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The Inequalities of Merris and Foregger for Permanents

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Abstract: Conjectures on permanents are well-known unsettled conjectures in linear algebra. Let A be an $n \times n$ matrix and S_n be the symmetric group on n element set. The permanent of A is defined as $\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$. The Merris conjectured that for all $n \times n$ doubly stochastic matrices (denoted by Ω_n), $n\text{per}A \geq \min_{1 \leq i \leq n} \sum_{j=1}^{n} \text{per}A(j|i)$, where $A(j|i)$ denotes the matrix obtained from A by deleting the jth row and ith column. Foregger raised a question whether $\text{per}(tJ_n + (1-t)A) \leq \text{per}A$ for $0 \leq t \leq \frac{n}{n-1}$ and for all $A \in \Omega_n$, where J_n is a doubly stochastic matrix with each entry $\frac{1}{n}$. The Merris conjecture is one of the well-known conjectures on permanents. This conjecture is still open for $n \geq 4$. In this paper, we prove the Merris inequality for some classes of matrices. We use the sub permanent inequalities to prove our results. Foregger's inequality is also one of the well-known inequalities on permanents, and it is not yet proved for $n \geq 5$. Using the concepts of elementary symmetric function and subpermanents, we prove the Foregger's inequality for $n = 5$ in [0.25, 0.6248]. Let $\sigma_k(A)$ be the sum of all subpermanents of order k. Holens and Dokovic proposed a conjecture (Holen–Dokovic conjecture), which states that if $A \in \Omega_n, A \neq J_n$ and k is an integer, $1 \leq k \leq n$, then $\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A)$. In this paper, we disprove the conjecture for $n = k = 4$.

Keywords: doubly stochastic matrices; permanent; Merris conjecture; Foregger's inequality

Citation: Udayan, D.K.; Somasundaram, K. The Inequalities of Merris and Foregger for Permanents. *Symmetry* **2021**, *13*, 1782. https://doi.org/10.3390/sym13101782

Academic Editors: Calogero Vetro and Juan Luis García Guirao

Received: 12 August 2021
Accepted: 18 September 2021
Published: 23 September 2021

1. Introduction

Let S_n be the symmetric group on n element set and let A be an $n \times n$ matrix. The permanent of A is defined as

$$\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

A matrix A is said to be doubly stochastic if it is a real non-negative matrix with each row sum and column sum equal to 1. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. For positive integers n and k with $(1 \leq k \leq n)$, $Q_{k,n}$ denotes the set $\{(i_1, \ldots, i_k)/1 \leq i_1 < \ldots < i_k \leq n\}$. For $\alpha, \beta \in Q_{k,n}$, let $A(\alpha/\beta)$ be the submatrix of A obtained by deleting the rows indexed by α and columns indexed by β and $A[\alpha/\beta]$ be the submatrix of A with rows and columns indexed by α and β, respectively.

For $1 \leq k \leq n$, the kth order subpermanent of A is defined by $\sigma_k(A) = \sum_{\alpha,\beta \in Q_{k,n}} \text{per}A[\alpha/\beta]$.

In this paper, we use the following results quoted by Minc [1]: If A and B are two $n \times n$ matrices and $1 \leq k \leq n$, then

$$\text{per}A = \sum_{\beta \in Q_{k,n}} \text{per}A[\alpha/\beta]\text{per}A(\alpha/\beta), \text{for } \alpha \in Q_{k,n},$$

and

$$\text{per}(A+B) = \sum_{k=0}^{n} S_k(A,B), \tag{1}$$

where $S_k(A,B) = \sum_{\alpha,\beta \in Q_{k,n}} \mathrm{per}A[\alpha/\beta]\mathrm{per}B(\alpha/\beta)$, $\mathrm{per}A[\alpha/\beta] = 1$ when $k = 0$ and per $B(\alpha/\beta) = 1$ when $k = n$.

Elliott H. Lieb [2] gave proofs of some conjectures on permanents. S G Hwang [3] proved that $J_n = (\frac{1}{n})_{n\times n}$ is the unique ϕ-maximizing matrix on K_n. Lih and Wang [4] proved the monotonicity conjecture for $n = 3$. A survey on conjectures on permanents are given in [5,6].

Merris [7] conjectured that if $A \in \Omega_n$ then $n\mathrm{per}A \geq \min_{1\leq i\leq n} \sum_{j=1}^{n} \mathrm{per}A(j|i)$. He also suggested a method to prove this conjecture. The conjecture is still open for $n \geq 4$. Subramanian and Somasundaram [8] have proved that if $A \in \Omega_n$ and the polynomial $\sum_{r=2}^{n} r\frac{(n-r)!}{n^{n-r}}\sigma_r(A - J_n)t^{r-2}$ has no root in $(0,1)$ then A satisfies Merris conjecture. Furthermore, they proved some sufficient conditions for matrices in Γ_k^n to satisfy the Merris conjecture, where Γ_k^n denote the set of $n \times n$ non-negative matrices with each row sum and column sum equal to k.

In Section 2, we prove the Merris inequality for all $n \times n$ non-negative matrices with minimum entry greater than or equal to $\frac{1}{n}$. We prove that if A is an $n \times n$ non-negative matrix with minimum entry greater than or equal to $\frac{1}{n}$ and maximum entry less than or equal to 1, then $n^2\ perA \geq \sum \lambda_i$, where $\lambda_i's$ are the eigenvalues of $[a_{ij}\ perA(i|j)]$. Furthermore, we give a sufficient condition for a doubly stochastic matrix A to satisfy the Merris conjecture.

Foregger [9] raised a question whether $\mathrm{per}(tJ_n + (1-t)S) \leq \mathrm{per}S$ for $0 \leq t \leq \frac{n}{n-1}$, and $S \in \Omega_n$. He proved in [9] that for $n = 3$, $\mathrm{per}(tJ_3 + (1-t)S \leq \mathrm{per}S$ for $0 \leq t \leq \frac{3}{2}$ for $S \in \Omega_3$ with equality iff $S = J_3$ or $t = \frac{3}{2}$ and S is (up to permutations of rows and columns) $\frac{1}{2}(I+P)$, where P is a full-cycle permutation matrix. In addition, he proved in [10] that if $S \in \Omega_4$ has all its off-diagonal entries less than or equal to $\frac{9}{20}$ and $t_0 < t \leq \frac{4}{3}$, where t_0 is the unique real root of $106t^3 - 418t^2 + 465t - 100$ then $\mathrm{per}(tJ_4 + (1-t)S) \leq \mathrm{per}S$ with equality if and only if $S = J_4$.

Subramanian and Somasundaram [8] proved that if $A \in \Omega_n, 2 \leq k \leq n$, and the polynomial $\sum_{r=2}^{k} rc_r\sigma_r(A - J_n)t^{r-2}$ has no root in $(0,1)$, where $c_r = \frac{(k-r)!}{n^{k-r}}\binom{n-r}{k-r}^2$, then $\sigma_k(tA + (1-t)J_n) \leq \sigma_k(A)$ for all $t \in [0,1]$. In Section 3, we prove that for all $S \in \Omega_5$ and all t such that $0.25 \leq t \leq 0.6248$, $\mathrm{per}(tJ_5 + (1-t)S) \leq \mathrm{per}S$.

Holens [11] and Dokovic [12] proposed a conjecture (Holen–Dokovic conjecture), which states that if $A \in \Omega_n, A \neq J_n$ and k is an integer, $1 \leq k \leq n$, then $\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A)$. S G Hwang [13] proved the conjecture for an $n-2$ dimensional face of Ω_n. Wanless [14] disproved this conjecture by providing a counterexample of order 22. The smallest order of a counterexample has not been established. In Section 3, we prove that the Holen–Dokovic conjecture fails for $n = k = 4$ and thus established that the smallest order of a counterexample to Holen–Dokovic conjecture is 4.

2. Merris Conjecture

Let Γ_k^n denote the set of $n \times n$ non-negative matrices with each row sum and column sum equal to k. Merris [7] conjectured that for all $n \times n$ doubly stochastic matrices,

$$n\mathrm{per}A \geq \min_{1\leq i\leq n} \sum_{j=1}^{n} \mathrm{per}A(j|i).$$

He also raised a question whether

$$n\mathrm{per}A \geq \max_{1\leq i\leq n} \sum_{j=1}^{n} \mathrm{per}A(j|i) \;\text{ for all }\; A \in \Omega_n.$$

The Merris conjecture is one of the well-known conjectures in linear algebra, in particular on permanent. The conjecture is still open for $n \geq 4$. There is not much progress in this conjecture. Subramanian and Somasundaram [8] have proved that if $A \in \Omega_n$ and the polynomial $\sum_{r=2}^{n} r\frac{(n-r)!}{n^{n-r}}\sigma_r(A-J_n)t^{r-2}$ has no root in $(0,1)$ then A satisfies the Merris conjecture, and they also proved some sufficient conditions for matrices in Γ_k^n to satisfy the Merris conjecture.

A matrix is said to be a positive matrix if all its entries are non-negative [15]. Let A_i be $k \times k$ matrix, $i = 1, 2, \ldots, n$. The direct sum of the matrices A_i is defined as follows: $\bigoplus_{i=1}^{n} A_i = \mathrm{diag}(A_1, A_2, \ldots, A_n) = \begin{pmatrix} A_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & A_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \\ \mathbf{0} & \mathbf{0} & \ldots & A_n \end{pmatrix}$, where $\mathbf{0}$ is the zero matrix.

Lemma 1. *If A is a $n \times n$ positive matrix with minimum entry greater than or equal to $\frac{1}{n}$, then*

1. $nperA \geq \max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} perA(j|i)$.
2. $nperA \geq \min\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} perA(j|i)$.
3. $n^2 perA \geq \sum\limits_{i,j=1}^{n} perA(i|j)$.

Proof.

1. We need to find $\max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} \mathrm{per}A(j|i)$. Let the maximum sum be attained in the kth column, i.e., $\max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} \mathrm{per}A(j|i) = \sum_{j=1}^{n} \mathrm{per}A(j|k)$, where $1 \leq k \leq n$. Let the entries of the kth column be $k_1, k_2, \ldots, k_n$. This implies that $\frac{1}{n} \leq k_l$ for each $l = 1, 2, \ldots, n$ and hence $1 \leq nk_l$ for each $l = 1, 2, \ldots, n$. Taking the permanent along the kth column, $\mathrm{per}A = \sum\limits_{i=1}^{n} k_i \mathrm{per}(i|k)$. Multiplying by n on both sides, $n\mathrm{per}A = \sum\limits_{i=i}^{n} nk_i \mathrm{per}A(i|k)$. Since $nk_l \geq 1$ for each $l = 1, 2, \ldots, n$ and since each of the subpermanents is non-negative, this implies that $n\mathrm{per}A \geq \sum\limits_{i=1}^{n} \mathrm{per}A(i|k)$. This implies that $n\mathrm{per}A \geq \max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} \mathrm{per}A(j|i)$.
2. From the inequality 1, $n\mathrm{per}A \geq \min\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} \mathrm{per}A(j|i)$.
3. From the proof of the inequality 1, $n\mathrm{per}A \geq \sum\limits_{j=1}^{n} \mathrm{per}A(j|i)$ for each $i = 1, 2, \ldots, n$. Taking summation over i, i running from 1 to n, $n^2\mathrm{per}A \geq \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \mathrm{per}A(j|i)$. $\Rightarrow$ $n^2\mathrm{per}A \geq \sum\limits_{i,j=1}^{n} \mathrm{per}A(i|j)$.

□

Theorem 1. *If A is a $n \times n$ positive matrix with constant columns and maximum entry greater than or equal to $\frac{1}{n}$ then A satisfies the inequality*

$$nperA \geq \min_{1 \leq i \leq n} \sum_{j=1}^{n} perA(j|i).$$

Proof. Suppose $A = (a_{ij}) = k_j$, for all i. Then per$A = n!k_1k_2 \ldots k_n$. $\sum_{j=1}^{n}$ per$A(j|i) = n(n-1)!k_1k_2 \ldots k_{i-1}k_{i+1} \ldots k_n = n!k_1k_2 \ldots k_{i-1}k_{i+1} \ldots k_n = \frac{\text{per}A}{k_i}$. $\min_{1\le i\le n} \sum_{j=1}^{n} \text{per}A(j|i) = \frac{\text{per}A}{k_l}$, where $k_l = \max\{k_1, k_2, \ldots k_n\}$. Since, $k_l \geq \frac{1}{n}$, we have $n\text{per}A - \min_{1\le i\le n} \sum_{j=1}^{n} perA(j|i) \geq \frac{\text{per}A}{k_l} - \min_{1\le i\le n} \sum_{j=1}^{n} \text{per}A(j|i) = 0$. Therefore, $n\text{per}A \geq \min_{1\le i\le n} \sum_{j=1}^{n} \text{per}A(j|i)$. □

Theorem 2. *If A is a $n \times n$ matrix whose minimum entry is greater than or equal to $\frac{1}{n}$ and maximum entry is less than or equal to 1 then $n^2 perA \geq \sum \lambda_i$, where λ_i is an eigenvalue of $[a_{ij}perA(i|j)]$.*

Proof. If A is an $n \times n$ non-negative matrix whose minimum entry is greater than or equal to $\frac{1}{n}$ then from Lemma 1, n^2 per$A \geq \sum_{i,j=1}^{n}$ per$A(i|j)$.
$\Rightarrow \text{per}A \geq \frac{1}{n^2} \sum_{i,j=1}^{n} \text{per}A(i|j)$.
Let a_{lm} be the maximum entry of A. Multiplying on both sides by a_{lm},
a_{lm} per$A \geq \frac{1}{n^2} \sum_{i,j=1}^{n} a_{lm}\text{per}A(i|j) \geq \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij}\text{per}A(i|j)$.
$\Rightarrow \text{per}A \geq \frac{1}{n^2 a_{lm}} \sum_{i,j=1}^{n} a_{ij}\text{per}A(i|j)$.
By the assumption $a_{lm} \leq 1, \Rightarrow \frac{1}{a_{lm}} \geq 1$.
$\Rightarrow \text{per}A \geq \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij}\text{per}A(i|j) \geq \frac{1}{n^2} \sum_{i=1}^{n} a_{ii}\text{per}A(i|i)$.
$\Rightarrow n^2 \text{ per}A \geq tr([a_{ij}\text{per}A(i|j)])$.
$\Rightarrow n^2\text{per}A \geq$ Sum of eigenvalues of $[a_{ij}\text{per}A(i|j)]$. □

Theorem 3. *Let $A \in \Omega_n$ and $P = (perA(i/j)) = (p_{ij})$. If kth row of P gives the* $\max_i \sum_{j=1}^{n} p_{ij}$ *and $min\{a_{kj}\} = \frac{1}{n}$ then*

$$nperA \geq \min_{1\le i\le n} \sum_{j=1}^{n} perA(i|j).$$

Proof. $\min_i \sum_{j=1}^{n} p_{ij} \leq \max_i \sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} p_{kj} \leq n \sum_{j=1}^{n} a_{kj}p_{kj} = n\text{per}A$, since $a_{kj} \geq \frac{1}{n}$. □

Example 1. $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \bigoplus \begin{pmatrix} \frac{6}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{6}{7} \end{pmatrix}$, *where $\bigoplus$ is the direct sum.*

It is easy to see that $P = \begin{pmatrix} \frac{74}{441} & \frac{74}{441} & \frac{74}{441} \\ \frac{74}{441} & \frac{74}{441} & \frac{74}{441} \\ \frac{74}{441} & \frac{74}{441} & \frac{74}{441} \end{pmatrix} \bigoplus \begin{pmatrix} \frac{4}{21} & \frac{2}{63} \\ \frac{2}{63} & \frac{4}{21} \end{pmatrix}$. *Maximum row sum of* $P = \frac{74}{147}$ *and the minimum element of the row corresponding to the maximum row sum of* $P = \frac{1}{3} > \frac{1}{5}$. *Therefore,* $5perA \geq \min_{1\le i\le 5} \sum_{j=1}^{5} perA(i|j)$.

3. Foregger's Inequality

Let J_n denote the $n \times n$ matrix with each entry equal to $\frac{1}{n}$. Several authors have considered the problem of finding an upper bound for the permanent of a convex combination of J_n and S, where $S \in \Omega_n$. Lih and Wang [16] discussed convexity inequality on the permanent of doubly stochastic matrices. For example, Marcus and Minc conjectured [17] that if $S \in \Omega_n, n \geq 2$, then $\text{per}(\frac{nJ_n - S}{n-1}) \leq \text{per}S$, equality holds when $n = 2$. If $n \geq 3$ then inequality holds iff $S = J_n$. They established in [17] that the conjecture is true for $n = 2$, or if S is positive semi-definite symmetric, or if S is in a sufficiently small neighborhood of J_n.

E.T.H.Wang conjectured [18] that $\text{per}(\frac{nJ_n + S}{n+1}) \leq \text{per}S$ and proved the Marcus and Minc conjecture for $n = 3$, with a revised statement of the case of equality.

Foregger [9] raised a question whether $\text{per}(tJ_n + (1-t)S) \leq \text{per}S$ for $0 \leq t \leq \frac{n}{n-1}$, and $S \in \Omega_n$. He proved in [9] that for $n = 3$, $\text{per}(tJ_3 + (1-t)S \leq \text{per}S$ for $0 \leq t \leq \frac{3}{2}$ for $S \in \Omega_3$ with equality iff $S = J_3$ or $t = \frac{3}{2}$ and S is (up to permutations of rows and columns) $\frac{1}{2}(I + P)$, where P is a full-cycle permutation matrix. In addition, he proved in [10] that if $S \in \Omega_4$ has all its off-diagonal entries less than or equal to $\frac{9}{20}$ and $t_0 < t \leq \frac{4}{3}$, where t_0 is the unique real root of $106t^3 - 418t^2 + 465t - 100$ then $\text{per}(tJ_4 + (1-t)S) \leq \text{per}S$ with equality iff $S = J_4$. This Foregger inequality is not yet proved for $n \geq 5$.

Subramanian and Somasundaram [8] proved that if $A \in \Omega_n, 2 \leq k \leq n$ and the polynomial $\sum_{r=2}^{k} rc_r\sigma_r(A - J_n)t^{r-2}$ has no root in $(0,1)$ where $c_r = \frac{(k-r)!}{n^{k-r}}\binom{n-r}{k-r}^2$ then $\sigma_k(tA + (1-t)J_n) \leq \sigma_k(A)$ for all $t \in [0,1]$. In this paper, we prove that for all $S \in \Omega_5$ and all t such that $0.25 \leq t \leq 0.6248$, $\text{per}(tJ_5 + (1-t)S) \leq \text{per}S$. The following theorem is from Ebelein (Theorem 1, [19]).

Theorem 4. *Let $\phi(x_1, x_2, \ldots, x_n)$ be a real symmetric polynomial of degree at most one in each variable defined for $0 \leq x_i \leq 1$ and $\sum_{i=1}^{n} x_i = \gamma$ (γ is a real constant), then the maximum and minimum of $\phi(x)$ on the set C = { $x| \sum_{i=1}^{n} x_i = \gamma$ and for $i = 1, 2, \ldots, n, x_i \in [\alpha_i, \beta_i]$, where $[\alpha_i, \beta_i]$ is any closed interval contained in $[0,1]$ } and is assumed at least among the points whose components which are not end points are all equal. Moreover, if the maximum or minimum is attained only in the interior of C then it is assumed uniquely at the point $(\frac{\gamma}{n}, \frac{\gamma}{n}, \ldots \frac{\gamma}{n})$.*

Let x be an n-dimensional vector. Then the elementary symmetric function of x denoted by $e_r(x)$ is the sum of products of coordinates of x taken r at a time. Let $x = (x_1, x_2, \ldots, x_n)$. Then $e_r(x) = e_r(x_1, x_2, \ldots, x_n), r = 1, 2, \ldots, n$.

Theorem 5. *Let $S \in \Omega_5$ have all its off-diagonal entries less than or equal to $\frac{9}{20}$ and $0.25 \leq t \leq 0.6248$. Then $\text{per}(tJ_5 + (1-t)S) \leq \text{per}S$.*

Proof. Let $S(t) = tJ_5 + (1-t)S$. Then by Eberlein and Mudholkar ([20], p. 393)

$$\text{per}S(t) = -9 + \sum_{T_1(S(t))} (-e_2 + e_3 - e_4 + 2e_5)(x) + \sum_{T_2(S(t)} (e_2 - e_3 + e_4 - 2e_5)(x),$$

where e_r is the rth elementary symmetric function and $T_r(B)$ is the set of sums of columns of B, taken r at a time. If $x \in T_1(S(t))$ then $x = t\frac{e}{5} + (1-t)s$ where $s \in T_1(S)$ and $e = [1,1,1,1,1]^T$. Hence,

$$\begin{aligned}
e_2(x) &= \tfrac{2}{5}t^2 + t(1-t)\tfrac{4}{5} + (1-t)^2 e_2(s), \\
e_3(x) &= \tfrac{2}{25}t^3 + \tfrac{6}{25}t^2(1-t) + \tfrac{1}{5}t(1-t)^2 e_2(s) + (1-t)^3 e_3(s), \\
e_4(x) &= \tfrac{1}{125}t^4 + \tfrac{4}{125}t^3(1-t) + \tfrac{1}{25}t^2(1-t)^2 e_2(s) + \tfrac{1}{5}t(1-t)^3 e_3(S) + (1-t)^4 e_4(s), \\
e_5(x) &= \tfrac{1}{3125}t^5 + \tfrac{4}{625}t^4(1-t) + \tfrac{1}{125}t^3(1-t)^2 e_2(s) + \tfrac{1}{25}t^2(1-t)^3 e_3(s) + \tfrac{1}{5}t(1-t)^4 e_4(s) \\
&\quad + (1-t)^5 e_5(s).
\end{aligned}$$

Similarly if $x \in T_2(S(t))$ then there exists $r \in T_2(S)$ such that $x = \frac{2}{5}te + (1-t)r$. Hence

$$\begin{aligned}
e_2(x) &= \tfrac{8}{5}t^2 + \tfrac{16}{5}t(1-t) + (1-t)^2 e_2(r), \\
e_3(x) &= \tfrac{16}{25}t^3 + \tfrac{48}{25}t^2(1-t) + \tfrac{2}{5}t(1-t)^2 e_2(r) + (1-t)^3 e_3(r),
\end{aligned}$$

$e_4(x) = \frac{16}{125}t^4 + \frac{64}{125}t^3(1-t) + \frac{4}{25}t^2(1-t)^2e_2(r) + \frac{2}{5}t(1-t)^3e_3(r) + (1-t)^4e_4(r),$
$e_5(x) = \frac{32}{3125}t^5 + \frac{128}{625}t^4(1-t) + \frac{8}{125}t^3(1-t)^2e_2(r) + \frac{4}{25}t^2(1-t)^3e_3(r) + \frac{2}{5}t(1-t)^4e_4(r)$
$+(1-t)^5e_5(r).$

After substitution and simplification we have

$$\text{per}S(t) = \text{per}S + p_1(t) + \sum_{T_1(S)} (p_2(t)e_2 + p_3(t)e_3 + p_4(t)e_4 + p_5(t)e_5)(x) +$$

$$\sum_{T_2(s)} (p_6(t)e_2 + p_7(t)e_3 + p_8(t)e_4 + p_9(t)e_5)(x),$$

where
$p_1(t) = \frac{7246}{3125}t^5 - \frac{2901}{625}t^4 + \frac{1426}{125}t^3 - \frac{448}{25}t^2 + \frac{76}{5}t,$
$p_2(t) = \frac{2}{125}t^5 - \frac{9}{125}t^4 + \frac{37}{125}t^3 - \frac{36}{25}t^2 + \frac{11}{5}t,$
$p_3(t) = \frac{-2}{25}t^5 + \frac{11}{25}t^4 - \frac{46}{25}t^3 + \frac{92}{25}t^2 - \frac{16}{5}t$
$p_4(t) = \frac{2}{5}t^5 - \frac{13}{5}t^4 + \frac{32}{5}t^3 - \frac{22}{5}t^2 + \frac{22}{5}t,$
$p_5(t) = -2t^5 + 10t^4 - 20t^3 + 20t^2 - 10t,$
$p_6(t) = \frac{-16}{125}t^5 + \frac{52}{125}t^4 - \frac{56}{125}t^3 + \frac{49}{25}t^2 - \frac{12}{5}t,$
$p_7(t) = \frac{8}{25}t^5 - \frac{34}{25}t^4 + \frac{79}{25}t^3 - \frac{113}{25}t^2 + \frac{17}{5}t$
$p_8(t) = \frac{-4}{5}t^5 + \frac{21}{5}t^4 - \frac{44}{5}t^3 + \frac{46}{5}t^2 - \frac{24}{5}t,$
$p_9(t) = 2t^5 - 10t^4 + 20t^3 - 20t^2 + 10t.$

Now use the identities ([20], p. 391)

$$\sum_{T_2(A)} e_2(x) = 3\sum_{T_1(A)} e_2(x) + 10 \text{ and } \sum_{T_2(A)} e_3(x) = \sum_{T_1(A)} e_3(x) + 3\sum_{T_1(A)} e_2(x)$$

to write

$$\text{per}S(t) = \text{per}S + \frac{10}{3}\alpha + p_1(t) + \sum_{T_1(S)} (p_2 + \alpha + \beta + 3\gamma)e_2 + (p_3 + \gamma + \frac{\beta}{3})e_3 + p_4(t)e_4 + p_5(t)e_5) +$$

$$\sum_{T_2(S)} ((p_6(t) - \frac{\alpha}{3})e_2 + (p_7(t) - \frac{\beta}{3} - \gamma)e_3) + p_8(t)e_4 + p_9(t)e_5))$$

for any polynomials α, β and γ.

$$\text{per}S(t) = \text{per}S + c(t) + \sum_{T_1(S)} f_t(s) + \sum_{T_2(S)} g_t(r),$$

where $f_t = p_5(t)e_5 + p_4(t)e_4 + (p_3 + \gamma + \beta/3)e_3 + (p_2 + \alpha + \beta + 3\gamma)e_2$, $g_t = (p_6(t) - \alpha/3)e_2 + (p_7(t) - \beta/3 - \gamma)e_3 + p_8(t)e_4 + p_9(t)e_5)$ and $c(t) = 10/3\alpha + p_1(t)$.

We assume that all vectors in T_1 satisfy the condition $0 \le x_1 \le 1, 0 \le x_i \le \frac{9}{20}$, $i = 2,3,4,5$. The functions f_t and g_t are linear combinations of elementary symmetric functions.

From Theorem 4, possible points of maximum of f_t are $[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}]$, $[\frac{11}{20}, \frac{9}{20}, 0, 0, 0]$, $[\frac{11}{40}. \frac{11}{40}, \frac{9}{20}, 0, 0]$, $[\frac{11}{60}, \frac{11}{60}, \frac{11}{60}\frac{9}{20}, 0]$, $[\frac{11}{80}, \frac{11}{80}, \frac{11}{80}, \frac{11}{80}, \frac{9}{20}]$, $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0]$, $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0]$, $[1, 0, 0, 0, 0]$, $[\frac{1}{10}, \frac{9}{20}, \frac{9}{20}, 0, 0,]$, $[\frac{1}{20}. \frac{9}{20}, \frac{9}{20}, \frac{1}{20}, 0]$, $[\frac{1}{30}, \frac{9}{20}, \frac{9}{20}, \frac{1}{30}, \frac{1}{30}]$ and possible points of maximum of g_t are $[\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}]$, $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0]$, $[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0]$, $[1, 1, 0, 0, 0]$, $[1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0]$, $[1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, $[1, \frac{1}{2}, \frac{1}{2}, 0, 0]$, $[\frac{1}{15}, \frac{9}{10}, \frac{9}{10}, \frac{1}{15}, \frac{1}{15}]$.

For each t, a set of linear inequalities must be satisfied in order for $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ to be a maximum for f_t and for $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$ to be a maximum for g_t. These inequalities are solved numerically for various values of t and then interpolated to find α and β (details are shown in Appendix A). Substituting the values of α and β we obtain the values for $f_t(s)$ and $g_t(s)$ at different points. We have shown the values of $f_t(s)$ for different values of s and $g_t(r)$ for different values of r are given in the next two tables, respectively.

s	$f_t(s)$
$[\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}]$	$8t^4/3125-7t^4/625+29/12500t^3-194/625t^2+75397/187500t-2609/375000$
$[\frac{11}{20}\cdot\frac{9}{20},0,0,0]$	$0.00396t^5-0.01782t^4+0.0647955t^3-0.3564t^2+0.38741175t-0.0043065$
$[\frac{11}{40},\frac{11}{40},\frac{9}{20},0,0]$	$979/400000t^5-6633/800000t^4+175813/8000000t^3-68013/20000t^2+127003657/320000000t-1798797/320000000$
$[\frac{11}{60},\frac{11}{60},\frac{11}{60},\frac{9}{20},0]$	$201/78125t^5-6061/625000t^4+717247/50000000t^3-5068/15625t^2+58906841/150000000t-1817611/300000000$
$[\frac{11}{80},\frac{11}{80},\frac{11}{80},\frac{11}{80},\frac{9}{20}]$	$0.0026t^5-0.0104t^4+0.0095t^3-0.311744t^2+0.3884t-0.0063$
$[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0]$	$8t^5/3125-241/25000t^4+1627/200000t^3-8179/25000t^2+969949/2400000t-3131/480000$
$[\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0]$	$8/3375t^5-26/3375t^4+2581/135000t^3-232/675t^2+32663/81000t-4697/810000$
$[\frac{1}{2},\frac{1}{2},0,0,0]$	$1/250t^5-9/500t^4+1309/20000t^3-9/25t^2+15653/40000t-87/20000$
$[1,0,0,0,0]$	0
$[\frac{1}{10},\frac{9}{20},\frac{9}{20},0,0]$	$191/62500t^5-379/31250t^4+78449/2000000t^3-5414/15625t^2+14733359t/37500000-1526647/300000000$
$[\frac{1}{20},\frac{9}{20},\frac{9}{20},\frac{1}{20},0]$	$-197/25000t^5+1953/50000t^4-275569/1000000t^3+2349/5000t^2-606491/2000000t-1281/250000$
$[\frac{1}{30},\frac{9}{20},\frac{9}{20},\frac{1}{30},\frac{1}{30}]$	$15669/5000000t^5-63953/5000000t^4+972461/25000000t^3-171569/500000t^2+117437989/300000000t-1543843/300000000$

r	$g_t(r)$
$[\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}]$	$-\frac{256}{3125}t^5+\frac{144}{625}t^4+\frac{1257}{3125}t^3+\frac{152}{125}t^2-\frac{86134}{46875}t+\frac{291}{31250}$
$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0]$	$-\frac{41}{500}t^5+\frac{413}{2000}t^4+\frac{3751}{10000}t^3+\frac{251}{200}t^2-\frac{112961}{60000}t+\frac{131}{15000}$
$[\frac{2}{3},\frac{2}{3},\frac{2}{3},0,0]$	$-\frac{11851}{156250}t^5+\frac{94803}{625000}t^4+\frac{17709461}{50000000}t^3+\frac{159249}{125000}t^2-\frac{191045273}{100000000}t+\frac{291289}{37500000}$
$[1,1,0,0,0]$	$-\frac{16}{125}t^5+\frac{52}{125}t^4-\frac{2183}{5000}t^3+\frac{49}{25}t^2-\frac{32827}{15000}t+\frac{7}{1200}$
$[1,\frac{1}{3},\frac{1}{3},\frac{1}{3},0]$	$-\frac{12771}{156250}t^5+\frac{128943}{625000}t^4+\frac{13137261}{50000000}t^3+\frac{63973}{50000}t^2-\frac{91805513}{50000000}t+\frac{2329571}{300000000}$
$[1,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}]$	$-\frac{2033}{25000}t^5+\frac{2711}{12500}t^4+\frac{55171}{200000}t^3+\frac{62519}{50000}t^2-\frac{4323233}{2400000}t+\frac{1281}{160000}$
$[1,\frac{1}{2},\frac{1}{2},0,0]$	$-\frac{2}{25}t^5+\frac{9}{50}t^4+\frac{977}{4000}t^3+\frac{33}{25}t^2-\frac{75423}{40000}t+\frac{437}{60000}$
$[\frac{1}{15},\frac{9}{10},\frac{9}{10},\frac{1}{15},\frac{1}{15}]$	$-\frac{31377}{312500}t^5+\frac{354931}{1250000}t^4-\frac{1128039}{50000000}t^3+\frac{49261}{31250}t^2-\frac{602407919}{300000000}t+\frac{129307}{18750000}$

In calculating the elementary symmetric functions and $f_t(s)$ and $g_t(r)$ at different points, MATLAB programs were used.

In the Appendix A, we have shown the curves $f_t(s)$ and $g_t(r)$ in Figures A1 and A2, respectively. From the figures, $f_t(s) \leq f_t(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$ in $(0.25, 0.98)$. Furthermore, $g_t(r) \leq g_t(1,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ in $(0.1, 0.65)$ and $g_t(r) \leq g_t(\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5})$ in $(0.65, 1)$. Therefore, in the interval (0.25, 0.65),

$$\mathrm{per}S(t) \leq \sum_{T_2(S)} g_t(1,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) + \sum_{T_1(S)} f_t(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}) + \mathrm{per}S + c(t).$$

Similarly, in the interval (0.65, 0.98), we have,

$$perS(t) \leq \sum_{T_2(S)} g_t(\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}) + \sum_{T_1(S)} f_t(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}) + perS + c(t).$$

Substituting the values, we obtain $\mathrm{per}S(t) \leq \mathrm{per}S$ in $(0.25, 0.6248)$. □

Holens [11] and Dokovic [12] proposed a conjecture (Holen–Dokovic conjecture), which states that if $A \in \Omega_n, A \neq J_n$ and k is an integer, $1 \leq k \leq n$, then $\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A)$. Dokovic proved that the conjecture is true for $k \leq 3$. Kopotun [?] proved that the conjecture is true for $k = 4$ and $n \geq 5$. Wanless [14] disproved this conjecture by providing a counterexample of order 22. The smallest order of a counterexample has not been established. In Theorem 6, we prove that the Holen–Dokovic conjecture fails for $n = k = 4$. Before that, we recall that Foregger [10] proved that if $A \in \Omega_4$ has all its off-diagonal entries less than or equal to $\frac{9}{20}$ and $t_0 < t \le \frac{4}{3}$, where t_0 is the unique real root of $106t^3 - 418t^2 + 465t - 100$, then $\mathrm{per}(tJ_4 + (1-t)A) \le \mathrm{per}S$ with equality if and only if $A = J_4$.

Theorem 6. *The Holen–Dokovic conjecture fails for $n = k = 4$.*

Proof. Let $\Delta(t) = \mathrm{per}A - \mathrm{per}(tJ_4 + (1-t)A)$.
Foregger [10] proved that $\Delta(t) \ge 0$ in $[t_0, \frac{4}{3}]$ and t_0 is the unique real root of $106t^3 - 418t^2 + 465t - 100$.
Now, $\Delta(t) = \mathrm{per}A - \sum_{r=0}^{4} c_r(1-t)^{4-r}t^r\sigma_r(A)$, where $c_r = \frac{(n-r)!}{n^{n-r}}$.
Here, $\Delta(1) = 0$ and
$\Delta'(1) = -\sum_{r=0}^{4} c_r\sigma_r(A)[(1-t)^{4-r}rt^{r-1} + t^r(1-t)^{3-r}(-1)]_{t=1}$
$= -4c_4\mathrm{per}A + c_3\sigma_3(A)$
$= -4\mathrm{per}A + \frac{1}{4}\sigma_3(A)$
$\Delta'(1) = -4\mathrm{per}A + \frac{1}{4}\sigma_3(A)$
$\Delta'(1) = \lim_{t\to 1} \frac{\Delta(t)}{t-1}$
$\Delta'(1) \le 0$ iff $\Delta(t) \ge 0$ for all $t \in [1-\epsilon, 1]$ and $\Delta(t) \le 0$ for all $t \in [1, 1+\epsilon]$, which is not the case since $\Delta(t) \ge 0$ for all $t \in [t_0, \frac{4}{3}]$.
Hence, for some $A \in \Omega_4$, $\sigma_k(A) < \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A)$. □

4. Conclusions

The Merris conjecture is one of the well-known conjectures in linear algebra and it is still open for $n \ge 4$. We proved the Merris inequality for all $n \times n$ non-negative matrices with minimum entry greater than or equal to $\frac{1}{n}$. Furthermore, we gave a sufficient condition for a doubly stochastic matrix A to satisfy the Merris conjecture. Secondly, we proved the Foregger's inequality. That is, for all $S \in \Omega_5$ with off-diagonal entries less than or equal to $\frac{9}{20}$ and all t such that $0.25 \le t \le 0.6248$, $\mathrm{per}(tJ_5 + (1-t)S) \le \mathrm{per}S$. Finally, we proved that the Holen–Dokovic conjecture fails for $n = k = 4$ and thus established that the smallest order of a counterexample to the Holen–Dokovic conjecture is $n = 4$.

Author Contributions: K.S. and D.K.U. conceived and designed the experiments, performed the experiments, analyzed the data, contributed reagents/materials/analysis tools and wrote the paper. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the anonymous reviewers for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In this Appendix, we have shown various calculations of α, β and γ values.

The inequalities $f_t(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \ge f_t(s)$ and $g_t(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}) \ge g_t(r)$ at $t = 1$
$\frac{61}{400}\alpha + \frac{43}{240}\beta + \frac{43}{80}\gamma \ge -0.10546$
$\frac{123}{1600}\alpha + \frac{8851}{96000}\beta + \frac{8851}{32000}\gamma \ge -0.06386625$
$\frac{31}{600}\alpha + 0.06115432099\beta + 0.183462963\gamma \ge -0.0678099537$
$\frac{5}{128}\alpha + \frac{139}{3072}\beta + \frac{139}{1024}\gamma \ge -0.03263519287$

$\frac{1}{40}\alpha + \frac{37}{1200}\beta + \frac{37}{400}\gamma \geq -0.02405375$
$\frac{1}{15}\alpha + \frac{164}{2025}\beta + \frac{164}{675}\gamma \geq -0.0566637037$
$\frac{3}{20}\alpha + \frac{53}{300}\beta + \frac{53}{100}\gamma \geq -0.10296$
$\frac{10}{25}\alpha + \frac{32}{75}\beta + \frac{32}{25}\gamma \geq -0.35296$
$\frac{43}{400}\alpha + \frac{1529}{12000}\beta + \frac{1529}{4000}\gamma \geq -0.08071$
$\frac{21}{200}\alpha + \frac{149}{1200}\beta + \frac{149}{400}\gamma \geq -0.08046$
$\frac{5}{48}\alpha + \frac{319}{2592}\beta + \frac{319}{864}\gamma \geq -0.0775237037$
$-\frac{4.166666667}{3}\alpha - \frac{0.2002037037}{3}\beta - 0.2002037037\gamma \geq -0.1736953067$
$-\frac{1}{30}\alpha - \frac{7}{150}\beta - \frac{7}{50}\gamma \geq 0.01284375$
$-\frac{4}{45}\alpha - \frac{232}{2025}\beta - \frac{232}{675}\gamma \geq -0.004136296296$
$-\frac{1}{5}\alpha - \frac{16}{75}\beta - \frac{16}{25}\gamma \geq -0.01984.$
$-\frac{1}{120}\alpha - \frac{13}{1200}\beta - \frac{13}{400}\gamma \geq -0.00070984375.$
$-0.1389\alpha - 0.1512\beta - 0.4537\gamma \geq -0.1070$
If $\gamma = s, \beta = -3s$, then $\alpha \geq -0.6864$.
We can take $\alpha = -0.6864, \gamma = 1, \beta = -3$.

The inequalities $f_t(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \geq f_t(s)$ and $g_t(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}) \geq g_t(r)$ at $t = \frac{1}{2}$
$\frac{61}{400}\alpha + \frac{43}{240}\beta + \frac{43}{80}\gamma \geq -0.0604625$
$\frac{123}{1600}\alpha + \frac{8851}{96000}\beta + \frac{8851}{32000}\gamma \geq -0.03212203125$
$\frac{31}{600}\alpha + 0.06115432099\beta + 0.183462963\gamma \geq -0.01752152778$
$\frac{5}{128}\alpha + \frac{139}{3072}\beta + \frac{139}{1024}\gamma \geq -0.01331022355$
$\frac{1}{40}\alpha + \frac{37}{1200}\beta + \frac{37}{400}\gamma \geq -0.0103815625$
$\frac{1}{15}\alpha + \frac{164}{2025}\beta + \frac{164}{675}\gamma \geq -0.02689111111$
$\frac{3}{20}\alpha + \frac{53}{300}\beta + \frac{53}{100}\gamma \geq -0.05853$
$\frac{2}{5}\alpha + \frac{32}{75}\beta + \frac{32}{25}\gamma \geq -0.25178$
$\frac{43}{400}\alpha + \frac{1529}{12000}\beta + \frac{1529}{4000}\gamma \geq -0.04359875$
$\frac{21}{200}\alpha + \frac{149}{1200}\beta + \frac{149}{400}\gamma \geq -0.0427715625$
$\frac{5}{48}\alpha + \frac{319}{2592}\beta + \frac{319}{864}\gamma \geq -0.04248444444$
$-0.00138888889\alpha - 0.0667345679\beta - 0.2002037037\gamma \geq -0.1962603704$
$-\frac{1}{30}\alpha - \frac{7}{150}\beta - \frac{7}{50}\gamma \geq -0.03498$
$-\frac{4}{45}\alpha - \frac{232}{2025}\beta - \frac{232}{675}\gamma \geq 0.10752$
$-\frac{1}{5}\alpha - \frac{16}{75}\beta - \frac{16}{25}\gamma \geq -0.17248$
$-\frac{1}{120}\alpha - \frac{13}{1200}\beta - \frac{13}{400}\gamma \geq -0.00748375$
$-0.1389\alpha - 0.1512\beta - 0.4537\gamma \geq -0.2793$
If $\gamma = s, \beta = -3s$, then $\alpha \geq -0.3391263441$.
We can take $\alpha = -0.3391263441, \gamma = 2, \beta = -6$

The inequalities $f_t(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \geq f_t(s)$ and $g_t(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}) \geq g_t(r)$ at $t = \frac{1}{4}$.
$\frac{61}{400}\alpha + \frac{43}{240}\beta + \frac{43}{80}\gamma \geq -0.02987980469$
$\frac{123}{1600}\alpha + \frac{8851}{96000}\beta + \frac{8851}{32000}\gamma \geq -0.01508300537$
$\frac{209}{600}\alpha + 0.3655123457\beta + 1.096537037\gamma \geq -0.1335164742$
$\frac{5}{128}\alpha + \frac{139}{3072}\beta + \frac{139}{1024}\gamma \geq -0.009529170096$
$\frac{1}{40}\alpha + \frac{37}{1200}\beta + \frac{37}{400}\gamma \geq -0.004418186035$
$\frac{1}{15}\alpha + \frac{164}{2025}\beta + \frac{164}{675}\gamma \geq -0.0121374537$
$\frac{3}{20}\alpha + \frac{53}{300}\beta + \frac{53}{100}\gamma \geq -0.02871890625$
$\frac{2}{5}\alpha + \frac{32}{75}\beta + \frac{32}{25}\gamma \geq -0.14480875$
$\frac{43}{400}\alpha + \frac{1529}{12000}\beta + \frac{1529}{4000}\gamma \geq -0.02107509766$
$\frac{21}{200}\alpha + \frac{149}{1200}\beta + \frac{149}{400}\gamma \geq -0.02128248291$
$\frac{8187}{21600}\alpha + 0.3979320988\beta + 1.193796296\gamma \geq -0.1489750101$
$-0.001388888889\alpha - \frac{0.2002037037}{3}\beta - 0.2002037037\gamma \geq -0.1169795821$
$-\frac{1}{30}\alpha - \frac{7}{150}\beta - \frac{7}{50}\gamma \geq -0.1006621875$
$-\frac{4}{45}\alpha - \frac{232}{2025}\beta - \frac{232}{675}\gamma \geq -0.2591237037$

$-\frac{1}{5}\alpha - \frac{16}{75}\beta - \frac{16}{25}\gamma \geq -0.60142$
$-\frac{1}{120}\alpha - \frac{13}{1200}\beta - \frac{13}{400}\gamma \geq -0.02454455811$
$-0.1389\alpha - 0.1512\beta - 0.4537\gamma \geq -0.0044$
If $\gamma = s, \beta = -3s$, then $\alpha \geq -0.1767274414$.
We can take $\alpha = -0.1767274414, \gamma = 3, \beta = -9$.

Interpolating the values of α, β, γ at $t = 1, \frac{1}{2}, \frac{1}{4}$ we obtain
$\alpha = -0.0342t^3 - 0.6346t - 0.0175$
$\gamma = 1.5238t^3 - 4.6667t + 4.1429$
$\beta = -4.5714t^3 + 14t - 12.4286$.

Figures A1 and A2 are showing $f_t(s)$ and $g_t(r)$ for different values of s and r.

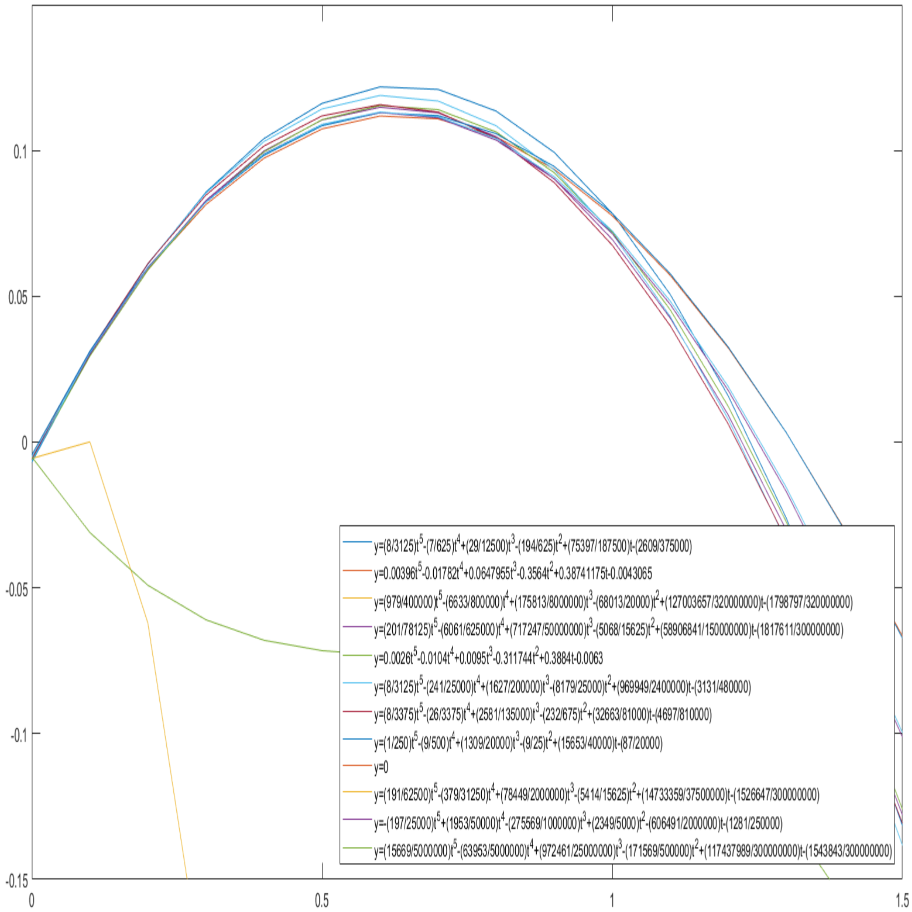

Figure A1. Curves $f_t(s)$.

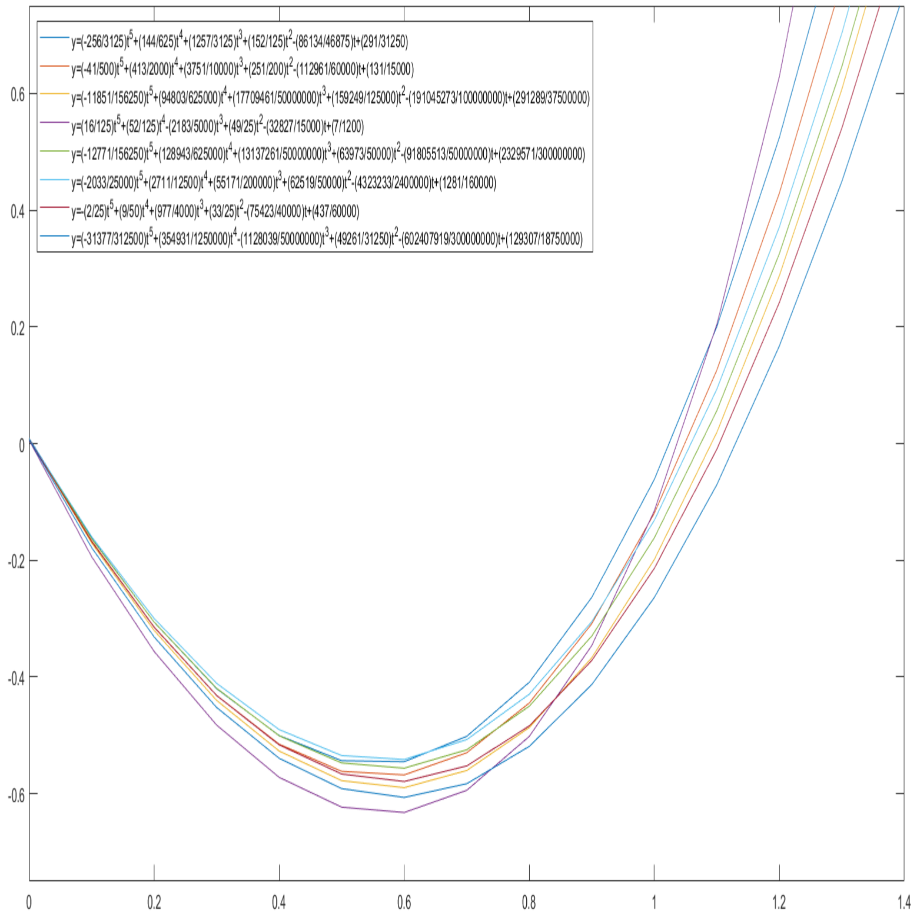

Figure A2. Curves $g_t(r)$.

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