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On the Total Neighbor Sum Distinguishing Index of IC-Planar Graphs

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Abstract: A proper total k -coloring ϕ of G with $\sum_{z \in E_G(u) \cup \{u\}} \phi(z) \neq \sum_{z \in E_G(v) \cup \{v\}} \phi(z)$ for each $uv \in E(G)$ is called a total neighbor sum distinguishing k -coloring, where $E_G(u) = \{uv \mid uv \in E(G)\}$. Pilśniak and Woźniak conjectured that every graph with maximum degree Δ exists a total neighbor sum distinguishing $(\Delta + 3)$ -coloring. In this paper, we proved that any IC-planar graph with $\Delta \geq 12$ satisfies this conjecture, which improves the result of Song and Xu.

Keywords: combinatorial nullstellensatz; IC-planar graphs; total neighbor sum distinguishing coloring; discharging

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. Set $T(G) = V(G) \cup E(G)$ and $E_G(u) = \{uv \mid uv \in E(G)\}$. A total neighbor sum distinguishing k -coloring (for short, *tnsd*) is a mapping $\phi : T(G) \rightarrow \{1, 2, \dots, k\}$ satisfying the following conditions:

- (i) $\phi(z_1) \neq \phi(z_2)$ for any two adjacent or incident elements z_1, z_2 in $T(G)$;
- (ii) $\sum_{z \in E_G(u) \cup \{u\}} \phi(z) \neq \sum_{z \in E_G(v) \cup \{v\}} \phi(z)$ for each $uv \in E(G)$.

The total neighbor sum distinguishing index of G , denoted by $\chi''_{\Sigma}(G)$, is the smallest integer k such that G has a *tnsd* k -coloring.

In 2015, Pilśniak and Woźniak [1] first studied the coloring and gave the conjecture about the *tnsd* index as follows:

Conjecture 1 ([1]). For every graph G , $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$.

Up to now, many special classes of graphs had been proved to satisfy Conjecture 1, including complete graphs, bipartite graphs, subcubic graphs [1], 2-degenerate graphs with maximum degree $\Delta \geq 6$ [2], K_4 -minor-free graphs [3], planar graphs with maximum degree $\Delta \geq 10$ [4], and planar graphs with maximum degree $\Delta \geq 7$ but without triangles [5].

A graph is *1-planar* if it can be embedded on the plane so that each edge is crossed at most once. In 1965, Ringel introduced the notion of 1-planar graphs to color a planar graph and its dual simultaneously. In particular, a 1-planar graph such that each vertex is incident with, at most, a crossing is called an *IC-planar* (independent crossing planar) graph, which was introduced by Albertson [6] while considering the relationship between the crossing number and the chromatic number.

Recently, the *tnsd* index on IC-planar graphs has been extensively studied. There are some recent results in the following.

Theorem 2. Each of the following classes of graphs satisfies Conjecture 1.

- (1) IC-planar graph with $\Delta \geq 7$ but without triangles ([7]).
- (2) IC-planar graph with $\Delta \geq 14$ but without 2-vertex incident with crossed edge ([8]).
- (3) IC-planar graph with $\Delta \geq 13$ ([9]).

There is also a result of list version about the tnsd index of IC-planar graphs, see [10].

In this paper, we discuss any IC-planar graph with maximum degree $\Delta \geq 12$ and obtain the following result, which extends the result of (3) in Theorem 2.

Theorem 3. *Let G be an IC-planar graph. Then, $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 15\}$.*

2. Preliminaries

Let G be a simple graph and uv be an edge of G . For a vertex $u \in V(G)$, we use $d_G(u)$ and $N_G(u)$ to denote the degree and neighbors of u in G , respectively. If $d_G(u) = \ell$ (resp., $d_G(u) \geq \ell$, $d_G(u) \leq \ell$), we call u an ℓ -vertex (resp., ℓ^+ -vertex, ℓ^- -vertex) and an ℓ -neighbor (resp., ℓ^+ -neighbor, ℓ^- -neighbor) of v . We use $n_G^\ell(u)$ (resp., $n_G^{\ell^+}(u)$, $n_G^{\ell^-}(u)$) to represent the number of ℓ -vertices (resp., ℓ^+ -vertices, ℓ^- -vertices) adjacent to u in G . For two subsets A and B of $V(G)$, we denote by $[A, B]$ the set of edges with one end in A and the other in B . In particular, for $u \in V(G)$, we write $[u, B]$ for $[\{u\}, B]$.

Theorem 4 ([11]). *Assume that $\mathbb{F}$ is an arbitrary field and $P \in \mathbb{F}[x_1, \dots, x_n]$ with degree $\deg(P) = \sum_{k=1}^n i_k$, where $i_k \geq 0$ is an integer. If the coefficient $c_P(x_1^{i_1}, \dots, x_n^{i_n})$ of the monomial $x_1^{i_1} \dots x_n^{i_n}$ in P is nonzero, and if $S_1, \dots, S_n$ are subsets of $\mathbb{F}$ with $|S_k| > i_k$, there are $s_1 \in S_1, \dots, s_n \in S_n$ such that $P(s_1, \dots, s_n) \neq 0$.*

The planar graph $G^\times$ obtained from a 1-planar graph G by turning all crossings of G into new 4-vertices on the plane is called the *associated planar graph* of G . Let $uv \in E(G^\times)$. Then, u is called a *false vertex* and a *false neighbor* of v if $u \in V(G^\times) \setminus V(G)$, and a *real vertex* and a *real neighbor* of v otherwise. We use $F(G^\times)$ to denote the set of faces in $G^\times$. For each face $f \in F(G^\times)$, we call f a *false face* if it is incident with one false vertex, and a *real face* otherwise. A face with degree ℓ is called an ℓ -face. Unless stated otherwise, an ℓ -face means a real or false ℓ -face throughout each section in the following.

Lemma 1 ([12]). *Assume that G is a 1-planar graph and $G^\times$ is its associated planar graph. If a real 3-vertex u is adjacent to a false vertex v in $G^\times$, then uv is not incident with two 3-faces in $G^\times$.*

3. Proof of Theorem 3

Let $V_i(G)$ ($V_{i-}(G)$, $V_{i+}(G)$, respectively) represent the set of vertices in G of degree i (at most i , at least i , respectively). For the graph G , a graph G' is *smaller* than it if one of the below conditions is satisfied.

- (i) $|E(G')| < |E(G)|$;
- (ii) In the lexicographic order, $(|V_t(G')|, |V_{t-1}(G')|, \dots, |V_1(G')|)$ precedes $(|V_t(G)|, |V_{t-1}(G)|, \dots, |V_1(G)|)$, where $t = \max\{\Delta(G'), \Delta(G)\}$.

We say a graph is *minimal* for some property if no smaller graph admits the property. To obtain a smaller graph, G' , than G , we will frequently delete some edges or split some vertices of G in the following.

From now on, let G be a minimal graph not satisfying Theorem 3 and $k = \max\{\Delta(G) + 3, 15\}$. Then, any graph G' that is smaller than G has a tnsd k -coloring ϕ' . By splitting a vertex v into two vertices v_1 and v_2 , we obtain a smaller graph G' than G , then G' has a tnsd k -coloring ϕ' . If we can obtain a tnsd k -coloring of G from the ϕ' by sticking v_1 and v_2 into v , we say that v_1 and v_2 can be *properly stuck*. In the following, we will often extend a tnsd k -coloring ϕ' of G' to a tnsd k -coloring ϕ of G to obtain a contradiction. Set $m_\phi(u) = \sum_{z \in E_G(u) \cup \{u\}} \phi(z)$. Assume that X is a subset of $T(G)$ and $\psi : X \rightarrow \{1, \dots, k\}$ is a mapping. For each $z \in T(G)$, set

$$S(z) = \{1, \dots, k\} \setminus \{\psi(x) \mid x \text{ is adjacent or incident to } z \text{ in } G\}.$$

Assume that u is a 4^- -vertex in G and $\phi' : T(G) \setminus \{u\} \rightarrow \{1, \dots, k\}$ is a mapping satisfying the conditions as follows:

- (i) $\phi'(z_1) \neq \phi'(z_2)$ for any adjacent or incident elements $z_1, z_2 \in T(G) \setminus \{u\}$;
- (ii) $\sum_{z \in E_G(z_1) \cup \{z_1\}} \phi'(z) \neq \sum_{z \in E_G(z_2) \cup \{z_2\}} \phi'(z)$ for any two adjacent vertices $z_1, z_2 \in V(G) \setminus \{u\}$.

Since $S(u) = \{1, \dots, k\} \setminus \{\phi'(x) \mid x \text{ is adjacent or incident to } u \text{ in } G\}$, $|S(u)| \geq k - 2d_G(u) > d_G(u)$. Thus, we can extend the ϕ' to a tnsd k -coloring ϕ of G , which is a contradiction. For simplicity, we will omit the colors of all 4^- -vertices in the following discussion.

In Figures 1 and 2, a black vertex represents that its neighbors are all depicted and a white vertex means that its neighbors may not be all depicted.

Claim 1. *The graph G contains no configuration in Figure 1.*

Proof. Suppose that G contains the configuration in Figure 1. By splitting the two 3-vertices v_2, v_4 of the configuration in Figure 1 into v_{21}, v_{22} and v_{41}, v_{42} , respectively, we obtain a smaller graph G' than G . Then, G' has a tnsd k -coloring ϕ' . Without loss of generality, set $\phi'(vv_1) = c_1$, $\phi'(vv_{22}) = c_2$, $\phi'(vv_3) = c_3$, $\phi'(vv_{42}) = c_4$, $\phi'(vv_5) = c_5$, $\phi'(v_1v_{21}) = c_6$, $\phi'(v_{21}v_3) = c_7$, $\phi'(v_3v_{41}) = c_8$, and $\phi'(v_{41}v_5) = c_9$ (see Figure 2).

If we can properly stick v_{21}, v_{41} with v_{22}, v_{42} , respectively, we can obtain a tnsd k -coloring ϕ of G , which is a contradiction. Thus, assume that at least one of v_{21} and v_{41} cannot be stuck properly below. Let $Y = T(G') \setminus \{v_{21}, v_{22}, v_{41}, v_{42}\}$.

Case 1. Suppose that one of v_{21} and v_{41} cannot be stuck properly. If v_{21} can not be stuck properly, $c_2 \in \{c_6, c_7\}$. Without loss of generality, set $c_2 = c_6$.

Set $c_8 \neq c_6$, $c_9 \neq c_6$, and $c_4 \neq c_7$. Then, we assign $\phi''(vv_{22}) = c_4$, $\phi''(vv_{42}) = c_2$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_{42}\}$.

Set $c_8 \neq c_6$, $c_9 \neq c_6$, and $c_4 = c_7$. Then, we assign $\phi''(vv_{22}) = c_4$, $\phi''(vv_{42}) = c_2$, $\phi''(v_{21}v_3) = c_8$, $\phi''(v_3v_{41}) = c_7$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_{42}, v_{21}v_3, v_3v_{41}\}$.

Set $c_8 = c_6$. If $c_9 \neq c_3$, we assign $\phi''(vv_{22}) = c_3$, $\phi''(vv_3) = c_2$, $\phi''(v_3v_{41}) = c_3$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_3, v_3v_{41}\}$. If $c_9 = c_3$ and $c_5 \neq c_7$, we assign $\phi''(vv_{22}) = c_5$, $\phi''(vv_3) = c_8$, $\phi''(v_3v_{41}) = c_3$, $\phi''(vv_5) = c_9$, $\phi''(v_{41}v_5) = c_5$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_3, v_3v_{41}, vv_5, v_{41}v_5\}$. If $c_9 = c_3$ and $c_5 = c_7$, we assign $\phi''(vv_1) = c_6$, $\phi''(v_1v_{21}) = c_1$, $\phi''(vv_{22}) = c_4$, $\phi''(vv_{42}) = c_1$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_1, v_1v_{21}, vv_{22}, vv_{42}\}$.

Set $c_9 = c_6$. If $c_8 \neq c_5$ and $c_5 \neq c_7$, we assign $\phi''(vv_{22}) = c_5$, $\phi''(vv_5) = c_2$, $\phi''(v_{41}v_5) = c_5$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_5, v_{41}v_5\}$. If $c_8 \neq c_5$ and $c_5 = c_7$, we assign $\phi''(vv_{22}) = c_4$, $\phi''(vv_5) = c_9$, $\phi''(v_{41}v_5) = c_5$, $\phi''(vv_3) = c_7$, $\phi''(v_{21}v_3) = c_3$, $\phi''(vv_{42}) = c_3$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, v_{21}v_3, vv_3, vv_{42}, vv_5, v_{41}v_5\}$. If $c_8 = c_5$, we assign $\phi''(vv_{22}) = c_3$, $\phi''(vv_5) = c_9$, $\phi''(v_{41}v_5) = c_5$, $\phi''(vv_3) = c_8$, $\phi''(v_3v_{41}) = c_3$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_{22}, vv_3, v_3v_{41}, vv_5, v_{41}v_5\}$.

In each case above, by sticking v_{i1} and v_{i2} into v_i for $i = 2, 4$, we obtain the graph G . Since both v_2 and v_4 are 3-vertices in G , we can obtain a tnsd k -coloring ϕ of G from ϕ'' . By symmetry, if v_{41} cannot be stuck properly, we can also obtain a tnsd k -coloring ϕ of G . It is a contradiction.

Case 2. Assume that both v_{21} and v_{41} cannot be stuck properly. Then, $c_2 = c_6 = c_8$ and $c_4 = c_7 = c_9$. Let $\phi''(vv_1) = c_6$, $\phi''(v_1v_{21}) = c_1$, $\phi''(vv_5) = c_9$, $\phi''(v_{41}v_5) = c_5$, $\phi''(vv_{22}) = c_5$, $\phi''(vv_{42}) = c_1$, and $\phi(z)'' = \phi'(z)$ for every $z \in Y \setminus \{vv_1, v_1v_{21}, vv_5, v_{41}v_5, vv_{22}, vv_{42}\}$. By sticking v_{i1} and v_{i2} into v_i for $i = 2, 4$, we obtain the graph G . Since both v_2 and v_4 are 3-vertices in G , we can obtain a tnsd k -coloring ϕ of G from ϕ'' , which is a contradiction. $\square$

Figure 1. No configuration in G .

Figure 2. No configuration with colors in G .

The below Claim 2 follows from (3) in Theorem 2.

Claim 2. $\Delta(G) \leq 12$.

Claim 3. If $u \in V_{6-}(G)$, then $|[u, V_{5-}(G)]| = 0$.

Proof. By contradiction, suppose that there is a vertex $u \in V_{6-}(G)$ and a vertex $v \in V_{5-}(G)$ such that $uv \in [u, V_{5-}(G)]$. Let $G' = G - uv$. Then, G' has a tnsd k -coloring ϕ' . We first erase the colors on u and v from ϕ' . Without loss of generality, set $d_G(u) = 6$ and $d_G(v) = 5$. Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (6 - 1) \geq 5, \\ |S(uv)| &\geq k - (6 - 1) - (5 - 1) \geq 6 \text{ and} \\ |S(v)| &\geq k - 2 \times (5 - 1) \geq 7. \end{aligned}$$

Let ϕ be a map on $T(G)$ obtained from ϕ' by assigning $\phi(u) = x_1$, $\phi(v) = x_2$, $\phi(uv) = y_1$, and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v, uv\}$. Set

$$\begin{aligned} P(x_1, x_2, y_1) &= \prod_{z \in N_G(u) \setminus \{v\}} (m_\phi(u) - m_\phi(z)) \prod_{z \in N_G(v) \setminus \{u\}} (m_\phi(v) - m_\phi(z)) \\ &\quad \cdot (m_\phi(u) - m_\phi(v))(x_1 - x_2)(x_1 - y_1)(x_2 - y_1). \end{aligned}$$

With the help of MATHEMATICA, we have $c_P(x_1^4 x_2^4 y_1^5) = -25$. By Theorem 4, there is a color $c_1 \in S(u)$, a color $c_2 \in S(v)$, and a color $c_3 \in S(uv)$ such that $P(c_1, c_2, c_3) \neq 0$. Therefore, we can extend the ϕ' to a tnsd k -coloring ϕ of G by recoloring u, v with colors c_1, c_2 and coloring uv with color c_3 , respectively. It is a contradiction. $\square$

Claim 4. If $u \in V_7(G)$, $|[u, V_{4-}(G)]| = 0$ and $|[u, V_5(G)]| \leq 1$.

Proof. We first prove $|[u, V_{4-}(G)]| = 0$ when $u \in V_7(G)$. By contradiction, suppose that there is a vertex $u \in V_7(G)$ and a vertex $v \in V_{4-}(G)$ such that $uv \in [u, V_{4-}(G)]$. Let

$G' = G - uv$. Then, G' has a tnsd k -coloring ϕ' . We first erase the colors on u and v from ϕ' . Note that v is a 4^- -vertex. Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (7 - 1) \geq 3 \text{ and} \\ |S(uv)| &\geq k - (7 - 1) - (4 - 1) \geq 6. \end{aligned}$$

Let ϕ be a map on $T(G) \setminus \{v\}$ obtained from ϕ' by assigning $\phi(u) = x_1$, $\phi(uv) = y_1$, and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v, uv\}$. Set

$$P(x_1, y_1) = \prod_{z \in N_G(u) \setminus \{v\}} (m_\phi(u) - m_\phi(z))(x_1 - y_1).$$

By applying MATHEMATICA, we have $c_P(x_1^2 y_1^5) = -9$. By Theorem 4, there is a color $c_1 \in S(u)$ and a color $c_2 \in S(uv)$ such that $P(c_1, c_2) \neq 0$.

Since v is a 4^- -vertex, we can obtain a tnsd k -coloring of G from the ϕ' by recoloring u with color c_1 and coloring uv with color c_2 , respectively. This is a contradiction.

Next, we show $|[u, V_5(G)]| \leq 1$ when $u \in V_7(G)$. Suppose, to the contrary, that there is a vertex $u \in V_7(G)$ and two vertices $v_1, v_2 \in V_5(G)$ such that $uv_1, uv_2 \in [u, V_5(G)]$. Let $G' = G - uv_1 - uv_2$. Then, G' has a tnsd k -coloring ϕ' . We first erase the colors on u, v_1 , and v_2 from ϕ' . Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (7 - 2) \geq 5, \\ |S(uv_i)| &\geq k - (7 - 2) - (5 - 1) \geq 6 \text{ and} \\ |S(v_i)| &\geq k - 2 \times (5 - 1) \geq 7 \text{ for } i = 1, 2. \end{aligned}$$

Let ϕ be a map on $T(G)$ obtained from ϕ' by assigning $\phi(u) = x_1$, $\phi(v_i) = x_{i+1}$, and $\phi(uv_i) = y_i$ for $i = 1, 2$, and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v_1, v_2, uv_1, uv_2\}$. Set

$$\begin{aligned} P(x_1, x_2, x_3, y_1, y_2) &= \prod_{z \in N_G(u) \setminus \{v_1, v_2\}} (m_\phi(u) - m_\phi(z)) \prod_{i=1}^2 (m_\phi(u) - m_\phi(v_i)) \\ &\cdot \prod_{i=1}^2 \prod_{z \in N_G(v_i) \setminus \{u\}} (m_\phi(v_i) - m_\phi(z)) \prod_{i=1}^2 (x_1 - x_{i+1}) \\ &\cdot (y_1 - y_2) \prod_{i=1}^2 (x_1 - y_i)(x_{i+1} - y_i). \end{aligned}$$

With the help of MATHEMATICA, we have $c_P(x_1^4 x_2^4 x_3^5 y_1^4 y_2^5) = 120$. By Theorem 4, there is a color $c_1 \in S(u)$, a color $c_{i+1} \in S(v_i)$, and a color $c_{i+3} \in S(uv_i)$ for $i = 1, 2$ such that $P(c_1, \dots, c_5) \neq 0$. Therefore, we can extend the ϕ' to a tnsd k -coloring ϕ of G by recoloring u, v_i with colors c_1, c_{i+1} and coloring uv_i with color c_{i+3} for $i = 1, 2$, respectively. This is a contradiction. $\square$

Claim 5. If $u \in V_\ell(G)$ with $8 \leq \ell \leq 9$, then,

- (1) $|[u, V_{4^-}(G)]| \leq \ell - 7$,
- (2) $|[u, V_{5^-}(G)]| \leq \ell - 7$ when $|[u, V_{4^-}(G)]| \geq 1$.

Proof. Suppose to be contrary that there is a vertex $u \in V_\ell(G)$, a vertex $v_1 \in V_{4^-}(G)$, and $(\ell - 7)$ vertices $v_2, \dots, v_{\ell-6} \in V_{5^-}(G)$ with $8 \leq \ell \leq 9$ such that $uv_1 \in [u, V_{4^-}(G)]$ and $uv_i \in [u, V_{5^-}(G)]$ for $i = 2, \dots, \ell - 6$. Let $G' = G - \{uv_i | i = 1, \dots, \ell - 6\}$. Then, G' has a

tnsd k -coloring ϕ' . We first erase the colors on u and v_i with $1 \leq i \leq \ell - 6$ from ϕ' . Without loss of generality, set $d_G(v_i) = 5$ with $2 \leq i \leq \ell - 6$. Note that v_1 is a 4^- -vertex. Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (\ell - (\ell - 6)) \geq 3, \\ |S(uv_1)| &\geq k - (\ell - (\ell - 6)) - (4 - 1) \geq 6, \\ |S(uv_i)| &\geq k - (\ell - (\ell - 6)) - (5 - 1) \geq 5 \text{ and} \\ |S(v_i)| &\geq k - 2 \times (5 - 1) \geq 7 \text{ for } i = 2, \dots, \ell - 6. \end{aligned}$$

Let ϕ be a map on $T(G) \setminus \{v_1\}$ obtained from ϕ' by assigning

$$\phi(u) = x_1, \phi(v_2) = x_2, \dots, \phi(v_{\ell-6}) = x_{\ell-6}, \phi(uv_1) = y_1, \dots, \phi(uv_{\ell-6}) = y_{\ell-6}$$

and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v_1, \dots, v_{\ell-6}, uv_1, \dots, uv_{\ell-6}\}$. Set

$$\begin{aligned} P(x_1, \dots, x_{\ell-6}, y_1, \dots, y_{\ell-6}) &= \prod_{z \in N_G(u) \setminus \{v_1, \dots, v_{\ell-6}\}} (m_\phi(u) - m_\phi(z)) \prod_{i=2}^{\ell-6} (m_\phi(u) - m_\phi(v_i)) \\ &\quad \cdot \prod_{i=2}^{\ell-6} \prod_{z \in N_G(v_i) \setminus \{u\}} (m_\phi(v_i) - m_\phi(z)) \prod_{i=2}^{\ell-6} (x_1 - x_i)(x_i - y_i) \\ &\quad \cdot \prod_{i=1}^{\ell-6} (x_1 - y_i) \prod_{1 \leq i < j \leq \ell-6} (y_i - y_j). \end{aligned}$$

When $\ell = 8, \ell - 6 = 2$, and we have $c_P(x_1^2 x_2^6 y_1^4 y_2^4) = 20$ by applying MATHEMATICA. When $\ell = 9, \ell - 6 = 3$, and we have $c_P(x_1^2 x_2^6 x_3^5 y_1^4 y_2^4 y_3^4) = -59$ by applying MATHEMATICA. By Theorem 4, there is a color $c_1 \in S(u)$, a color $c_i \in S(v_i)$ with $2 \leq i \leq \ell - 6$, and a color $c_{i+\ell-6} \in S(uv_i)$ with $1 \leq i \leq \ell - 6$ such that $P(c_1, \dots, c_{2\ell-12}) \neq 0$ when $8 \leq \ell \leq 9$. Since v_1 is a 4^- -vertex, we can extend the ϕ' to a tnsd k -coloring of G by recoloring $u, v_2, \dots, v_{\ell-6}$ with colors $c_1, \dots, c_{\ell-6}$ and coloring $uv_1, \dots, uv_{\ell-6}$ with colors $c_{\ell-5}, \dots, c_{2\ell-12}$, respectively. This is a contradiction. $\square$

Claim 6. If $u \in V_\ell(G)$ with $10 \leq \ell \leq 11$, then

- (1) $|[u, V_{3^-}(G)]| \leq \ell - 7$,
- (2) $|[u, V_{4^-}(G)]| \leq \ell - 7$ when $|[u, V_{3^-}(G)]| \geq 1$,
- (3) $|[u, V_5(G)]| = 0$ when $|[u, V_{3^-}(G)]| \geq 1$ and $|[u, V_{4^-}(G)]| = \ell - 7$.

Proof. By contradiction, suppose that there is a vertex $u \in V_\ell(G)$, a vertex $v_1 \in V_{5^-}(G)$, a vertex $v_2 \in V_{3^-}(G)$, $(\ell - 8)$ vertices $v_3, \dots, v_{\ell-6} \in V_{4^-}(G)$ with $10 \leq \ell \leq 11$ such that $uv_1 \in [u, V_{5^-}(G)]$, $uv_2 \in [u, V_{3^-}(G)]$, and $uv_i \in [u, V_{4^-}(G)]$ for $i = 3, \dots, \ell - 6$. Let $G' = G - \{uv_i \mid i = 1, \dots, \ell - 6\}$. Then, G' has a tnsd k -coloring ϕ' . We first erase the colors on u and v_i with $1 \leq i \leq \ell - 6$ from ϕ' . Without loss of generality, set $d_G(v_1) = 5$. Note that v_2 is a 3^- -vertex and v_i is a 4^- -vertex with $3 \leq i \leq \ell - 6$. Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (\ell - (\ell - 6)) \geq 3, \\ |S(v_1)| &\geq k - 2 \times (5 - 1) \geq 7, \\ |S(uv_1)| &\geq k - (\ell - (\ell - 6)) - (5 - 1) \geq 5, \\ |S(uv_2)| &\geq k - (\ell - (\ell - 6)) - (3 - 1) \geq 7 \text{ and} \\ |S(uv_i)| &\geq k - (\ell - (\ell - 6)) - (4 - 1) \geq 6 \text{ for } i = 3, \dots, \ell - 6. \end{aligned}$$

Let ϕ be a map on $T(G) \setminus \{v_2, \dots, v_{\ell-6}\}$ obtained from ϕ' by assigning

$$\phi(u) = x_1, \phi(v_1) = x_2, \phi(uv_1) = y_1, \dots, \phi(uv_{\ell-6}) = y_{\ell-6}$$

and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v_1, \dots, v_{\ell-6}, uv_1, \dots, uv_{\ell-6}\}$. Set

$$\begin{aligned} P(x_1, x_2, y_1, \dots, y_{\ell-6}) &= \prod_{z \in N_G(u) \setminus \{v_1, \dots, v_{\ell-6}\}} (m_\phi(u) - m_\phi(z)) \prod_{z \in N_G(v_1) \setminus \{u\}} (m_\phi(v_1) - m_\phi(z)) \\ &\quad \cdot (m_\phi(u) - m_\phi(v_1))(x_1 - x_2)(x_2 - y_1) \\ &\quad \cdot \prod_{i=1}^{\ell-6} (x_1 - y_i) \prod_{1 \leq i < j \leq \ell-6} (y_i - y_j). \end{aligned}$$

When $\ell = 10$, $\ell - 6 = 4$, and we have $c_P(x_1^2 x_2^6 y_1^4 y_2^6 y_3^4 y_4) = -20$ with the help of MATHEMATICA. When $\ell = 11$, $\ell - 6 = 5$, and we have $c_P(x_1^2 x_2^6 y_1^4 y_2^6 y_3^5 y_4^4 y_5) = -4$ with the help of MATHEMATICA. By Theorem 4, there is a color $c_1 \in S(u)$, a color $c_2 \in S(v_1)$, and a color $c_{i+2} \in S(uv_i)$ with $1 \leq i \leq \ell - 6$ such that $P(c_1, \dots, c_{\ell-4}) \neq 0$ when $10 \leq \ell \leq 11$. Since v_i is a 4^- -vertex with $2 \leq i \leq \ell - 6$, we can obtain a tnsd k -coloring of G from the ϕ' by recoloring u, v_1 with colors c_1, c_2 and coloring $uv_1, \dots, uv_{\ell-6}$ with colors $c_3, \dots, c_{\ell-4}$, respectively. This is a contradiction. $\square$

Claim 7. If $u \in V_{12}(G)$, then

- (1) $|[u, V_{2^-}(G)]| \leq 5$,
- (2) $|[u, V_{3^-}(G)]| \leq 5$ when $|[u, V_{2^-}(G)]| \geq 1$,
- (3) $|[u, V_{4^-}(G)]| \leq 5$ when $|[u, V_{2^-}(G)]| \geq 1$ and $|[u, V_{3^-}(G)]| \geq 2$,
- (4) $|[u, V_5(G)]| = 0$ when $|[V_{2^-}(G), u]| \geq 1$, $|[V_{3^-}(G), u]| \geq 2$ and $|[u, V_{4^-}(G)]| = 5$.

Proof. Suppose to the contrary that there is a vertex $u \in V_{12}(G)$, a vertex $v_1 \in V_{5^-}(G)$, a vertex $v_2 \in V_{2^-}(G)$, a vertex $v_3 \in V_{3^-}(G)$, and three vertices $v_4, v_5, v_6 \in V_{4^-}(G)$ such that $uv_1 \in [u, V_{5^-}(G)]$, $uv_2 \in [u, V_{2^-}(G)]$, $uv_3 \in [u, V_{3^-}(G)]$, and $uv_i \in [u, V_{4^-}(G)]$ for $i = 4, 5, 6$. Let $G' = G - \{uv_i \mid i = 1, \dots, 6\}$. Then, G' has a tnsd k -coloring ϕ' . We first erase the colors on u and v_i with $1 \leq i \leq 6$ from ϕ' . Without loss of generality, set $d_G(v_1) = 5$. Note that v_2 is a 2^- -vertex, v_3 is a 3^- -vertex, and v_i is a 4^- -vertex with $4 \leq i \leq 6$. Then,

$$\begin{aligned} |S(u)| &\geq k - 2 \times (12 - 6) \geq 3, \\ |S(v_1)| &\geq k - 2 \times (5 - 1) \geq 7, \\ |S(uv_1)| &\geq k - (12 - 6) - (5 - 1) \geq 5, \\ |S(uv_2)| &\geq k - (12 - 6) - (2 - 1) \geq 8, \\ |S(uv_3)| &\geq k - (12 - 6) - (3 - 1) \geq 7 \text{ and} \\ |S(uv_i)| &\geq k - (12 - 6) - (4 - 1) \geq 6 \text{ for } i = 4, 5, 6. \end{aligned}$$

Let ϕ be a map on $T(G) \setminus \{v_2, \dots, v_6\}$ obtained from ϕ' by assigning

$$\phi(u) = x_1, \phi(v_1) = x_2, \phi(uv_1) = y_1, \dots, \phi(uv_6) = y_6$$

and $\phi(z) = \phi'(z)$ for every $z \in T(G) \setminus \{u, v_1, \dots, v_6, uv_1, \dots, uv_6\}$. Set

$$\begin{aligned} P(x_1, x_2, y_1, \dots, y_6) &= \prod_{z \in N_G(u) \setminus \{v_1, \dots, v_6\}} (m_\phi(u) - m_\phi(z)) \prod_{z \in N_G(v_1) \setminus \{u\}} (m_\phi(v_1) - m_\phi(z)) \\ &\quad \cdot (m_\phi(u) - m_\phi(v_1))(x_1 - x_2)(x_2 - y_1) \\ &\quad \cdot \prod_{i=1}^6 (x_1 - y_i) \prod_{1 \leq i < j \leq 6} (y_i - y_j). \end{aligned}$$

With the help of MATHEMATICA, we have $c_P(x_1^2 x_2^6 y_1^4 y_2^7 y_3^6 y_4^5 y_5^4) = -4$. By Theorem 4, there is a color $c_1 \in S(u)$, a color $c_2 \in S(v_1)$, and a color $c_{i+2} \in S(uv_i)$ with $1 \leq i \leq 6$ such that $P(c_1, \dots, c_8) \neq 0$. Since v_i is a 4^- -vertex with $2 \leq i \leq 6$, we can obtain a tnsd

k -coloring of G from the ϕ' by recoloring u, v_1 with colors c_1, c_2 and coloring $uv_1, \dots, uv_6$ with colors $c_3, \dots, c_8$, respectively. This is a contradiction. $\square$

Claim 8. Let $C_3 = v_1v_2v_3v_1$ be a 3-cycle in G . Then, $(d_G(v_1), d_G(v_2), d_G(v_3)) \neq (6^-, 6^-, 7^-)$.

The following Equation (1) follows from Claims 3~7.

$$\begin{cases} n_G^{5-}(u) = 0 & \text{if } d_G(u) \leq 6; \\ n_G^5(u) \leq 1 & \text{if } d_G(u) = 7; \\ n_G^{4-}(u) \leq d_G(u) - 7 & \text{if } 8 \leq d_G(u) \leq 9; \\ n_G^{3-}(u) \leq d_G(u) - 7 & \text{if } 10 \leq d_G(u) \leq 11; \\ n_G^{2-}(u) \leq 5 & \text{if } d_G(u) = 12. \end{cases} \quad (1)$$

Let $H = G - V_{2-}(G)$. Then, $d_H(u) = d_G(u) - n_G^{2-}(u)$ for each $u \in V(H)$.

Claim 9. If $u \in V(H)$, then,

- (1) $d_H(u) \geq 3$,
- (2) $d_H(u) = d_G(u)$ if $3 \leq d_G(u) \leq 7$,
- (3) $d_H(u) \geq 7$ if $8 \leq d_G(u) \leq 12$,
- (4) $n_H^{5-}(u) = 0$ if $d_H(u) = 7$ and $8 \leq d_G(u) \leq 12$,
- (5) $n_H^{5-}(u) = 0$ if $d_H(u) \leq 6$.

Proof. Note that $3 \leq d_G(u) \leq 12$ by the definition of H and Claim 2 for every $u \in V(H)$. By the definition of H , (1), (2), (3), and (4) follow from Equation (1). Below, we prove (5).

(5) Suppose that $n_H^{5-}(u) \geq 1$ when $d_H(u) \leq 6$. Without loss of generality, set $d_H(u) = 6$. Then, $d_G(u) = 6$ by (2). Thus, $n_G^{5-}(u) = 0$ by Claim 3, and u has at least one 7^+ -neighbor v in G with $d_H(v) \leq 5$ by (2). Hence, $d_H(v) = d_G(v) - n_G^{2-}(v) \geq 7$ by (3) and the definition of H , which is a contradiction. Therefore, the statement (5) holds. $\square$

By a similar discussion to that of Claim 9, we can obtain the following Claims 10~12.

Claim 10. If $d_H(u) = 7$, then $n_H^5(u) \leq 1$.

Claim 11. If $d_H(u) = \ell$ with $8 \leq \ell \leq 9$, then,

- (1) $n_H^{4-}(u) \leq \ell - 7$,
- (2) $n_H^{5-}(u) \leq \ell - 7$ when $n_H^{4-}(u) \geq 1$.

Claim 12. If $d_H(u) = \ell$ with $10 \leq \ell \leq 11$, then,

- (1) $n_H^{3-}(u) \leq \ell - 7$,
- (2) $n_H^{4-}(u) \leq \ell - 7$ when $n_H^{3-}(u) \geq 1$,
- (3) $n_H^5(u) = 0$ when $n_H^{3-}(u) \geq 1$ and $n_H^{4-}(u) = \ell - 7$.

By Claims 8 and 9, the below claim is immediate.

Claim 13. Let $C_3 = v_1v_2v_3v_1$ be a 3-cycle in H . Then,

- (1) $(d_H(v_1), d_H(v_2), d_H(v_3)) \neq (6^-, 6^-, 6^-)$,
- (2) $n_H^{5-}(v_3) = 0$ when $(d_H(v_1), d_H(v_2), d_H(v_3)) = (6, 6, 7)$.

From now on, we always assume that H has been embedded on the plane such that the number of crossings is as small as possible. Let $H^\times$ be the associated planar graph of H and $uv \in E(H^\times)$. If uv is incident with two 3-faces and u is a real ℓ -vertex in $H^\times$, we call u a *bad* ℓ -neighbor of v . If uv is incident with exactly one 3-face and u is a real 3-vertex in $H^\times$, we call u a *special* 3-neighbor of v . Let $n_{H^\times}^{\ell b}(v)$ ($n_{H^\times}^{3s}(v)$) denote the number of bad ℓ -neighbors (special 3-neighbors) of v in $H^\times$. Unless stated otherwise, an ℓ -vertex

represents a real ℓ -vertex in the following. A 4-face is called a *good* 4-face if it is incident at most a real 5^- -vertex and a *bad* 4-face otherwise. By Claims 1 and 9, the below Claims 14 and 15 are immediate.

Claim 14. If $d_H(u) = 12$, then,

- (1) $n_{H^\times}^{3b}(u) \leq 4$, $n_{H^\times}^{3s}(u) \leq 6$, $n_{H^\times}^{4b}(u) \leq 6$ and $n_{H^\times}^{5b}(u) \leq 6$,
- (2) $n_{H^\times}^{3s}(u) + n_{H^\times}^{4b}(u) + n_{H^\times}^{5b}(u) = 0$ when $n_{H^\times}^{3b}(u) = 4$.

Claim 15. Every k -face is incident with at most $\lfloor \frac{k}{2} \rfloor$ real 5^- -vertices in $H^\times$.

Let $\gamma(H^\times)$ denote the number of connected components of $H^\times$. Note that each plane graph has exactly one unbounded face. Then, we have

$$\sum_{f \in F(H^\times)} d_{H^\times}(f) = 2|E(H^\times)|$$

and Euler's formula

$$|V(H^\times)| - |E(H^\times)| + |F(H^\times)| = 1 + \gamma(H^\times).$$

Below, we will apply the discharging method [13] on the planar graph $H^\times$ to complete the proof of Theorem 3. Set $\omega(v) = d_{H^\times}(v) - 6$ for any $v \in V(H^\times)$ and $\omega(f) = 2d_{H^\times}(f) - 6$ for any $f \in F(H^\times)$. By Euler's formula,

$$|V(H^\times)| - |E(H^\times)| + |F(H^\times)| = 1 + \gamma(H^\times)$$

and

$$\sum_{v \in V(H^\times)} d_{H^\times}(v) = \sum_{f \in F(H^\times)} d_{H^\times}(f) = 2|E(H^\times)|.$$

One can obtain

$$\sum_{v \in V(H^\times)} (d_{H^\times}(v) - 6) + \sum_{f \in F(H^\times)} (2d_{H^\times}(f) - 6) = -6(1 + \gamma(H^\times)).$$

In order to redistribute charges on $T(H^\times)$ and keep the total charges unchanged, we make some discharging rules as follows.

(R1) Let uv be an edge of $H^\times$.

(R1.1) If u is a bad 3-neighbor of v , u receives 1 from v .

(R1.2) If u is a special 3-neighbor of v , u receives $\frac{1}{2}$ from v .

(R1.3) If u is a bad 4-neighbor of v , u receives $\frac{1}{2}$ from v .

(R1.4) If u is a bad 5-neighbor of v , u receives $\frac{1}{5}$ from v .

(R2) Each real ℓ -vertex with $3 \leq \ell \leq 5$ receives 1 from every 4^+ -face incident with it.

(R3) Let uv be an edge and u be a false 4-vertex of $H^\times$.

(R3.1) If $d_{H^\times}(v) = 7$, u receives 1 when v is not adjacent to any real 5^- -vertex and $\frac{4}{5}$ otherwise from v .

(R3.2) If $d_{H^\times}(v) = \ell$ with $8 \leq \ell \leq 12$, u receives 2 when $n_{H^\times}^3(v) < n_H^3(v)$ and 1 otherwise from v .

(R3.3) If f is a good 4-face or 5^+ -face incident with u , u receives 1 from f .

For each $z \in V(H^\times) \cup F(H^\times)$, we use $\omega'(z)$ to represent the new charge after applying the rules. Then, one can obtain

$$\sum_{z \in V(H^\times) \cup F(H^\times)} \omega'(z) = \sum_{v \in V(H^\times)} (d_{H^\times}(v) - 6) + \sum_{f \in F(H^\times)} (2d_{H^\times}(f) - 6) = -6(1 + \gamma(H^\times)) < 0.$$

Thus, there exists a vertex or a face whose charge is negative.

Firstly, it is easy to verify that the new charge of every face is non-negative by Claim 15 and the discharging rules.

Next, we show that the new charge of every real vertex is non-negative. Pick arbitrarily a real vertex v from $V(H)$. Note that v is adjacent to at most a false 4-vertex as G (and thus, H) is an IC-planar graph. Note also that $3 \leq d_H(v) = d_{H^\times}(v) \leq 12$ and $n_{H^\times}^3(v) \leq n_H^3(v)$ by the definition of $H^\times$ and Claim 9.

(1) For $v \in V_3(H)$. If v is incident with three 4^+ -faces, then $\omega'(v) = 3 - 6 + 3 \times 1 = 0$ by (R2). If v is incident with two 4^+ -faces, it is a common special 3-neighbor of two neighbors, and so $\omega'(v) = 3 - 6 + 2 \times 1 + 2 \times \frac{1}{2} = 0$ by (R1.2) and (R2). If v is incident with one 4^+ -face, then it is a special 3-neighbor of one neighbor and a common bad 3-neighbor of the other two neighbors, respectively, and so $\omega'(v) = 3 - 6 + 1 \times 1 + 1 \times 1 + 2 \times \frac{1}{2} = 0$ by (R1.1), (R1.2), and (R2). If v is not incident with any 4^+ -faces, it is a common bad 3-neighbor of three neighbors; so, $\omega'(v) = 3 - 6 + 3 \times 1 = 0$ by (R1.1).

(2) For $v \in V_4(H)$. If v is incident with at least two 4^+ -faces, $\omega'(v) \geq 4 - 6 + 2 \times 1 = 0$ by (R2). If v is incident with one 4^+ -face, it is a common bad 4-neighbor of two neighbors; so, $\omega'(v) = 4 - 6 + 1 \times 1 + 2 \times \frac{1}{2} = 0$ by (R1.3) and (R2). If v is not incident with any 4^+ -faces, it is a common bad 4-neighbor of four neighbors; so, $\omega'(v) = 4 - 6 + 4 \times \frac{1}{2} = 0$ by (R1.3).

(3) For $v \in V_5(H)$. If v is incident with at least one 4^+ -faces, $\omega'(v) \geq 5 - 6 + 1 \times 1 = 0$ by (R2). If v is not incident with any 4^+ -faces, it is a common bad 5-neighbor of five neighbors; so, $\omega'(v) = 5 - 6 + 5 \times \frac{1}{5} = 0$ by (R1.4).

(4) For $v \in V_6(H)$. $\omega'(v) = 6 - 6 = 0$ as no rule is applied to it.

(5) For $v \in V_7(H)$. Note that v is adjacent to at most a real 5-vertex in H (and so, in $H^\times$) by Claim 10. If v is not adjacent to any real 5-vertex in $H^\times$, $\omega'(v) \geq 7 - 6 - 1 \times 1 = 0$ by (R3.1). If v is adjacent to a real 5-vertex in $H^\times$, $\omega'(v) \geq 7 - 6 - 1 \times \frac{4}{5} - 1 \times \frac{1}{5} = 0$ by (R1.4) and (R3.1).

(6) For $v \in V_\ell(H)$ with $8 \leq \ell \leq 9$. If $n_{H^\times}^3(v) = n_H^3(v)$, $\omega'(v) \geq \ell - 6 - 1 \times 1 - \max\{(\ell - 7) \times 1, \lfloor \frac{\ell}{2} \rfloor \times \frac{1}{5}\} = 0$ by (R1), (R3.2), and Claim 11. If $n_{H^\times}^3(v) < n_H^3(v)$, $n_H^3(v) \geq 1$; so, v is adjacent to at most $(\ell - 8)$ real 5^- -vertices in $H^\times$ by Claim 11 and the definition of $H^\times$. Thus, $\omega'(v) \geq \ell - 6 - 1 \times 2 - (\ell - 8) \times 1 = 0$ by (R1) and (R3.2).

(7) For $v \in V_\ell(H)$ with $10 \leq \ell \leq 11$. If $n_{H^\times}^3(v) = n_H^3(v)$, $\omega'(v) \geq \ell - 6 - 1 \times 1 - \max\{(\ell - 7) \times 1, \lfloor \frac{\ell}{2} \rfloor \times \frac{1}{2}\} = 0$ by (R1), (R3.2), and Claim 12. If $n_{H^\times}^3(v) < n_H^3(v)$, $n_H^3(v) \geq 1$; so, v is adjacent to at most $(\ell - 8)$ real 4^- -vertices in $H^\times$ by Claim 12 and the definition of $H^\times$. Thus, $\omega'(v) \geq \ell - 6 - 1 \times 2 - \max\{(\ell - 8) \times 1, \lfloor \frac{\ell}{2} \rfloor \times \frac{1}{5}\} = 0$ by (R1) and (R3.2).

(8) For $v \in V_{12}(H)$. By Claim 14, $\omega'(v) \geq 12 - 6 - 1 \times 2 - \max\{4 \times 1, \lfloor \frac{\ell}{2} \rfloor \times \frac{1}{2}\} = 0$ by (R1) and (R3.3).

Hence, the new charge of every real vertex is non-negative.

Finally, we prove the new charge of every false 4-vertex is non-negative. Choose arbitrarily a false 4-vertex u from $V(H^\times) \setminus V(H)$. Note that u is adjacent to at most two 5^- -vertices in $H^\times$ by the definition of $H^\times$ and Claim 9. Assume that the edges v_1v_2 and v_3v_4 intersect at u in H . Then, $N_{H^\times}(u) = \{v_1, v_2, v_3, v_4\}$. The subgraph of $H^\times$ induced by $N_{H^\times}(u) \cup \{u\}$ is denoted by $H^\times[N_{H^\times}(u) \cup \{u\}]$. Note that if v_i (v_{i+2}) is a 5^- -vertex with $i = 1, 2$, v_{i+2} (v_i) is a 7^+ -vertex and v_{i+2} (v_i) is not adjacent to any real 5^- -vertex when v_{i+2} (v_i) is a 7-vertex in $H^\times$ by the definition of $H^\times$ and Claim 9.

(9) Let $H^\times[N_{H^\times}(u) \cup \{u\}]$ contain no 3-face (see Figure 3). Then, u is incident with at least three faces, each of which either is a good 4-face or a 5^+ -face since $n_{H^\times}^{5^-}(u) \leq 2$ by Claim 9; so, $\omega'(u) \geq 4 - 6 + 3 \times 1 = 1$ by (R3.3).

(10) Let $H^\times[N_{H^\times}(u) \cup \{u\}]$ contain exactly one 3-face (see Figure 4). Note that $n_{H^\times}^{5^-}(u) \leq 2$ and at most one of $d_{H^\times}(v_1)$ and $d_{H^\times}(v_2)$ is equal to 3 by Claim 9. If $d_{H^\times}(v_1) \neq 3$ and $d_{H^\times}(v_2) \neq 3$, $\omega'(u) \geq 4 - 6 + 2 \times 1 = 0$ by (R3.3) since u is incident with at least two faces, each of which either is a good 4-face or a 5^+ -face. If $d_{H^\times}(v_1)$ is equal to 3, $\omega'(u) \geq 4 - 6 + 2 \times 1 + 1 \times 1 - 1 \times \frac{1}{2} = \frac{1}{2}$ by (R1.2) and (R3) since u is incident

with at least two faces, each of which either is a good 4-face or a 5^+ -face. By symmetry, $\omega'(u) \geq \frac{1}{2}$ when $d_{H^\times}(v_2)$ is equal to 3.

Figure 3. $H^\times[\{u\} \cup N_{H^\times}(u)]$ contains no 3-face.

Figure 4. $H^\times[\{u\} \cup N_{H^\times}(u)]$ contains one 3-face.

(11) Let $H^\times[N_{H^\times}(u) \cup \{u\}]$ contain two 3-faces (see Figure 5). Note that $n_{H^\times}^{5^-}(u) \leq 2$ by Claim 9 and v_1 of (b) in Figure 5 is not a 3-vertex by Lemma 1. If $n_{H^\times}^{5^-}(u) = 0$, $\omega'(u) \geq 4 - 6 + 2 \times 1 = 0$ by (R3.3) since u is incident with two faces, each of which either is a good 4-face or a 5^+ -face. If $n_{H^\times}^{5^-}(u) = 1$, $\omega'(u) \geq 4 - 6 + 2 \times 1 + 1 \times 1 - 1 \times \frac{1}{2} = \frac{1}{2}$ by (R1.2) and (R3.3), since u is incident with two faces, each of which either is a good 4-face or a 5^+ -face. If $n_{H^\times}^{5^-}(u) = 2$, $\omega'(u) \geq 4 - 6 + 1 \times 1 + 2 \times 1 - 2 \times \frac{1}{2} = 0$ by (R1.2) and (R3.3) since u is incident with one face, which either is a good 4-face or a 5^+ -face.

Figure 5. $H^\times[\{u\} \cup N_{H^\times}(u)]$ contains two 3-faces; (a) two nonadjacent 3-faces; (b) two adjacent 3-faces.

(12) Let $H^\times[N_{H^\times}(u) \cup \{u\}]$ contain three 3-faces (see Figure 6). Note that $n_{H^\times}^{5^-}(u) \leq 2$ by Claim 9 and neither v_1 nor v_2 is a 3-vertex by Lemma 1 in Figure 6. If $n_{H^\times}^{5^-}(u) = 0$, at least one neighbor of u is a 7^+ -vertex, and when one neighbor of u is 7-vertex, it is not adjacent to any real 5^- -vertex by Claim 13; so, $\omega'(u) \geq 4 - 6 + 1 \times 1 + 1 \times 1 = 0$ by (R1) and (R3.3) since u is incident with one face, which either is a good 4-face or a 5^+ -face. If $n_{H^\times}^{5^-}(u) = 1$ and $n_{H^\times}^3(u) = 0$, u is adjacent to at least two 7^+ -vertices; so, $\omega'(u) \geq 4 - 6 + 1 \times 1 + 2 \times \frac{4}{5} - 1 \times \frac{1}{2} = \frac{1}{10}$ by (R1.3), (R1.4), and (R3.3) since u is incident with one face, which either

is a good 4-face or a 5^+ -face. If $n_{H^\times}^{5^-}(u) = 1$ and $n_{H^\times}^3(u) = 1$, $d_{H^\times}(v_3) = 3$ or $d_{H^\times}(v_4) = 3$. Since v_3 and v_4 are symmetrical in Figure 6, we can set $d_{H^\times}(v_3) = 3$. Then, $d_{H^\times}(v_1) \geq 8$, $d_{H^\times}(v_2) \geq 8$, and $n_{H^\times}^3(v_1) < n_H^3(v_1)$; so, $\omega'(u) \geq 4 - 6 + 1 \times 1 + 1 \times 2 - 1 \times \frac{1}{2} = \frac{1}{2}$ by (R1.2), (R3.2), and (R3.3) since u is incident with one face, which either is a good 4-face or a 5^+ -face. If $n_{H^\times}^{5^-}(u) = 2$, $d_{H^\times}(v_3) \leq 5$, $d_{H^\times}(v_4) \leq 5$, $d_{H^\times}(v_1) \geq 8$, and $d_{H^\times}(v_2) \geq 8$ by Claims 9 and 10. If $n_{H^\times}^3(u) = 0$, $\omega'(u) \geq 4 - 6 + 2 \times 1 = 0$ by (R3.2). If $n_{H^\times}^3(u) = 1$, then we may as well set $d_{H^\times}(v_3) = 3$ since v_3 and v_4 are symmetrical in Figure 6. Thus, $n_{H^\times}^3(v_1) < n_H^3(v_1)$; so, $\omega'(u) \geq 4 - 6 + 1 \times 1 + 1 \times 2 - 1 \times \frac{1}{2} = \frac{1}{2}$ by (R1.2) and (R3.2). If $n_{H^\times}^3(u) = 2$, $d_{H^\times}(v_3) = d_{H^\times}(v_4) = 3$. Thus, $n_{H^\times}^3(v_1) < n_H^3(v_1)$ and $n_{H^\times}^3(v_2) < n_H^3(v_2)$. So, $\omega'(u) \geq 4 - 6 + 2 \times 2 - 2 \times \frac{1}{2} = 1$ by (R1.2) and (R3.2).

Figure 6. $H^\times[\{u\} \cup N_{H^\times}(u)]$ contains three 3-faces.

(13) Let $H^\times[N_{H^\times}(u) \cup \{u\}]$ contain four 3-faces (see Figure 7). Then, $d_{H^\times}(v_i) \neq 3$ for $i = 1, 2, 3, 4$ by Lemma 1. Note that $n_{H^\times}^{5^-}(u) \leq 1$ by Claim 9. If $n_{H^\times}^{5^-}(u) = 0$ and $n_{H^\times}^{8^+}(u) = 0$, then u is adjacent to at least two 7-vertices by Claim 13. When u is adjacent to two 7-vertices, each of the two 7-vertices is not adjacent to any real 5^- -vertex by Claim 13, and so $\omega'(u) \geq 4 - 6 + 2 \times 1 = 0$ by (R3.1). When u is adjacent to at least three 7-vertices, $\omega'(u) \geq 4 - 6 + 3 \times \frac{4}{5} = \frac{2}{5}$ by (R3.1). If $n_{H^\times}^{5^-}(u) = 0$ and $n_{H^\times}^{8^+}(u) = 1$, we may as well set $d_{H^\times}(v_1) \geq 8$ by symmetry. Then, at least one of v_2, v_3 and v_4 is a 7-vertex by Claim 13. When one of v_2, v_3 and v_4 is a 7-vertex, this 7-vertex is not adjacent to any real 5^- -vertex by Claim 13, and so $\omega'(u) \geq 4 - 6 + 1 \times 1 + 1 \times 1 = 0$ by (R3.1) and (R3.2). When at least two of v_2, v_3 and v_4 are 7-vertices, $\omega'(u) \geq 4 - 6 + 2 \times \frac{4}{5} + 1 \times 1 = \frac{3}{5}$ by (R3.1) and (R3.2). If $n_{H^\times}^{5^-}(u) = 0$ and $n_{H^\times}^{8^+}(u) \geq 2$, $\omega'(u) \geq 4 - 6 + 2 \times 1 = 0$ by (R3.2). If $n_{H^\times}^{5^-}(u) = 1$ and $n_{H^\times}^4(u) = 1$, u is adjacent to three 8^+ -vertices by Claim 9; so, $\omega'(u) \geq 4 - 6 + 3 \times 1 - \frac{1}{2} = \frac{1}{2}$ by (R1.3) and (R3.2). If $n_{H^\times}^{5^-}(u) = 1$ and $n_{H^\times}^4(u) = 0$, u is adjacent to three 7^+ -vertices by Claim 9; so, $\omega'(u) \geq 4 - 6 + 3 \times \frac{4}{5} - \frac{1}{5} = \frac{1}{5}$ by (R1.4), (R3.1), and (R3.2).

Figure 7. $H^\times[\{u\} \cup N_{H^\times}(u)]$ contains four 3-faces.

Hence, the new charge of every false 4-vertex is non-negative.

By the above analysis, we obtain that the new charge of every member in $V(H^\times) \cup F(H^\times)$ is non-negative. This is a contradiction. The proof of Theorem 3 is completed.

4. Conclusions

In this paper, we proved that any IC-planar graph with $\Delta \geq 12$ satisfies the tnsd index conjecture by the discharging method and Combinatorial Nullstellensatz. Whether the bound $\Delta \geq 12$ can be improved is still open for future work. The difficulty in improving this bound is how to reduce the number of bad 3-neighbors of one vertex. Therefore, finding suitable technology to reduce the number of bad 3-neighbors of one vertex is our main future work in the study of the tnsd index of IC-planar graph.

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