni I \mapsto \bigcup I \in \tau_X$  is an isomorphism of lattices, where  $\mathfrak{I}(CO(X))$  is the lattice of all ideals in CO(X),
- (3) for each  $p \in PJ(CO(X))$  there exists a unique  $x_p \in X$  such that

 $\bigcup p = \operatorname{int}(X \setminus \{x_p\}),$ 

(4) the mapping  $H : \mathfrak{PI}(CO(X)) \ni p \mapsto x_p \in X$  is a homeomorphism, where the topology in  $\mathfrak{PI}(CO(X))$  is defined as in Theorem 3.

**Proposition 6.** For a topological space  $(X, \tau_X)$ , the following conditions are equivalent:

- (1)  $(X, \tau_X)$  is up-spectral,
- (2)  $(X, \tau_X)$  satisfies the conditions in the assumption of Theorem 4.

**Proof.** (1)  $\implies$  (2) Since the other conditions are obvious, we prove (\*). One may assume  $F \neq \emptyset$  and  $\emptyset \neq \emptyset \subseteq CO(X)$  is centered on *F*. Choose  $C \in \emptyset$ . Then  $F \cap C$  is patch compact and members of  $\emptyset \cap_1 C$  are patch closed in *C*. Since finite subfamilies of  $\emptyset \cap_1 C$  meet  $F \cap C$ , the set  $F \cap \bigcap \emptyset$  is nonempty.

(2)  $\implies$  (1) Assume *V* is a proper irreducible open subset of *X*. Then  $I(V) = \{A \in CO(X) \mid A \subseteq V\}$  is a prime ideal in CO(X). By (3) of Theorem 4, there exists a unique  $x_V$  such that  $\bigcup I(V) = V = int(X \setminus \{x_V\})$ . Hence *X* as well as all members of CO(X) are sober. The other conditions are obvious.  $\Box$ 

**Definition 20.** A subset  $X_d \subseteq X$  in a up-spectral space  $(X, \tau_X)$  will be called decent if any of the two equivalent conditions is satisfied:

(1) for  $A, B \in CO(X)$  if  $A \neq B$ , then  $A \cap X_d \neq B \cap X_d$ ,

(2)  $R_d : CO(X) \ni A \mapsto A \cap X_d \in CO(X)_d$  is an isomorphism of lattices.

**Definition 21** (cf. [14,30]). *The* patch topology *of an up-spectral space*  $(X, \tau_X)$  *is the topology on X with a basis CO*(*X*)  $\setminus_1$  *CO*(*X*).

**Proposition 7.** In an up-spectral space  $(X, \tau_X)$  the decent subsets are exactly the patch dense subsets.

**Proof.** The same as the proof of Proposition 1.  $\Box$ 

**Example 6.** The spaces  $(\mathfrak{PF}(\mathcal{L}_{lom}), \tau(\widehat{\mathcal{L}_{lom}}))$  and  $(\mathfrak{PF}(\mathcal{L}_{lrom}), \tau(\widehat{\mathcal{L}_{lrom}}))$  are up-spectral and are homeomorphic to open patch dense subspaces in the respective spectral spaces  $(\mathfrak{PF}(\mathcal{L}_{lom}^{o}), \tau(\widehat{\mathcal{L}_{lom}^{o}}))$  and  $(\mathfrak{PF}(\mathcal{L}_{lom}^{o}), \tau(\widehat{\mathcal{L}_{lom}^{o}}))$  that are known from Example 4.

**Definition 22.** A mapping  $g : (X, \tau_X) \to (Y, \tau_Y)$  between up-spectral spaces will be called spectral *if the following conditions are satisfied:* 

(1) g is bounded: g(CO(X)) refines CO(Y),

(2) *g* is s-continuous:  $g^{-1}(ICO(Y)) \subseteq ICO(X)$ .

**Remark 6.** In [30], all mappings satisfying (2) are called spectral, but this condition is too weak in our context of up-spectral spaces.

The following proposition gives a better understanding of spectral maps between up-spectral spaces.

**Proposition 8.** If  $g : (X, \tau_X) \to (Y\tau_Y)$  is a mapping between up-spectral spaces, then the following conditions are equivalent:

- (1) g is spectral,
- (2) *g* is bounded and locally spectral (i.e., for any  $A \in CO(X)$ ,  $B \in CO(Y)$  such that  $g(A) \subseteq B$ , the restriction  $g_A^B : A \to B$  is a spectral mapping between spectral spaces).

**Proof.** (1)  $\implies$  (2) If *g* is spectral and *A*, *B* are as in the statement, then

 $CO(A) = ICO(A), \quad CO(B) = ICO(B) \subseteq ICO(Y).$ 

Now  $(g_A^B)^{-1}(CO(B)) \subseteq ICO(X) \cap_1 A \subseteq CO(A)$ , so *g* is locally spectral. (2)  $\implies$  (1) If *g* is bounded and locally spectral, then, for  $D \in ICO(Y)$  and  $A \in CO(X)$ , we have

$$g^{-1}(D) \cap A = (g^B_A)^{-1}(D \cap B) \in CO(X),$$

with some  $B \in CO(Y)$  such that  $g(A) \subseteq B$ . This means g is s-continuous.  $\Box$ 

**Definition 23.** By **uSpec** we shall denote the category of up-spectral spaces and spectral mappings between them.

**Definition 24.** By **SpecB** we shall denote the full subcategory in **SpecBD** generated by objects  $((Y, \tau_Y), CO_s(Y), Y_d)$  satisfying  $Y_d = \bigcup CO_s(Y)$ .

**Theorem 5.** The categories **uSpec** and **SpecB** are equivalent.

**Proof.** Let  $LSS_0(uSpec)$  be the full subcategory of  $LSS_0$  generated by those objects  $(X, \mathcal{L}_X)$  whose topology  $\tau(\mathcal{L}_X)$  is up-spectral and whose family of smops  $\mathcal{L}_X$  coincides with the compact open subsets in this topology.

We have a concrete isomorphism of constructs (see [32])  $\overline{I}$  : **uSpec**  $\rightarrow$  **LSS**<sub>0</sub>(**uSpec**) given by the formula  $\overline{I}(X, \tau_X) = (X, CO(X))$ ; notice that the spectral mappings between upspectral spaces are exactly the continuous bounded mappings between the corresponding objects of **LSS**<sub>0</sub>(**uSpec**).

We show, using Theorem 1, that the functor  $\overline{SA}$  transforms  $LSS_0(uSpec)$  into SpecB. If (X, CO(X)) is an object of  $LSS_0(uSpec)$ , then it embeds into the spectral space  $\mathcal{PF}(ICO(X))$  with topology  $\tau(ICO(X))$ , the distinguished bornology  $\widetilde{CO(X)}$  and the distinguished decent lump  $\hat{X}$ . Obviously  $\hat{X} \subseteq \bigcup \widetilde{CO(X)}$ , so we are to show  $\bigcup \widetilde{CO(X)} \subseteq \hat{X}$ . Let  $F \in \widetilde{B}$  with  $B \in CO(X)$ . Then  $F \cap CO(B) \in \mathcal{PF}(CO(B))$ . Since B is spectral, there exists  $x \in B$  such that  $\hat{x}^B = \{A \in CO(B) : x \in A\} = F \cap CO(B)$ . Then  $F = \hat{x} \in \hat{X}$ . (If  $C \in F \triangle \hat{x}$ , then  $C \cap B \in (F \cap CO(B)) \triangle \hat{x}^B$ . Contradiction.) Since both subcategories are full, we are done.

Applying the restriction functor  $\overline{R}$  to some object  $((Y, \tau_Y), CO_s(Y), \bigcup CO_s(Y))$  of **SpecB**, we get the up-spectral space with the induced topology whose family of compact open sets is equal to

$$CO(Y) \cap \mathcal{P}(\bigcup CO_s(Y)) = CO(Y) \cap_1 CO_s(Y) = CO_s(Y),$$

so  $(\bigcup CO_s(Y), CO_s(Y))$  is an object of **LSS**<sub>0</sub>(**uSpec**). Again, there is no problem with morphisms since both subcategories are full. This means  $\overline{R}$  transforms **SpecB** into **LSS**<sub>0</sub>(**uSpec**).

We get a pair of functors  $\bar{S}\bar{A}\bar{I}$  : **uSpec**  $\rightarrow$  **SpecB** and  $\bar{I}^{-1}\bar{R}$  : **SpecB**  $\rightarrow$  **uSpec** giving an equivalence between **uSpec** and **SpecB**.  $\Box$ 

**Corollary 2.** Each spectral mapping between up-spectral spaces is a restriction of a spectral mapping between spectral spaces to some open patch dense subsets.

**Definition 25.** A mapping  $g : (X, \tau_X) \to (Y, \tau_Y)$  between up-spectral spaces is strongly spectral *if the following conditions are satisfied:* 

- (1) *g* is bounded,
- (2) *g* is strongly continuous:  $g^{-1}(CO(Y)) \subseteq CO(X)$ .

**Definition 26.** By **uSpec**<sup>*s*</sup> we shall denote the category of up-spectral spaces and strongly spectral mappings between them.

## 7. The Category ZLat

This section introduces the category of distributive lattices with zeros and dominating homomorphisms between them as well as it states a version of Stone Duality for this category (Theorem 6).

**Definition 27.** For a homomorphism of lattices  $h : L \to M$ , we say that:

(1) *h* is dominating or satisfies the condition of domination, if

 $\forall a \in M \exists b \in L \quad a \lor h(b) = h(b),$ 

(2) *h* is proper ([31]) if the preimage of any prime ideal in M is a prime ideal in L.

The following fact follows from the proof of Lemma 4 in [31].

Fact 4. Each proper homomorphism between distributive lattices with zeros respects the zero.

**Example 7.** Not each proper and respecting the zero lattice homomorphism is dominating: take  $\mathcal{L}id_{\mathbb{R}} : \mathcal{L}_{lom} \to \mathcal{L}_{slom}$ .

**Proposition 9.** *Each dominating and respecting the zero homomorphism between distributive lattices with zeros is proper.* 

**Proof.** Let  $h : L \to M$  be such a homomorphism and I be a prime ideal in M. Then  $h^{-1}(I)$  contains the zero. Assume  $a \in M \setminus I$ . Then  $h(b) \ge a$  for some  $b \in L$ . However,  $h(b) \notin I$ , so  $h^{-1}(I)$  is a proper subset of L. A standard checking proves that the conditions

(i)  $b_1 \in h^{-1}(I)$  and  $b_2 \in h^{-1}(I)$  if and only if  $b_1 \lor b_2 \in h^{-1}(I)$ ,

(ii)  $b_1 \in h^{-1}(I)$  or  $b_2 \in h^{-1}(I)$  if and only if  $b_1 \land b_2 \in h^{-1}(I)$ 

for a prime ideal are satisfied.  $\Box$ 

**Definition 28.** By **ZLat** we denote the category of distributive lattices with zeros and dominating and respecting zeros homomorphisms of lattices.

**Theorem 6.** The categories **uSpec**<sup>s</sup> and **ZLat** are dually equivalent.

**Proof. Step 1:** Defining functor  $\widehat{Co}$  : **uSpec**<sup>*s*</sup>  $\rightarrow$  **ZLat**<sup>*op*</sup>.

For an object  $(X, \tau_X)$  of **uSpec**<sup>*s*</sup>, we set  $\widehat{Co}(X, \tau_X) = (CO(X), \cup, \cap, \emptyset)$ . For a morphism  $g : (X, \tau_X) \to (Y, \tau_Y)$  of **uSpec**<sup>*s*</sup>, we set  $\widehat{Co}(g) = (\mathcal{L}g)^{op} : CO(X) \to CO(Y)$ , where

 $(\mathcal{L}g)(W) = g^{-1}(W)$  for  $W \in CO(Y)$  defines a morphism  $\mathcal{L}g$  of **ZLat**. Hence  $\widehat{Co}$  is a well defined functor.

**Step 2:** Defining functor  $\widehat{Sp}$  : **ZLat**<sup>*op*</sup>  $\rightarrow$  **uSpec**<sup>*s*</sup>.

For an object  $L = (L, \lor, \land, 0)$  of **ZLat**, we put  $\widehat{Sp}(L) = (\mathfrak{PF}(L), \tau(\tilde{L}))$ , where  $\tilde{L} = \{\tilde{a} \mid a \in L\}$ , which is an up-spectral space by Theorem 3. For a morphism  $h : L \to M$  of **ZLat**, we set  $\widehat{Sp}(h^{op}) = h^{\bullet}$ , which is a strongly spectral mapping.

Boundedness of  $h^{\bullet}$ : since  $h^{\bullet}(\widehat{h(L)})$  refines  $\widetilde{L}$  and  $\widehat{h(L)} = {\widetilde{h(a)} | a \in L}$  dominates in  $\widetilde{M}$ , hence  $h^{\bullet}(\widetilde{M})$  refines  $\widetilde{L}$ .

Strong continuity of  $h^{\bullet}$ : for any  $\widetilde{a} \in CO(\mathfrak{PF}(L)) = \{ \widetilde{a} \mid a \in L \}$  (see Theorem 3), we have  $(h^{\bullet})^{-1}(\widetilde{a}) = \widetilde{h(a)} \in CO(\mathfrak{PF}(M))$ .

Hence  $\widehat{Sp}$  is a well-defined functor.

**Step 3:** The functor  $\widehat{CoSp}$  is naturally isomorphic to  $Id_{\mathbf{ZLat}^{op}}$ .

Define a natural transformation  $\alpha$  from  $Id_{\mathbf{ZLat}^{op}}$  to  $\widehat{CoSp}$  by  $\alpha_M(a) = \tilde{a} \in \widetilde{M}$  for any object M of  $\mathbf{ZLat}$ . Then each  $\alpha_M : M \to \widetilde{M}$  is an isomorphism of  $\mathbf{ZLat}$  (injectivity follows from ([3], Theorem 6)). For a morphism  $h : L \to M$  in  $\mathbf{ZLat}$ , one has  $(\alpha_M \circ h)(a) = \widetilde{h(a)} = (h^{\bullet})^{-1}(\widetilde{a}) = (\mathcal{L}h^{\bullet} \circ \alpha_L)(a)$ , so  $\alpha_L \circ h^{op} = (\mathcal{L}h^{\bullet})^{op} \circ \alpha_M$ . Hence  $\alpha$  is a natural isomorphism. **Step 4:** The functor  $\widehat{SpCo}$  is naturally isomorphic to  $Id_{\mathbf{uSpec}^s}$ .

Define a natural transformation  $\beta$  from  $Id_{\mathbf{uSpec}^s}$  to SpCo by  $\beta_X(x) = \hat{x}$  for any object  $(X, \tau_X)$  of  $\mathbf{uSpec}^s$ , where  $\hat{x} = \{V \in CO(X) \mid x \in V\}$ . (We have  $\hat{X} = \{\hat{x} \mid x \in X\} = \mathcal{PF}(CO(X))$  by the dual of Theorem 4). For a morphism  $g: (X, \tau_X) \to (Y, \tau_Y)$  of  $\mathbf{uSpec}^s$ , we have  $(\beta_Y \circ g)(x) = \widehat{g(x)} = (\mathcal{L}g)^{\bullet}(\hat{x}) = ((\mathcal{L}g)^{\bullet} \circ \beta_X)(x)$  for  $x \in X$ . Now  $\beta_X$  is an isomorphism, since  $\beta_X(CO(X)) = CO(\hat{X})$ , where  $CO(\hat{X}) = \{\tilde{A} \mid A \in CO(X)\}$  and  $\tilde{A} = \{\hat{x} \mid x \in A\}$ . Hence  $\beta$  is a natural isomorphism.  $\Box$ 

**Remark 7.** The co-equivalence mentioned in the above theorem is a restriction of the co-equivalence from Corollary 4 of [31] between the category of up-spectral spaces and their strongly continuous mappings (denoted there  $\mathfrak{US}$ ) and the category of distributive lattices with zeros and proper homomorphisms between them (denoted there  $\mathfrak{D}_0$ ).

# 8. Stone Duality for LSS<sup>s</sup><sub>0</sub>

This section presents a version of Stone Duality for locally small spaces with bounded strongly continuous mappings (Theorem 7).

# **Definition 29.** The category $\mathbf{uSpecD}^s$ has

- (1) pairs  $((X, \tau_X), X_d)$  where  $(X, \tau_X)$  is an up-spectral space and  $X_d$  is a distinguished decent subset of X as objects,
- (2) strongly spectral mappings respecting the decent subsets as morphisms.

## Definition 30. The category ZLatD has

- (1) pairs  $(L, \mathbf{D}_L)$  where L is a distributive lattice with zero and  $\mathbf{D}_L$  is a distinguished decent set of prime filters in  $\mathcal{PF}(L)$  as objects,
- (2) homomorphisms of lattices with zeros respecting the decent sets of prime filters and satisfying the condition of domination as morphisms.

**Definition 31.** *The category*  $LSS_0^s$  *is a subcategory of*  $LSS_0$  *with the same objects and bounded strongly continuous mappings as morphisms.* 

**Example 8.** Let  $\pi : \mathbb{R}_{lom} \sqcup \mathbb{R}_{lom} \to \mathbb{R}_{lom}$  be the natural projection from the disjoint union of two copies of the real locally o-minimal line to the real locally o-minimal line. This finite covering mapping is a morphism of  $\mathbf{LSS}_0^s$  that is not an isomorphism.

**Theorem 7.** The categories  $LSS_0^s$ ,  $ZLatD^{op}$  and  $uSpecD^s$  are equivalent.

**Proof.** Similar to the proof of Theorem 1, using Theorem 6 instead of the classical Stone Duality, with no necessity to mention explicitly the ambient bounded lattice  $\mathcal{L}_X^o$  of  $\mathcal{L}_X$ , with an object *L* of **ZLat** playing the role of  $L_s$  and  $CO(X)_d$  playing the role of  $CO_s(X)_d$  in Theorem 1, restricting to the appropriate classes of morphisms.  $\Box$ 

**Example 9.** The mapping  $(\mathcal{L}\pi)^{\bullet}$ :  $\mathfrak{PF}(\mathcal{L}_{lom} \oplus \mathcal{L}_{lom}) \to \mathfrak{PF}(\mathcal{L}_{lom})$ , where  $\pi : \mathbb{R}_{lom} \sqcup \mathbb{R}_{lom} \to \mathbb{R}_{lom}$  is as in Example 8, is the natural projection from the disjoint union of two copies of  $\mathfrak{PF}(\mathcal{L}_{lom})$  to  $\mathfrak{PF}(\mathcal{L}_{lom})$ . It is a strongly spectral mapping between up-spectral spaces. Moreover,  $\mathbb{R}(\mathcal{L}_{lom})$  is a patch dense set in  $\mathfrak{PF}(\mathcal{L}_{lom})$  and  $\mathbb{R} \sqcup \mathbb{R}(\mathcal{L}_{lom} \oplus \mathcal{L}_{lom})$  is a patch dense set in  $\mathfrak{PF}(\mathcal{L}_{lom} \oplus \mathcal{L}_{lom})$ . The morphism  $(\mathcal{L}\pi)^{\bullet}$  of  $\mathbf{uSpecD}^{s}$  corresponds to the morphism  $\pi$  of  $\mathbf{LSS}_{0}^{s}$  and may be understood as an extention of  $\pi$ .

## 9. Spectralifications

This section introduces a notion of a spectralification of a topological space and discusses it in connection with similar notions and from the point of view of the main new version of Stone Duality.

**Definition 32.** By a spectralification of a topological space  $(X, \tau_X)$  we shall understand a pair  $(e, (Y, \tau_Y))$  where  $(Y, \tau_Y)$  is a spectral topological space,  $e : (X, \tau_X) \to (Y, \tau_Y)$  is a topological embedding and e(X) is a patch dense set in  $(Y, \tau_Y)$ .

**Remark 8.** The morphism e will often be treated as containing information about the space  $(Y, \tau_Y)$  and called a spectralification. On the other hand, when the embedding e is obvious, we shall say that the topological space  $Y = (Y, \tau_Y)$  is a spectralification of  $X = (X, \tau_X)$ . We can also treat  $(X, \tau_X)$  as a topological subspace of  $(Y, \tau_Y)$ .

**Remark 9.** Since a spectral map between spectral spaces is continuous in the patch topologies, any spectralification  $(e, (Y, \tau_Y))$  in our sense has the following uniqueness property: for any spectral mappings  $g_1, g_2$  from  $(Y, \tau_Y)$  to some spectral space  $(Z, \tau_Z)$  if  $g_1 \circ e = g_2 \circ e$ , then  $g_1 = g_2$ .

**Remark 10.** (1) When the Hochster spectralification (see ([14], Theorem 8)) exists, which takes place when this space is semispectral,  $T_0$  and ICO sets form a basis of the topology, then it is a spectralification in our sense.

(2) When the H-spectralification (see [30]) of a hemispectral space exists, which takes place when this space is  $T_0$  and ICO sets form a basis of the topology ([30], Lemma 3.7), then it is a spectralification in our sense.

(3) When X is  $T_0$ , then the spectral reflection  $S_X : X \to S(X)$  (see ([10], Chapter 11)) is a spectralification in our sense.

The tilde operator known in semialgebraic geometry ([12], Chapter 7) gives examples of spectralifications. Other examples appear as spaces of types in model theory ([18], p. 112).

**Example 10.** The space  $(\mathbb{R}, \tau(\mathbb{B}))$  from Example 3 is a spectralification of the real line (with the natural topology), homeomorphic to  $(\mathfrak{PF}(\mathcal{L}_{om}), \tau(\widetilde{\mathcal{L}_{om}}))$ . The corresponding patch (or: constructible) topology on the same set  $\mathbb{R}$  gives an example of a Hausdorff spectralification of the discrete real line. This latter spectralification is homeomorphic to the space of types of the ordered field of reals with the usual topology on this space.

**Example 11.** Consider  $\mathcal{L}_{rom}$  from Example 1. The points of  $\mathcal{PF}(\mathcal{L}_{rom})$  are:

$$\hat{r} = \{A \in \mathcal{L}_{rom} \mid r \in A\}$$
 for  $r \in \mathbb{R}$ ,  
 $\hat{q}^- = \{A \in \mathcal{L}_{rom} \mid (l,q) \subseteq A \text{ for some } l < q\} \text{ for } q \in \mathbb{Q}$ ,

$$\hat{q}^{+} = \{A \in \mathcal{L}_{rom} \mid (q, l) \subseteq A \text{ for some } l > q\} \text{ for } q \in \mathbb{Q},$$
$$\widehat{-\infty} = \{A \in \mathcal{L}_{rom} \mid (-\infty, l) \subseteq A \text{ for some } l \in \mathbb{Q}\},$$
$$\widehat{+\infty} = \{A \in \mathcal{L}_{rom} \mid (l, +\infty) \subseteq A \text{ for some } l \in \mathbb{Q}\}.$$

Here  $\mathcal{L}_{rom} = \{\widetilde{A} \mid A \in \mathcal{L}_{rom}\}, \quad \widetilde{A} = \{F \in \mathfrak{PF}(\mathcal{L}_{rom}) \mid A \in F\}.$  We can see that  $(\mathfrak{PF}(\mathcal{L}_{rom}), \tau(\widetilde{\mathcal{L}_{rom}}))$  is another spectralification of the real line with the natural topology, homeomorphic to the space of types over  $\mathbb{Q}$  of the theory  $Th(\mathbb{R}, <)$  with the spectral (Harrison) topology (compare ([10], Section 14.2)), obtained without using the language of model theory.

**Example 12.** For the patch topology on the "same" set of points as in the previous example, one takes as the new  $\mathcal{L}_X$  the Boolean algebra  $\mathbb{B}_{rom}$  generated by  $\mathcal{L}_{rom}$ . This changes the topology on  $\mathbb{R}$ : the rational points become isolated. The space  $(\mathbb{PF}(\mathbb{B}_{rom}), \tau(\overline{\mathbb{B}_{rom}}))$  is a Hausdorff spectralification of this modified real line, and is identified with the (usual in model theory, compare ([17], Section 4.2)) space of types over  $\mathbb{Q}$  of the theory  $Th(\mathbb{R}, <)$ .

**Proposition 10.** For a topological space, being  $T_0$  is equivalent to admitting a spectralification.

**Proof.** For  $(X, \tau_X)$  a  $T_0$  topological space, choose a basis  $\mathcal{L}_X$  of  $\tau_X$  that is a sublattice containing  $\emptyset$ . By Step 4 of the proof of Theorem 1,  $(X, \tau_X)$  embeds into the spectral space  $(\mathcal{PF}(\mathcal{L}_X^o), \tau(\widetilde{\mathcal{L}_X}))$  with the image  $\hat{X}$  of the embedding patch dense in  $(\mathcal{PF}(\mathcal{L}_X^o), \tau(\widetilde{\mathcal{L}_X}))$ . Since a subspace of a  $T_0$  space is  $T_0$ , only  $T_0$  topological spaces can have spectralifications.  $\Box$ 

**Remark 11.** The proposition above allows producing many spectralifications of a Kolmogorov topological space  $(X, \tau_X)$  by taking many different bases  $\mathcal{L}_X$  of the topology  $\tau_X$ . That is why it is more versatile than Theorem 11.1.3(ii) of [10] concerning the spectral reflection of a Kolmogorov topological space. By Theorem 11.1.3(x) of [10], only localic spectral spaces are codomains of spectral reflections, while any spectral spaces are codomains of spectralifications obtained by using Theorem 1.

## **10. Conclusions**

Several goals have been achieved. The categories of Kolmogorov small and locally small spaces were introduced (Definition 7). We proved new versions of Stone Duality (Theorems 1, 2, 6 and 7) and gave an equivalent description of the category of up-spectral spaces and their spectral mappings (Theorem 5), giving new instances of symmetry on the category theory level.

By giving new versions of Stone Duality, we have developed some theory of locally small spaces (often used in the literature without naming these structures) and important classes of mappings between them, which is a contribution to a new chapter in general topology. In particular, Kolmogorov locally small spaces have been considered as patch dense subsets in spectral or up-spectral spaces (Theorems 1 and 7), while morphisms between them were seen as restrictions of spectral or strongly spectral mappings (Corollaries 1 and 2 and a similar corollary from Theorem 7). The special case of Kolmogorov small spaces was covered separately (Theorem 2).

We have also distinguished the interesting class of strongly spectral mappings respecting the decent subsets between up-spectral spaces (Definition 29) and the class of dominating and respecting the decent set of prime filters homomorphisms of distributive lattices with zeros (Definition 30) as those that correspond to the class of bounded strongly continuous mappings between Kolmogorov locally small spaces (Theorem 7).

In consequence, we have also widened the method of taking spectra of algebraic structures (known from algebraic and analytic geometry) or spaces of types (known from model theory or o-minimality). Taking spectra of small or locally small spaces (using functor  $\bar{S}\bar{A}$  from the proof of Theorem 1) is an extension of this method. Spectralifications of topological spaces form an interesting topic for further research as a sort of non-Hausdorff compactifications.

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