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On Eigenfunctions and Eigenvalues of a Nonlocal Laplace Operator with Multiple Involution

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Abstract: We study the eigenfunctions and eigenvalues of the boundary value problem for the nonlocal Laplace equation with multiple involution. An explicit form of the eigenfunctions and eigenvalues for the unit ball are obtained. A theorem on the completeness of the eigenfunctions of the problem under consideration is proved.

Keywords: nonlocal Laplace operator; multiple involution; Dirichlet problem; eigenfunctions; eigenvalues

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1. Introduction and the Problem Statement

The notion of a nonlocal operator and the related notions of a nonlocal differential equation appeared relatively recently in the theory of differential equations. In [1], loaded equations, equations containing fractional derivatives of the unknown function,and equations with deviating arguments are considered. Equations in which the unknown function and its derivatives enter for different values of arguments are called nonlocal differential equations.

Special place among nonlocal differential equations, is occupied by equations in which the deviation of arguments has an involutive character. An involution is called a function that is its own inverse $S^2(x) = S(S(x)) = x$. Differential equations containing an involutive deviation in the unknown function or its derivative are some model equations with an alternating deviation of the argument. Such equations can be classified as functional differential equations.

Mathematicians have been studying differential equations with involution for a long time. For example, in 1816, Babbage [2] considered algebraic and differential equations with involution. The monographs of D. Przeworska-Rolewicz [3] and J. Wiener [4] are devoted to the theory of solvability of various differential equations with involution. In papers [5–14], spectral problems for differential operators of the first and second orders with involution were studied. In [15–22], the results of studying spectral problems with involution are used to solve inverse problems. A series of works by the authors Alberto Cabada and F. Adrian F. Tojo are devoted to the creation of the theory of the Green's function for one-dimensional differential equations with involution (see, for example, Refs [23,24] as well as the bibliography in these papers). The papers [25–28] are devoted to questions of the theory of solvability of some partial differential equations with involution. Elliptic functional differential equations with mappings of compression and extension type are considered in [29–31]. In addition, in [32–34], some classes of functional differential equations with deviating arguments are investigated. In [35], for the following ODE:

$$y''(t) + ay''(-t) = \lambda y(t), \quad -\pi < t < \pi$$

the boundary value problem with Dirichlet conditions $y(-\pi) = y(\pi) = 0$ is studied. It is shown that the eigenfunctions and eigenvalues of this problem have the form:

$$y_k(t) = \sin kt,\ \lambda_k = -(1+a)k^2;\ y_m(t) = \cos\left(m - \frac{1}{2}\right)t,\ \lambda_m = -(1+a)\left(m - \frac{1}{2}\right)^2,$$

where $k, m \in \mathbb{N}$. This system is complete in $L_2[-\pi, \pi]$. Note that the eigenfunctions of this problem for $a = 0$ coincide with the eigenfunctions of the classical equation and differ only in eigenvalues.

In the present paper, generalizing the problems considered in [36], to the case of multiple involution, we introduce the concept of a nonlocal analogue of the Laplace operator. In Section 2, matrices of a special form arising in this operator are investigated. Then, in Section 3, we study the structure of the eigenfunctions and eigenvalues of the Dirichlet problem. In Section 4, the eigenfunctions and eigenvalues of the Dirichlet problem for the nonlocal Laplace equation in the unit ball are constructed in an explicit form and the completeness of the system of eigenfunctions is proved.

Let $\Omega = \{x \in \mathbb{R}^l : |x| < 1\}$ be the unit ball in $\mathbb{R}^l$, $l \geq 2$, and $\partial\Omega = \{x \in \mathbb{R}^l : |x| = 1\}$ be the unit sphere. Let also $S_1, \ldots, S_n$, be a set of real symmetric commutative matrices $S_i S_j = S_j S_i$ such that $S_i^2 = I$. Note that since $|x|^2 = (S_i{}^2 x, x) = (S_i x, S_i x) = |S_i x|^2$, then $x \in \Omega \Rightarrow S_i x \in \Omega$ and $y \in \partial\Omega \Rightarrow S_i y \in \partial\Omega$. For example, matrix S_1 can be a matrix of the following linear mapping $S_1 x = (-x_1, x_2, \ldots, x_l)$, because:

$$S_1 = \begin{pmatrix} -1 & \mathbf{0}_{1\times(l-1)} \\ \mathbf{0}_{(l-1)\times 1} & I_{l-1} \end{pmatrix}.$$

Let $n \in \mathbb{N}_0$ and $a_0, a_1, a_2, a_3, \ldots, a_{2^n-1}$ be a set of real numbers. If we write the summation index i in the binary number system $(i_n \ldots i_1)_2 \equiv i$, where $i_k = 0, 1$ for $k = 1, \ldots, n$, then the coefficients a_k can be written as $a_{(0\ldots00)_2}$, $a_{(0\ldots01)_2}$, $a_{(0\ldots10)_2}$, $a_{(0\ldots11)_2}$, $\ldots$, $a_{(1\ldots11)_2}$.

Let us introduce the following nonlocal differential operator:

$$L_n u \equiv \sum_{i=0}^{2^n-1} a_i \Delta u(S_n^{i_n} \ldots S_1^{i_1} x)$$

and consider the following boundary value problem.

Problem S. *Find a function $u(x) \neq 0$ from the class $u \in C(\bar{\Omega}) \cap C^2(\Omega)$, satisfying the conditions:*

$$L_n u(x) + \lambda u(x) = 0,\ x \in \Omega, \tag{1}$$

$$u(x) = 0,\ x \in \partial\Omega, \tag{2}$$

where $\lambda \in \mathbb{R}$.

If $n > 0$, $a_0 = 1$, $a_j = 0$, $j = 1, \ldots, 2^n - 1$, then this problem coincides with the spectral Dirichlet problem for the classical Laplace operator.

2. Preliminaries

To study the above problems (1) and (2), we need some auxiliary statements. Let us introduce the function:

$$v(x) = \sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2} a_i u(S_n^{i_n} \ldots S_1^{i_1} x), \tag{3}$$

where the summation is taken in the ascending order with respect to the index i. From this equality it is easy to conclude that functions of the form $v(S_n^{j_n} \ldots S_1^{j_1} x)$, where $j = 0, \ldots, 2^n - 1$ can be linearly expressed in terms of functions $u(S_n^{i_n} \ldots S_1^{i_1} x)$. If we consider the following vectors $U(x) = \left(u(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1}$, $V(x) = \left(v(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1}$ of order 2^n, then this dependence can be expressed in the matrix form:

$$V(x) = A_n U(x), \tag{4}$$

where $A_n = \left(a_{i,j}\right)_{i,j=0,\dots,2^n-1}$ is the matrix of order $2^n \times 2^n$.

Let us investigate the structure of matrices of the form A_n.

Theorem 1. *The matrix A_n from the equality (4) can be represented in the form:*

$$A_n = \left(a_{i,j}\right)_{i,j=0,\dots,2^n-1} = \left(a_{i\oplus j}\right)_{i,j=0,\dots,2^n-1}, \tag{5}$$

where the operation in the subscript of the matrix coefficients is understood in the following sense $i \oplus j \equiv (i)_2 \oplus (j)_2 = ((i_n + j_n \bmod 2) \dots (i_1 + j_1 \bmod 2))_2$, where $(i)_2 = (i_n \dots i_1)_2$ is a representation of the index in the binary number system. The linear combination of matrices of the form (5) is a matrix of the form (5).

Proof. Let $n = 1$, then we have:

$$A_1 = \begin{pmatrix} a_{0\oplus 0} & a_{0\oplus 1} \\ a_{1\oplus 0} & a_{1\oplus 1} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix},$$

and if $n = 2$, then we get:

$$A_2 = \begin{pmatrix} a_{(00)_2\oplus(00)_2} & a_{(00)_2\oplus(01)_2} & a_{(00)_2\oplus(10)_2} & a_{(00)_2\oplus(11)_2} \\ a_{(01)_2\oplus(00)_2} & a_{(01)_2\oplus(01)_2} & a_{(01)_2\oplus(10)_2} & a_{(01)_2\oplus(11)_2} \\ a_{(10)_2\oplus(00)_2} & a_{(10)_2\oplus(01)_2} & a_{(10)_2\oplus(10)_2} & a_{(10)_2\oplus(11)_2} \\ a_{(11)_2\oplus(00)_2} & a_{(11)_2\oplus(01)_2} & a_{(11)_2\oplus(10)_2} & a_{(11)_2\oplus(11)_2} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}.$$

Consider the function $v(S_n^{i_n} \dots S_1^{i_1} x)$, whose coefficients at $u(S_n^{j_n} \dots S_1^{j_1})$ make up the $i \equiv (i_n \dots i_1)_2$th row of the matrix A_n:

$$\begin{aligned} v(S_n^{i_n} \dots S_1^{i_1} x) &= \sum_{j\equiv(j_n\dots j_1)_2=0}^{2^n-1=(1\dots1)_2} a_{(j_n\dots j_1)_2} u(S_n^{j_n} \dots S_1^{j_1} S_n^{i_n} \dots S_1^{i_1} x) \\ &= \sum_{j\equiv(j_n\dots j_1)_2=0}^{(1\dots1)_2} a_{(j_n\dots j_1)_2} u(S_n^{j_n+i_n \bmod 2} \dots S_1^{j_1+i_1 \bmod 2} x). \end{aligned} \tag{6}$$

Here, the following properties $S_j^2 x = x$ and $S_j S_i x = S_i S_j x$ of the matrices $S_1, \dots, S_n$ are taken into account. Let's replace the index $i \oplus j = l$. Then $l \oplus i = i \oplus j \oplus i = j$, and the correspondence $j \sim l$ is one-to-one. Replacement $j \to l$ of the index changes only the order of summation in the sum (6). For example, if $i = 1$, then the sequence $j : 0, 1, 2, 3, 4, 5, \dots$ goes to $l = 1 \oplus j : 1, 0, 3, 2, 5, 4, \dots$. After replacing the index, we get:

$$v(S_n^{i_n} \dots S_1^{i_1} x) = \sum_{l=0}^{(1\dots1)_2} a_{(i_n+l_n \bmod 2 \dots i_1+l_1 \bmod 2)_2} u(S_n^{l_n} \dots S_1^{l_1} x),$$

whence $a_{i,l} = a_{(i_n+l_n \bmod 2\dots i_1+l_1 \bmod 2)_2} = a_{i\oplus l}$ which proves (4).

It is clear that if α, β are constants, then:

$$\alpha \left(a_{i\oplus j}\right)_{i,j=0,\dots,2^n-1} + \beta \left(b_{i\oplus j}\right)_{i,j=0,\dots,2^n-1} = \left(\alpha a_{i\oplus j} + \beta b_{i\oplus j}\right)_{i,j=0,\dots,2^n-1}.$$

The theorem is proved. □

We present important information for the further analysis corollaries of Theorem 1.

Corollary 1. *The matrix A_n is uniquely determined by its first row* $(a_0, a_1, \ldots, a_{2^n-1})$.

Indeed, the ith row of the matrix A_n can be written through its 1st row in the form $\left(a_{i\oplus 0}, a_{i\oplus 1}, \ldots, a_{i\oplus(2^n-1)}\right)$.

This property of the matrix A_n we denote by the equality $A_n \equiv A_n(a_0, \ldots, a_{2^n-1})$.

Corollary 2. *The matrix A_n has the symmetry property:*

$$\left(a_{i,j}\right)_{i,j=0,\ldots,2^n-1} = \left(a_{j,i}\right)_{i,j=0,\ldots,2^{n-1}} \tag{7}$$

and it can be written as:

$$A_n = \begin{pmatrix} A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}) & A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}) \\ A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}) & A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}) \end{pmatrix}, \tag{8}$$

or more generally in the form of a block matrix A_{n-m} consisting of matrices A_m:

$$A_n = A_{n-m}\left(A_m^{(0\ldots0)_2}, \ldots, A_m^{(k_n\ldots k_{m+1})_2}, \ldots, A_m^{(1\ldots1)_2}\right), \tag{9}$$

where $A_m^{(k_n\ldots k_{m+1})_2}(a_{(k_n\ldots k_{m+1}0\ldots0)_2}, \ldots, a_{(k_n\ldots k_{m+1}1\ldots1)_2})$ is a matrix of the form (4) of order 2^m.

Proof. Indeed, since the binary operation $i \oplus j$ is commutative:

$$i \oplus j = (i_n + j_n \bmod 2 \ldots i_1 + j_1 \bmod 2)_2 = (j_n + i_n \bmod 2 \ldots j_1 + i_1 \bmod 2)_2 = j \oplus i,$$

then property (7) holds true, and:

$$\left(a_{i,j}\right)_{i,j=0,\ldots,2^n-1} = \left(a_{i\oplus j}\right)_{i,j=0,\ldots,2^n-1} = \left(a_{j\oplus i}\right)_{i,j=0,\ldots,2^n-1} = \left(a_{j,i}\right)_{i,j=0,\ldots,2^n-1}.$$

Further, it is easy to see the validity of the equalities:

$$\left(a_{(0i_{n-1}\ldots i_1)_2\oplus(0j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = \left(a_{(1i_{n-1}\ldots i_1)_2\oplus(1j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} \tag{10}$$

and:

$$\left(a_{(0i_{n-1}\ldots i_1)_2\oplus(1j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = \left(a_{(1i_{n-1}\ldots i_1)_2\oplus(0j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1}, \tag{11}$$

from which the property (8) follows. Indeed, if we divide the matrix A_n into four equally sized square blocks and consider the lower right block, then its indices are located in the range $(10\ldots0)_2 \le i,j \le (11\ldots1)_2$, which means that this block, by virtue of (10), has the form:

$$\begin{aligned}&\left(a_{(1i_{n-1}\ldots i_1)_2\oplus(1j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} \\ &\qquad = \left(a_{(0i_{n-1}\ldots i_1)_2\oplus(0j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}),\end{aligned}$$

i.e., the diagonal blocks of the matrix A_n are of the form $A_{n-1}(a_0, \ldots, a_{2^{n-1}-1})$. Similarly, the top right block of A_n has the indices in the range $(00\ldots0)_2 \le i \le (01\ldots1)_2$, $(10\ldots0)_2 \le j \le (11\ldots1)_2$, which means this block has the form:

$$\left(a_{(0i_{n-1}\ldots i_1)_2\oplus(1j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}).$$

By equality (10), the lower left block of A_n has the form:

$$\left(a_{(1i_{n-1}\ldots i_1)_2\oplus(0j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = \left(a_{(0i_{n-1}\ldots i_1)_2\oplus(1j_{n-1}\ldots j_1)_2}\right)_{i,j=0,\ldots,2^{n-1}-1} = A_{n-1}(a_{2^{n-1}},\ldots,a_{2^n-1}).$$

Equality (8) is proved. Now consider a block matrix of the form:

$$A_{n-m}\left(A_m^{(0\ldots0)_2},\ldots,A_m^{(k_n\ldots k_{m+1})_2},\ldots,A_m^{(1\ldots1)_2}\right) = \left(A_m^{(i_n\ldots i_{m+1})_2\oplus(j_n\ldots j_{m+1})_2}\right)_{i,j=0,\ldots,2^{n-m}-1}.$$

The elements of its block matrix with the number $(k_n\ldots k_{m+1})_2$ can be written as:

$$A_m^{(k_n\ldots k_{m+1})_2}(a_{(k_n\ldots k_{m+1}0\ldots0)_2},\ldots,a_{(k_n\ldots k_{m+1}1\ldots1)_2}) = \left(a_{(k_n\ldots k_{m+1}(i_m\ldots i_1)_2\oplus(j_m\ldots j_1)_2)}\right)_{i,j=0,\ldots,2^m-1}.$$

Consider the element $a_{i,j}$ of the block matrix:

$$A_{n-m}\left(A_m^{(0\ldots0)_2},\ldots,A_m^{(k_n\ldots k_{m+1})_2},\ldots,A_m^{(1\ldots1)_2}\right).$$

It is located in the block with indices $(i_n\ldots i_{m+1})_2$, $(j_n\ldots j_{m+1})_2$, and this means it is in the block $A_m^{(i_n\ldots i_{m+1})_2\oplus(j_n\ldots j_{m+1})_2}$, and therefore has the form:

$$a_{i,j} = a_{((i_n\ldots i_{m+1})_2\oplus(j_n\ldots j_{m+1})_2(i_m\ldots i_1)_2\oplus(j_m\ldots j_1)_2)_2} = a_{i\oplus j}.$$

This coincides with Formula (5). Therefore, the corollary is proved. □

Example 1. *Property (8) of the matrix A_n can be seen on the example of matrices A_1, A_2 and A_3:*

$$A_2(a_0,a_1,a_2,a_3) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_1(a_0,a_1) & A_1(a_2,a_3) \\ A_1(a_2,a_3) & A_1(a_0,a_1) \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & a_3 & a_0 & a_1 & a_6 & a_7 & a_4 & a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & a_7 & a_4 & a_5 & a_2 & a_3 & a_0 & a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_2(a_0,a_1,a_2,a_3) & A_2(a_4,a_5,a_6,a_7) \\ A_2(a_4,a_5,a_6,a_7) & A_2(a_0,a_1,a_2,a_3) \end{pmatrix}$$

and property (9) is written as:

$$A_3 = A_2\left(A_1^{(0,0)_2}(a_0,a_1), A_1^{(0,1)_2}(a_2,a_3), A_1^{(1,0)_2}(a_4,a_5), A_1^{(1,1)_2}(a_6,a_7)\right) \equiv A_2\left(A_1^0,A_1^1,A_1^2,A_1^3\right) = \begin{pmatrix} A_1^0 & A_1^1 & A_1^2 & A_1^3 \\ A_1^1 & A_1^0 & A_1^3 & A_1^2 \\ A_1^2 & A_1^3 & A_1^0 & A_1^1 \\ A_1^3 & A_1^2 & A_1^1 & A_1^0 \end{pmatrix}.$$

Let us investigate the product of matrices of the form (5).

Theorem 2. *Multiplication of matrices of the form (5) is commutative. The product of matrices of the form (5) is again a matrix of the form (5).*

Proof. For $n = 1$ we have:

$$A_1 B_1 = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 \\ b_1 & b_0 \end{pmatrix} = \begin{pmatrix} a_0 b_0 + a_1 b_1 & a_0 b_1 + a_1 b_0 \\ a_1 b_0 + a_0 b_1 & a_1 b_1 + a_0 b_0 \end{pmatrix} = B_1 A_1.$$

Assuming that the multiplication of matrices A_{n-1} and B_{n-1} of the order $n-1$ is commutative, using the property (8) and equalities similar to the above, it is easy to obtain $A_n B_n = B_n A_n$.

Thus, it is not hard to see that:

$$AB = \left(a_{i\oplus j}\right)_{i,j=0,\ldots,2^n-1} \left(b_{i\oplus j}\right)_{i,j=0,\ldots,2^n-1} = \left(\sum_{k=0}^{2^n-1} a_{i\oplus k} b_{k\oplus j}\right)_{i,j=0,\ldots,2^n-1}.$$

In the sum, from the formula above, let us change the index $k \to l$, as in Theorem 1, according to equality $i \oplus k = l$. Then $l \oplus i = i \oplus k \oplus i = i \oplus i \oplus k = k$, and it means that the correspondence $k \sim l$ is one-to-one. Replacement of the index $k \to l$ changes only the order of summation in the sum. By virtue of the associativity of the operation $\oplus$, we have:

$$AB = \left(\sum_{l=0}^{2^n-1} a_l b_{(l\oplus i)\oplus j}\right)_{i,j=0,\ldots,2^n-1} = \left(\sum_{l=0}^{2^n-1} a_l b_{l\oplus(i\oplus j)}\right)_{i,j=0,\ldots,2^n-1}.$$

The first row of the matrix AB is:

$$(AB)_{i=0} = \left(\sum_{k=0}^{2^n-1} a_k b_{k\oplus j}\right)_{j=0,\ldots,2^n-1},$$

and hence, the matrix C of the form (5), constructed by the first row of AB, is written in the form coinciding with AB:

$$C \equiv \left(\sum_{k=0}^{2^n-1} a_k b_{k\oplus(i\oplus j)}\right)_{j=0,\ldots,2^n-1} = AB.$$

The theorem is proved. $\square$

The following theorem gives an idea of eigenvectors and eigenvalues of matrices of the form (5).

Theorem 3. *The eigenvectors of the matrix $A_n(a_0, \ldots, a_{2^n-1})$ can be chosen in the form:*

$$\mathbf{a}_n^k = \left(\mathbf{a}_{n-1}^k, \pm\mathbf{a}_{n-1}^k\right)^T, \, k = 0, \ldots, 2^{n-1} - 1,$$

where $\mathbf{a}_{n-1}^k$ is the eigenvector of the matrix $A_{n-1}(a_0, \ldots, a_{2^{n-1}-1})$, $k = 0, \ldots, 2^{n-1} - 1$ besides for $n = 1$ we have $\mathbf{a}_1^0 = (1,1)^T$, $\mathbf{a}_1^1 = (1,-1)^T$. The eigenvectors of the matrix A_n are orthogonal. The eigenvalues of the matrix A_n are of the form:

$$\mu_n^{k,\pm} = \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k, \, k = 0, \ldots, 2^{n-1} - 1,$$

where μ_{n-1}^k and $\hat{\mu}_{n-1}^k$ are eigenvalues of the matrices:

$$A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}) \quad \text{and} \quad \hat{A}_{n-1} = A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}),$$

respectively, corresponding to the eigenvector $\mathbf{a}_{n-1}^k$, besides $\mu_1^0 = a_0 + a_1$, $\mu_1^1 = a_0 - a_1$.

Proof. Let us carry out the proof by induction on n. Suppose that the eigenvectors of the matrix $A_n(a_0, \ldots, a_{2^n-1})$ are independent on numbers $a_0, \ldots, a_{2^n-1}$. For $n = 1$, it is obvious that the eigenvectors of the matrix $A_1(a_0, a_1)$ can be chosen in the form $\mathbf{a}_1^+ = (1,1)^T$, $\mathbf{a}_1^- = (1,-1)^T$, and the eigenvalues corresponding to them have the form $\mu_1^+ = a_0 + a_1$, $\mu_1^- = a_0 - a_1$. For the matrix:

$$A_2 = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_1(a_0,a_1) & A_1(a_2,a_3) \\ A_1(a_2,a_3) & A_1(a_0,a_1) \end{pmatrix}$$

eigenvectors are:

$$\mathbf{a}_2^{(+,+)} = (\mathbf{a}_1^+, \mathbf{a}_1^+)^T,\ \mathbf{a}_2^{(-,+)} = (\mathbf{a}_1^-, \mathbf{a}_1^-)^T,\ \mathbf{a}_2^{(+,-)} = (\mathbf{a}_1^+, -\mathbf{a}_1^+)^T,\ \mathbf{a}_2^{(-,-)_2} = (\mathbf{a}_1^-, -\mathbf{a}_1^-)^T,$$

or briefly $\mathbf{a}_2^{(\pm_1,\pm_2)} = (\mathbf{a}_1^{\pm_1}, \pm_2\, \mathbf{a}_1^{\pm_1})^T$. Signs $+$ and $-$ in the expressions $\pm_1$ and $\pm_2$ are taken values independently of each other. Indeed, the equalities:

$$\begin{aligned} A_2\mathbf{a}_2^{(\pm_1,\pm_2)} &= \begin{pmatrix} A_1(a_0,a_1) & A_1(a_2,a_3) \\ A_1(a_2,a_3) & A_1(a_0,a_1) \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^{\pm_1} \\ \pm_2\mathbf{a}_1^{\pm_1} \end{pmatrix} \\ &= \begin{pmatrix} A_1(a_0,a_1)\mathbf{a}_1^{\pm_1} \pm_2 A_1(a_2,a_3)\mathbf{a}_1^{\pm_1} \\ A_1(a_2,a_3)\mathbf{a}_1^{\pm_1} \pm_2 A_1(a_0,a_1)\mathbf{a}_1^{\pm_1} \end{pmatrix} = \begin{pmatrix} (a_0 \pm_1 a_1)\mathbf{a}_1^{\pm_1} \pm_2 (a_2 \pm_1 a_3)\mathbf{a}_1^{\pm_1} \\ (a_2 \pm_1 a_3)\mathbf{a}_1^{\pm_1} \pm_2 (a_0 \pm_1 a_1)\mathbf{a}_1^{\pm_1} \end{pmatrix} \\ &= (a_0 \pm_1 a_1 \pm_2 (a_2 \pm_1 a_3)) \begin{pmatrix} \mathbf{a}_1^{\pm_1} \\ \pm_2\mathbf{a}_1^{\pm_1} \end{pmatrix} = (a_0 \pm_1 a_1 \pm_2 (a_2 \pm_1 a_3))\mathbf{a}_2^{(\pm_1,\pm_2)} \end{aligned}$$

are true and hence $\left(\mathbf{a}_1^{\pm_1}, \pm_2\mathbf{a}_1^{\pm_1}\right)^T$, are the eigenvectors for four different combinations of signs $\pm_1$ and $\pm_2$. It is seen that the eigenvectors $\mathbf{a}_2^{(\pm_1,\pm_2)} = (1, \pm_1 1, \pm_2 1, \pm_2 \pm_1 1)^T$, of the matrix $A_2(a_0, a_1, a_2, a_3)$, do not depend on the numbers $\{a_k\}$.

Furthermore, assuming that the eigenvectors $\mathbf{a}_{n-1}^0, \ldots, \mathbf{a}_{n-1}^{2^{n-1}-1}$, of the matrix $A_{n-1} = A_{n-1}\ (a_0, \ldots, a_{2^{n-1}-1})$, do not depend on its coefficients, we prove that this property is also true for the matrix $A_n = A_n(a_0, \ldots, a_{2^n-1})$. Let $\mu_{n-1}^0, \ldots, \mu_{n-1}^{2^{n-1}-1}$ be the eigenvalues corresponding to the above eigenvectors of the matrix $A_{n-1}(a_0, \ldots, a_{2^{n-1}-1})$, independent of its coefficients, then vectors of the form $\mathbf{a}_n^k = \left(\mathbf{a}_{n-1}^k, \pm\mathbf{a}_{n-1}^k\right)^T$, where $k = 0, \ldots, 2^{n-1} - 1$, are the eigenvectors of the matrix $A_n(a_0, \ldots, a_{2^n-1})$. Indeed, we have:

$$\begin{aligned} A_n\mathbf{a}_n^k &= A_n\left(\mathbf{a}_{n-1}^k, \pm\mathbf{a}_{n-1}^k\right)^T \\ &= \begin{pmatrix} A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}) & A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}) \\ A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1}) & A_{n-1}(a_0, \ldots, a_{2^{n-1}-1}) \end{pmatrix} \begin{pmatrix} \mathbf{a}_{n-1}^k \\ \pm\mathbf{a}_{n-1}^k \end{pmatrix} \\ &= \begin{pmatrix} A_{n-1}(a_0, \ldots, a_{2^{n-1}-1})\mathbf{a}_{n-1}^k \pm A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1})\mathbf{a}_{n-1}^k \\ A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1})\mathbf{a}_{n-1}^k \pm A_{n-1}(a_0, \ldots, a_{2^{n-1}-1})\mathbf{a}_{n-1}^k \end{pmatrix} \\ &= \begin{pmatrix} \mu_{n-1}^k\mathbf{a}_{n-1}^k \pm \hat{\mu}_{n-1}^k\mathbf{a}_{n-1}^k \\ \hat{\mu}_{n-1}^k\mathbf{a}_{n-1}^k \pm \mu_{n-1}^k\mathbf{a}_{n-1}^k \end{pmatrix} = \left(\mu_{n-1}^k \pm \hat{\mu}_{n-1}^k\right) \begin{pmatrix} \mathbf{a}_{n-1}^k \\ \pm\mathbf{a}_{n-1}^k \end{pmatrix} = \left(\mu_{n-1}^k \pm \hat{\mu}_{n-1}^k\right)\mathbf{a}_n^k, \end{aligned}$$

where $\hat{\mu}_{n-1}^k$ is the eigenvalue of the matrix $A_{n-1}(a_{2^{n-1}}, \ldots, a_{2^n-1})$, corresponding to the eigenvector $\mathbf{a}_{n-1}^k$. Obviously, there are 2^n vectors of the form $\mathbf{a}_n^k = \left(\mathbf{a}_{n-1}^k, \pm\mathbf{a}_{n-1}^k\right)^T$. Therefore, all eigenvalues of the matrix $A_n(a_0, \ldots, a_{2^n-1})$, are $\mu_n^{k,\pm} = \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k$.

Orthogonality. It is obvious that the eigenvectors $\mathbf{a}_1^+ = (1,1)^T$, and $\mathbf{a}_1^- = (1,-1)^T$, of the matrix $A_1(a_0,a_1)$, are orthogonal. If the eigenvectors $\mathbf{a}_{n-1}^k$, $k = 0,\dots,2^{n-1}-1$ of the matrix $A_{n-1}(a_{2^{n-1}},\dots,a_{2^n-1})$ are chosen orthogonal, then the eigenvectors $\mathbf{a}_n^k = \left(\mathbf{a}_{n-1}^k, \pm\mathbf{a}_{n-1}^k\right)^T$ of the matrix $A_n(a_0,\dots,a_{2^n-1})$ are also orthogonal:

$$\mathbf{a}_n^{k_1}\mathbf{a}_n^{k_2} = \left(\mathbf{a}_{n-1}^{k_1}, \pm\mathbf{a}_{n-1}^{k_1}\right)^T\left(\mathbf{a}_{n-1}^{k_2}, \pm\mathbf{a}_{n-1}^{k_2}\right)^T = \mathbf{a}_{n-1}^{k_1}\mathbf{a}_{n-1}^{k_2} + \mathbf{a}_{n-1}^{k_1}\mathbf{a}_{n-1}^{k_2} = 0,\ k_1 \neq k_2$$

and $\left(\mathbf{a}_{n-1}^k, \mathbf{a}_{n-1}^k\right)^T\left(\mathbf{a}_{n-1}^k, -\mathbf{a}_{n-1}^k\right)^T = 0$. The theorem is proved. □

Let us give important consequences from Theorem 3 that allow us to build eigenvectors and eigenvalues of the matrix A_n.

Corollary 3. *Let $k = (k_n,\dots,k_1)_2$, $k_i = 0,1$, then the eigenvector of the matrix A_n, numbered by k, can be written in the form:*

$$\mathbf{a}_n^k = (1,(-1)^{k_1},(-1)^{k_2},(-1)^{k_2+k_1},(-1)^{k_3},(-1)^{k_3+k_1},(-1)^{k_3+k_2},\\ (-1)^{k_3+k_2+k_1},(-1)^{k_4},\dots,(-1)^{k_n+\dots+k_1})^T = \left((-1)^{k\otimes m}\right)_{m=0,\dots,2^n-1}, \quad (12)$$

where $k\otimes i \equiv (k_n\dots k_1)_2 \otimes (i_n\dots i_1)_2 = k_n i_n + \dots + k_1 i_1$ is a "scalar" product of the indexes $(k)_2$ and $(i)_2$. The eigenvalue corresponding to the eigenvector $\mathbf{a}_n^{(k_n\dots k_1)_2}$ can be written in a similar form:

$$\mu_n^k \equiv \mu_n^{(k_n\dots k_1)_2} = \sum_{i=0}^{2^n-1}(-1)^{k\otimes i}a_i = \sum_{i=0}^{2^n-1}(-1)^{k_n i_n+\dots+k_1 i_1}a_{(i_n\dots i_1)_2}. \quad (13)$$

Proof. Let us prove (12). For $n=1$ we have $\mathbf{a}_1^+ = \mathbf{a}_1^{(0)_2} = ((-1)^{0\otimes 0},(-1)^{0\otimes 1})^T$, $\mathbf{a}_1^- = \mathbf{a}_1^{(1)_2} = ((-1)^{1\otimes 0},(-1)^{1\otimes 1})^T$ and (12) is true. If Formula (12) is true for the vector $\mathbf{a}_{n-1}^{(k_{n-1}\dots k_1)_2}$, then by Theorem 3 we have:

$$\begin{aligned}\left(\mathbf{a}_{n-1}^{(k_{n-1}\dots k_1)_2}, \pm\mathbf{a}_{n-1}^{(k_{n-1}\dots k_1)_2}\right)^T &= \left(\mathbf{a}_{n-1}^{(k_{n-1}\dots k_1)_2}, (-1)^{k_n}\mathbf{a}_{n-1}^{(k_{n-1}\dots k_1)_2}\right)^T \\ &= \left(\left((-1)^{(k_n k_{n-1}\dots k_1)_2\otimes(0m_{n-1}\dots m_1)_2}\right)_{m=0,\dots,2^{n-1}-1},\right. \\ &\qquad \left.\left((-1)^{(k_n k_{n-1}\dots k_1)_2\otimes(1m_{n-1}\dots m_1)_2}\right)_{m=0,\dots,2^{n-1}-1}\right)^T \\ &= \left(\left((-1)^{k\otimes m}\right)_{m=0,\dots,2^{n-1}-1}, \left((-1)^{k\otimes m}\right)_{m=2^{n-1},\dots,2^n-1}\right)^T \\ &= \left(\left((-1)^{k\otimes m}\right)_{m=0,\dots,2^n-1}\right)^T = \mathbf{a}_n^{(k_n k_{n-1}\dots k_1)_2}\end{aligned}$$

and hence the Formula (12) is also true for the vector $\mathbf{a}_n^{(k_n k_{n-1}\dots k_1)_2} = \mathbf{a}_n^k$.

Let us prove (13). For $n=1$ we have:

$$\mu_1^{k_1} = a_0 + (-1)^{k_1}a_1 = (-1)^0 a_{(0)_2} + (-1)^{k_1\cdot 1}a_{(1)_2},$$

where $k_1 = 0, 1$. Assume that the Formula (13) is valid for $n = n-1$ and prove its validity for n. By Theorem 3, changing the notation $\pm = (-1)^{k_n}$, we write:

$$\mu_n^{k,\pm} = \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k = \sum_{i=0}^{2^{n-1}-1} (-1)^{k_n \cdot 0 + k_{n-1} i_{n-1} + \ldots + k_1 i_1} a_{(i_{n-1} \ldots i_1)_2}$$
$$+ \sum_{i=0}^{2^{n-1}-1} (-1)^{k_n \cdot 1 + k_{n-1} i_{n-1} + \ldots + k_1 i_1} a_{(1 i_{n-1} \ldots i_1)_2} = \sum_{i=0}^{2^{n-1}-1} (-1)^{k_n i_n + k_{n-1} i_{n-1} + \ldots + k_1 i_1} a_{(i_n i_{n-1} \ldots i_1)_2}$$
$$+ \sum_{i=2^{n-1}}^{2^n-1} (-1)^{k_n i_n + k_{n-1} i_{n-1} + \ldots + k_1 i_1} a_{(i_n i_{n-1} \ldots i_1)_2} = \sum_{i=0}^{2^n-1} (-1)^{k_n i_n + \ldots + k_1 i_1} a_{(i_n i_{n-1} \ldots i_1)_2},$$

which proves (13). The corollary is proved. □

Example 2. *For $n = 2$ the matrix:*

$$A_2(a_0, a_1, a_2, a_3) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix},$$

according to Corollary 3, has the following four eigenvectors:

$$\mathbf{a}_2^k = \left(1, (-1)^{k_1}, (-1)^{k_2}, (-1)^{k_2+k_1}\right)^T,$$

where $k = (00)_2, (01)_2, (10)_2, (11)_2$, or:

$$\mathbf{a}_2^0 = \mathbf{a}_2^{(00)_2} = (1,1,1,1)^T, \qquad \mathbf{a}_2^1 = \mathbf{a}_2^{(01)_2} = (1,-1,1,-1)^T,$$
$$\mathbf{a}_2^2 = \mathbf{a}_2^{(10)_2} = (1,1,-1,-1)^T, \qquad \mathbf{a}_2^3 = \mathbf{a}_2^{(11)_2} = (1,-1,-1,1)^T$$

and the following eigenvalues:

$$\begin{aligned} \mu_0 = \mu_{(00)_2} &= a_0 + a_1 + a_2 + a_3, \quad \mu_1 = \mu_{(01)_2} = a_0 - a_1 + a_2 - a_3, \\ \mu_2 = \mu_{(10)_2} &= a_0 + a_1 - a_2 - a_3, \quad \mu_3 = \mu_{(11)_2} = a_0 - a_1 - a_2 + a_3, \end{aligned} \tag{14}$$

where, for convenience, we transfer the superscript of the eigenvalue to the subscript as $n = 2$ is fixed. For the matrix $A_3(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ from the Formula (11) we obtain eigenvectors in the form:

$$\mathbf{a}_3^k = \left(1, (-1)^{k_1}, (-1)^{k_2}, (-1)^{k_2+k_1}, (-1)^{k_3}, (-1)^{k_3+k_1}, (-1)^{k_3+k_2}, (-1)^{k_3+k_2+k_1}\right)^T.$$

For example, for $(101)_2 = 5$ we have an eigenvector of the form:

$$\mathbf{a}_3^5 = \mathbf{a}_3^{(101)_2} = (1, -1, 1, -1, -1, 1, -1, 1)^T.$$

The eigenvalue corresponding to the eigenvector $\mathbf{a}_3^5 = \mathbf{a}_3^{(101)_2}$ is written in a similar form:

$$\mu_3^{(101)_2} = \sum_{i=0}^{7} (-1)^{k_3 i_3 + k_2 i_2 + k_1 i_1} a_{(i_3 i_2 i_1)_2} = a_{(0)_2} - a_{(1)_2} + a_{(10)_2} - a_{(11)_2} - a_{(100)_2}$$
$$+ a_{(101)_2} - a_{(110)_2} + a_{(111)_2} = a_0 - a_1 + a_2 - a_3 - a_4 + a_5 - a_6 + a_7.$$

3. The Main Problem S

To study the Problem S, the following statement is required.

Lemma 1 ([36] (Lemma 3.1))**.** *Let S be an orthogonal matrix, then the operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ commute $\Delta I_S u(x) = I_S \Delta u(x)$ on functions $u \in C^2(\Omega)$. The operator $\Lambda = \sum_{i=1}^{n} x_i u_{x_i}(x)$ and operator I_S also commute $\Lambda I_S u(x) = I_S \Lambda u(x)$ on functions $u \in C^1(\bar{\Omega})$ and the equality $\nabla I_S = I_S S^T \nabla$ is valid.*

Corollary 4. *Equation (1) generates a matrix equation equivalent to it:*

$$A_n \Delta U(x) + \lambda U(x) = 0, \tag{15}$$

where $U(x) = \left(u(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1}$.

Proof. Let $u(x)$ satisfy the Equation (1). We denote:

$$v(x) = \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i u(S_n^{i_n} \dots S_1^{i_1} x),$$

and $V(x) = \left(v(S_n^{j_n} \dots S_1^{j_1} x)\right)^T_{j=0,\dots,2^n-1}$. The function $v(x)$ generates the equality (4). Let us apply the Laplace operator to equality (4). Since the matrices of the form $S_n^{i_n} \dots S_1^{i_1}$ are symmetric and orthogonal, and therefore $\left(S_n^{i_n} \dots S_1^{i_1}\right)^2 = I$, then by virtue of Lemma 1, we can write:

$$\begin{aligned}
\Delta V(x) &= \left(\Delta I_{S_n^{j_n} \dots S_n^{j_1}} v(x)\right)^T_{j=0,\dots,2^n-1} = \left(I_{S_n^{j_n} \dots S_n^{j_1}} \Delta v(x)\right)^T_{j=0,\dots,2^n-1} \\
&= \left(I_{S_n^{j_n} \dots S_n^{j_1}} \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i I_{S_n^{i_n} \dots S_1^{i_1}} \Delta u(x)\right)^T_{j=0,\dots,2^n-1} \\
&= \left(\sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i I_{S_n^{j_n+i_n} \dots S_1^{j_1+i_1}} \Delta u(x)\right)^T_{j=0,\dots,2^n-1} \\
&= \left(\sum_{l \equiv (l_n \dots l_1)_2 = 0}^{(1\dots1)_2} a_{j \oplus l} I_{S_n^{l_n} \dots S_1^{l_1}} \Delta u(x)\right)^T_{l=0,\dots,2^n-1} \\
&= \left(\sum_{l \equiv (l_n \dots l_1)_2 = 0}^{(1\dots1)_2} a_{j \oplus l} \Delta u(S_n^{l_n} \dots S_1^{l_1} x)\right)^T_{j=0,\dots,2^n-1} \\
&= A_n \Delta U(x).
\end{aligned}$$

Hence, using the equality $\Delta v(S_n^{j_n} \dots S_1^{j_1} x) + \lambda u(S_n^{j_n} \dots S_1^{j_1} x) = 0$, we obtain Equation (15). The corollary is proved. □

Basing on Lemma 1, we prove the following statement about necessary conditions for the existence of eigenvalues of problem S.

Theorem 4. *Let the function $u(x) \neq 0$ be an eigenfunction of the problem S, and λ be its eigenvalue, then the function $w(x) = (U(x), \mathbf{a}_n^k)$, where $U(x) = \left(u(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1}$ and $\mathbf{a}_n^k$ is an eigenvector of the matrix $A_n(a_0, \dots, a_{2^n-1})$, is a solution to the Dirichlet problem:*

$$\Delta w(x) + \mu w(x) = 0, \ x \in \Omega, \tag{16}$$

$$w(x) = 0, \ x \in \partial\Omega, \tag{17}$$

where $\mu = \lambda/\mu_n^k$ *and* $\mu_n^k \neq 0$ *is the eigenvalue of the matrix* $A_n(a_0, \ldots, a_{2^n-1})$ *corresponding to the vector* $\mathbf{a}_n^k$.

Proof. Let λ be the eigenvalue of the problem S and $u(x) \neq 0$ be its eigenfunction. By Corollary 4, the equality (15) is true. Let's multiply it scalar by the vector $\mathbf{a}_n^k$. Then, we have:

$$\left(A_n \Delta U(x), \mathbf{a}_n^k\right) + \lambda\left(U(x), \mathbf{a}_n^k\right) = 0,$$

whence, using the symmetry of the matrix $A_n(a_0, \ldots, a_{2^n-1})$ (see Corollary 2) and the properties of the vector $\mathbf{a}_n^k$, we find:

$$\Delta\left(U(x), A_n \mathbf{a}_n^k\right) + \lambda\left(U(x), \mathbf{a}_n^k\right) = 0,$$

whence follows:

$$\mu_n^k \Delta w(x) + \lambda w(x) = 0$$

and since $\lambda = \mu_n^k \mu$, and $\mu_n^k \neq 0$, we get (16):

$$0 = \mu_n^k(\Delta w(x) + \mu w(x)) \Rightarrow \Delta w(x) + \mu w(x) = 0.$$

Finally, since $u(x) = 0$, $x \in \partial\Omega$, and $x \in \partial\Omega \Rightarrow S_n^{i_n} \ldots S_1^{i_1} x \in \partial\Omega$, then $U(x) = 0$, and therefore $w(x) = (U(x), \mathbf{a}_n^k) = 0$, $x \in \partial\Omega$. The theorem is proved. □

The following converse statement to Theorem 4 is important, which allows us to construct solutions to Problem S.

Theorem 5. *Let the function* $w(x) \neq 0$ *be a solution to the problem* (16) *and* (17)*:*

$$\Delta w(x) + \mu w(x) = 0, \; x \in \Omega,$$
$$w(x) = 0, \; x \in \partial\Omega$$

for some μ*, then the function:*

$$u_k(x) = (W(x), \mathbf{a}_n^k), \tag{18}$$

where $W(x) = \left(w(S_n^{i_n} \ldots S_1^{i_1} x)\right)^T_{i=0,\ldots,2^n-1}$ *and* $\mathbf{a}_n^k$ *is an eigenvector of the matrix* $A_n = A_n(a_0, \ldots, a_{2^n-1})$ *with an eigenvalue* $\mu_n^k \neq 0$ *is a solution to the Dirichlet problem* (1) *and* (2) *for* $\lambda = \mu_n^k \mu$.

Proof. Let $w(x) \neq 0$ be a solution to problem (16) and (17). Consider the vector $W(x) = \left(w(S_n^{i_n} \ldots S_1^{i_1} x)\right)^T_{i=0,\ldots,2^n-1}$ and compose the function $u_k(x) = (W(x), \mathbf{a}_n^k)$, where $x \in \Omega$. It is easy to see that, according to Corollary 3, we have in Ω:

$$\begin{aligned} u_k(S_n^{j_n} \ldots S_1^{j_1} x) &= (W(S_n^{j_n} \ldots S_1^{j_1} x), \mathbf{a}_n^k) \\ &= \left(\left(w(S_n^{i_n+j_n} \ldots S_1^{i_1+j_1} x)\right)^T_{i=0,\ldots,2^n-1}, \left((-1)^{k\otimes i}\right)_{i=0,\ldots,2^n-1}\right) \\ &= \left(\left(w(S_n^{l_n} \ldots S_1^{l_1} x)\right)^T_{l=0,\ldots,2^n-1}, \left((-1)^{k\otimes(l\oplus j)}\right)_{l=0,\ldots,2^n-1}\right) \\ &= (-1)^{k\otimes j}\left(W(x), \mathbf{a}_n^k\right) = (-1)^{k\otimes j} u_k(x), \end{aligned}$$

and therefore:

$$U_k(x) = \left(u_k(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1} = \left((-1)^{k\otimes j} u_k(x)\right)^T_{i=0,\dots,2^n-1}$$
$$= u_k(x)\left((-1)^{k\otimes j}\right)^T_{i=0,\dots,2^n-1} = u_k(x)\mathbf{a}_n^k.$$

Thus,

$$\Delta U_k(x) = \Delta u_k(x)\mathbf{a}_n^k$$

and hence, since by Lemma 1:

$$\Delta W(x) = \left(\Delta w(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1} = \left(-\mu w(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1} = -\mu W(x),$$

we get:

$$A_n \Delta U_k(x) = \Delta u_k(x) A_n \mathbf{a}_n^k = (\Delta W(x), \mathbf{a}_n^k)\mu_n^k \mathbf{a}_n^k = -\mu(W(x), \mathbf{a}_n^k)\mu_n^k \mathbf{a}_n^k$$
$$= -\mu\mu_n^k u_k(x)\mathbf{a}_n^k = -\mu\mu_n^k U_k(x).$$

Separating the first components of this vector equality, we obtain:

$$\sum_{i=0}^{2^n-1} a_i \Delta u_k(S_n^{i_n} \dots S_1^{i_1} x) = -\mu\mu_n^k u_k(x),\ x \in \Omega,$$

which means that $u_k(x)$ is a solution to Equation (1). Let us check the boundary conditions (2) of the problem S. Since $x \in \partial\Omega \Rightarrow S_n^{i_n} \dots S_1^{i_1} x \in \partial\Omega$, then for $x \in \partial\Omega$ we get:

$$u_k(x) = \left(\left(w(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1}, \mathbf{a}_n^k\right) = \left(\mathbf{0}, \mathbf{a}_n^k\right) = 0.$$

The theorem is proved. □

Example 3. *Let $n = 2$. According to Example 2, the eigenvectors of the matrix $A_2(a_0, a_1, a_2, a_3)$ have the form:*

$$\mathbf{a}_2^0 = (1,1,1,1)^T,\ \mathbf{a}_2^1 = (1,-1,1,-1)^T,\ \mathbf{a}_2^2 = (1,1,-1,-1)^T,\ \mathbf{a}_2^3 = (1,-1,-1,1)^T$$

and by Theorem 5 the eigenfunctions of the problem corresponding to the eigenvalue μ and the eigenfunction $w_\mu(x)$ of problem (16) and (17) can be taken in the form $u_k(x) = (W(x), \mathbf{a}_n^k)$, $k = 0,1,2,3$ or:

$$\begin{aligned}
u_0(x) &= w_\mu(x) + w_\mu(S_1 x) + w_\mu(S_2 x) + w_\mu(S_1 S_2 x),\\
u_1(x) &= w_\mu(x) - w_\mu(S_1 x) + w_\mu(S_2 x) - w_\mu(S_1 S_2 x),\\
u_2(x) &= w_\mu(x) + w_\mu(S_1 x) - w_\mu(S_2 x) - w_\mu(S_1 S_2 x),\\
u_3(x) &= w_\mu(x) - w_\mu(S_1 x) - w_\mu(S_2 x) + w_\mu(S_1 S_2 x).
\end{aligned}$$

In what follows, it will be necessary to expand the polynomials into the sum of the "generalized parity" polynomials.

Lemma 2. *Let $H(x)$ be some function on Ω. We denote:*

$$H^{(k_n \dots k_1)_2}(x) = \frac{1}{2^n} \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1 \dots 1)_2} (-1)^{k \otimes i} H(S_n^{i_n} \dots S_1^{i_1} x),\ x \in \Omega.$$

Then the function $H^{(k_n \ldots k_1)_2}(x)$ has the "generalized parity" property:

$$H^{(k_n \ldots k_1)_2}(S_i x) = (-1)^{k_i} H^{(k_n \ldots k_1)_2}(x) \tag{19}$$

and besides, the following equality:

$$\sum_{i \equiv (i_n \ldots i_1)_2 = 0}^{(1 \ldots 1)_2} (-1)^{m \otimes i} H^{(k_n \ldots k_1)_2}(S_n^{i_n} \ldots S_1^{i_1} x) = \begin{cases} H^{(k_n \ldots k_1)_2}(x) & m = k \\ 0 & m \neq k \end{cases} \tag{20}$$

holds true. Moreover, the function $H(x)$, $x \in \Omega$ can be represented as:

$$H(x) = \sum_{k \equiv (k_n \ldots k_1)_2 = 0}^{(1 \ldots 1)_2} H^{(k_n \ldots k_1)_2}(x),\ x \in \Omega. \tag{21}$$

Proof. It is not hard to see that:

$$H^{(k_n \ldots k_1)_2}(S_i x) = \frac{1}{2^n} \sum_{j \equiv (j_n \ldots j_1)_2 = 0}^{(1 \ldots 1)_2} (-1)^{k \otimes j} H(S_n^{j_n} \ldots S_i^{j_i + 1} \ldots S_1^{j_1} x)$$
$$= \frac{1}{2^n} \sum_{j \equiv (j_n \ldots j_1)_2 = 0}^{(1 \ldots 1)_2} (-1)^{k_n j_n + \ldots + k_1 j_1 + k_i} H(S_n^{j_n} \ldots S_i^{j_i} \ldots S_1^{j_1} x) = (-1)^{k_i} H^{(k_n \ldots k_1)_2}(x),$$

where a change of variables is made under the sum sign, as in Theorem 2. Equality (19) is proved.

Consider now, equality (21). It is easy to see that for $x \in \Omega$:

$$\sum_{k=0}^{(1 \ldots 1)_2} H^{(k_n \ldots k_1)_2}(x) = \sum_{k=0}^{(1 \ldots 1)_2} \frac{1}{2^n} \sum_{i=0}^{(1 \ldots 1)_2} (-1)^{k \otimes i} H(S_n^{i_n} \ldots S_1^{i_1} x)$$
$$= \sum_{i=0}^{(1 \ldots 1)_2} H(S_n^{i_n} \ldots S_1^{i_1} x) \frac{1}{2^n} \sum_{k=0}^{(1 \ldots 1)_2} (-1)^{k \otimes i}. \tag{22}$$

Let us calculate the inner sum from the right-hand side of equalities (22). It is clear that $i \neq 0 \Rightarrow \exists j\, i_j \neq 0$, and then:

$$\sum_{k=0}^{(1 \ldots 1)_2} (-1)^{k \otimes i} = \sum_{k_j=0}^{1} (-1)^{k_j i_j} \left(\sum_{k_n=0}^{1} \cdots \sum_{k_1=0}^{1} (-1)^{k_n i_n + \ldots + k_1 i_1} \right)$$
$$= (-1)^0 \left(\sum_{k_n=0, \ldots, k_1=0, \cancel{k_j}}^{1, \ldots, 1} (-1)^{k_n i_n + \ldots + k_1 i_1} \right) + (-1)^{i_j} \left(\sum_{k_n=0, \ldots, k_1=0, \cancel{k_j}}^{1, \ldots, 1} (-1)^{k_n i_n + \ldots + k_1 i_1} \right) = 0,$$

If $i = 0$, then $\sum_{k=0}^{(1 \ldots 1)_2} (-1)^{k \otimes 0} = 2^n$, i.e., $\frac{1}{2^n} \sum_{k=0}^{(1 \ldots 1)_2} (-1)^{k \otimes i} = \delta_{k,0}$. Therefore, (22) implies (21). Now let us prove (20). It is not hard to see that:

$$
\begin{aligned}
&\sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2}(-1)^{m\otimes i}H^{(k_n\ldots k_1)_2}(S_n^{i_n}\ldots S_1^{i_1}x)\\
&=\sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2}(-1)^{m\otimes i}\frac{1}{2^n}\sum_{j\equiv(j_n\ldots j_1)_2=0}^{(1\ldots1)_2}(-1)^{k\otimes j}H(S_n^{i_n+j_n}\ldots S_1^{i_1+j_1}x)\\
&=\sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2}(-1)^{m\otimes i}\frac{1}{2^n}\sum_{l\equiv(l_n\ldots l_1)_2=0}^{(1\ldots1)_2}(-1)^{k\otimes(l\oplus i)}H(S_n^{l_n}\ldots S_1^{l_1}x)\\
&=\sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2}(-1)^{m\otimes i+k\otimes i}\frac{1}{2^n}\sum_{l\equiv(l_n\ldots l_1)_2=0}^{(1\ldots1)_2}(-1)^{k\otimes l}H(S_n^{l_n}\ldots S_1^{l_1}x)\\
&=H^{(k_n\ldots k_1)_2}(x)\frac{1}{2^n}\sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2}(-1)^{(m\oplus k)\otimes i}=H^{(k_n\ldots k_1)_2}(x)\delta_{k,m}
\end{aligned}
$$

Here it is taken into account that $m\oplus k=0\Leftrightarrow m=k$. The lemma is proved. □

Example 4. *Let $l=2$ and $n=2$, $S_1x=(-x_1,x_2)$, $S_2x=(x_1,-x_2)$. Then, according to Lemma 2, the generalized parity components for the function $H(x)$ from expansion (21) have the form:*

$$
\begin{aligned}
H^0(x)&=H^{(00)_2}(x)=\frac{1}{4}(H(x_1,x_2)+H(-x_1,x_2)+H(x_1,-x_2)+H(-x_1,-x_2)),\\
H^1(x)&=H^{(01)_2}(x)=\frac{1}{4}(H(x_1,x_2)+(-1)^{(01)_2\otimes(01)_2}H(-x_1,x_2)\\
&\quad+(-1)^{(01)_2\otimes(10)_2}H(x_1,-x_2)+(-1)^{(01)_2\otimes(11)_2}H(-x_1,-x_2))\\
&=\frac{1}{4}(H(x_1,x_2)-H(-x_1,x_2)+H(x_1,-x_2)-H(-x_1,-x_2)),\\
H^2(x)&=H^{(10)_2}(x)=\frac{1}{4}(H(x_1,x_2)+H(-x_1,x_2)-H(x_1,-x_2)-H(-x_1,-x_2)),\\
H^3(x)&=H^{(11)_2}(x)=\frac{1}{4}(H(x_1,x_2)-H(-x_1,x_2)-H(x_1,-x_2)+H(-x_1,-x_2)).
\end{aligned}
$$

If, for example, the function $H(x)$ is even in x_1 then its components of generalized parity 1 and 3 is zero $H^1(x)=0$, $H^3(x)=0$.

Let $H(x)=H_m(x)$ be homogeneous harmonic polynomial of degree m. Then, if (r,φ) are polar coordinates of $x=(x_1,x_2)$, then:

$$
H_m(x)=\alpha\,\mathrm{Re}\,(x_1+ix_2)^m+\beta\,\mathrm{Im}\,(x_1+ix_2)^m=r^m(\alpha\cos m\varphi+\beta\sin m\varphi)
$$

and:

$$
\begin{aligned}
&H_m(-x_1,x_2)=\alpha\,\mathrm{Re}\,(-x_1+ix_2)^m+\beta\,\mathrm{Im}\,(-x_1+ix_2)^m=(-r)^m(\alpha\cos m\varphi-\beta\sin m\varphi),\\
&H_m(x_1,-x_2)=\alpha\,\mathrm{Re}\,(x_1-ix_2)^m+\beta\,\mathrm{Im}\,(x_1-ix_2)^m=r^m(\alpha\cos m\varphi-\beta\sin m\varphi),\\
&H_m(-x_1,-x_2)=(-r)^m(\alpha\cos m\varphi+\beta\sin m\varphi).
\end{aligned}
$$

From these equalities we get:

$$
\begin{aligned}
H_m^0(x)&=\frac{r^m}{2}\alpha(1+(-1)^m)\cos m\varphi, & H_m^1(x)&=\frac{r^m}{2}\alpha(1-(-1)^m)\cos m\varphi,\\
H_m^2(x)&=\frac{r^m}{2}\alpha(1-(-1)^m)\sin m\varphi, & H_m^3(x)&=\frac{r^m}{2}\alpha(1+(-1)^m)\sin m\varphi.
\end{aligned}
$$

Therefore, for $m \in \mathbb{N}_0$:

$$H^0_{2m}(x) = \alpha r^{2m}\cos 2m\varphi,\ H^1_{2m}(x) = 0,\ H^2_{2m}(x) = \beta r^{2m}\sin 2m\varphi,\ H^3_{2m}(x) = 0,$$
$$H^0_{2m-1}(x) = 0,\ H^1_{2m}(x) = \alpha r^{2m+1}\cos(2m+1)\varphi,$$
$$H^2_{2m}(x) = 0, H^3_{2m}(x) = \beta r^{2m+1}\sin(2m+1)\varphi.$$

4. Eigenfunctions and Eigenvalues of Problem S

Let us transform the result of Theorem 5 to a simpler form.

Theorem 6. *The eigenfunctions and eigenvalues of the Dirichlet problem (1) and (2) from Theorem 5 can be represented as:*

$$u^k_n(x) = \sum_{i\equiv(i_n\ldots i_1)_2=0}^{(1\ldots1)_2} (-1)^{k\otimes i} w_\mu(S_n^{i_n}\ldots S_1^{i_1}x), \quad \lambda_{\mu,k} = \mu\sum_{i=0}^{2^n-1}(-1)^{k\otimes i}a_i, \tag{23}$$

where the function $w_\mu(x)$ is a solution to the problem (16) and (17):

$$\Delta w(x) + \mu w(x) = 0,\ x\in\Omega; \quad w(x) = 0,\ x\in\partial\Omega$$

for some $\mu \in \mathbb{R}_+$. Functions $u^k_n(x)$, for $k = 0,\ldots,2^n-1$ are orthogonal in $L_2(\Omega)$.

Proof. We prove Formula (23) by induction on n. For $n = 1$ from (18), taking into account the equalities $\mathbf{a}^0_1 = (1,1)^T$, $\mathbf{a}^1_1 = (1,-1)^T$ from Theorem 3, we obtain:

$$u_0(x) = (W(x), \mathbf{a}^0_1) = w_\mu(x) + w_\mu(S_1x) \equiv u^0_1(x)$$
$$u_1(x) = (W(x), \mathbf{a}^1_1) = w_\mu(x) - w_\mu(S_1x) \equiv u^1_1(x).$$

We shifted the subscript of the functions $u_k(x)$ from (18) to the top to make room for the n subscript. Suppose that Formula (23) is valid for $n = n-1$ and prove its validity for n. In accordance with Theorems 3 and 5, we have $\mathbf{a}^k_n = \left(\mathbf{a}^k_{n-1}, \pm\mathbf{a}^k_{n-1}\right)^T$ and $u_k(x) = (W(x), \mathbf{a}^k_n)$ and hence the function:

$$u_{(k_nk_{n-1}\ldots k_1)_2}(x) = u_{(k_{n-1}\ldots k_1)_2}(x) + (-1)^{k_n}u_{(k_{n-1}\ldots k_1)_2}(S_nx)$$

is an eigenfunction of the Dirichlet problem (1) and (2). Using the induction hypothesis, we transform this function:

$$\begin{aligned} u_{(k_nk_{n-1}\ldots k_1)_2}(x) &= \sum_{i\equiv(0i_{n-1}\ldots i_1)_2=0}^{(01\ldots1)_2} (-1)^{k_n0+(k_{n-1}\ldots k_1)_2\otimes(i_{n-1}\ldots i_1)_2} w_\mu(S_n^0S_{n-1}^{i_{n-1}}\ldots S_1^{i_1}x) \\ &+ \sum_{i\equiv(1i_{n-1}\ldots i_1)_2=0}^{(11\ldots1)_2} (-1)^{k_n1+(k_{n-1}\ldots k_1)_2\otimes(i_{n-1}\ldots i_1)_2} w_\mu(S_nS_{n-1}^{i_{n-1}}\ldots S_1^{i_1}x) \\ &= \sum_{i\equiv(i_ni_{n-1}\ldots i_1)_2=0}^{(11\ldots1)_2} (-1)^{(k_nk_{n-1}\ldots k_1)_2\otimes(i_ni_{n-1}\ldots i_1)_2} w_\mu(S_n^{i_n}S_{n-1}^{i_{n-1}}\ldots S_1^{i_1}x) \equiv u_n^{(k_n\ldots k_1)_2}(x), \end{aligned}$$

which proves Formula (23). The eigenvalues of the Dirichlet problem (1) and (2) corresponding to eigenfunction $u^k_n(x)$, by Corollary 3, have the form:

$$\lambda_{\mu,k} = \mu\mu^k_n = \mu\sum_{i=0}^{2^n-1}(-1)^{k\otimes i}a_i.$$

Now let us prove that the functions $u_n^k(x) = u_n^{(k_n\dots k_1)_2}(x)$ for different k are orthogonal in $L_2(\Omega)$. Indeed, if $k \neq m$, then there exists i such that $k_i \neq m_i$ and hence $k_i + m_i \neq 0 \bmod 2$. According to Lemma 4.1 from [37] the following equality holds true for $g \in C(\Omega)$:

$$\int_\Omega g(S_i\xi)\, d\xi = \int_\Omega g(\xi)\, d\xi.$$

Therefore using equality (19) from Lemma 2 we get:

$$\begin{aligned}\int\limits_\Omega u_n^{(k_n\dots k_1)_2}(x) u_n^{(m_n\dots m_1)_2}(x)\, dx &= \int\limits_\Omega u_n^{(k_n\dots k_1)_2}(S_i x) u_n^{(m_n\dots m_1)_2}(S_i x)\, dx = \\ = (-1)^{k_i+m_i} \int\limits_\Omega u_n^{(k_n\dots k_1)_2}(x) u_n^{(m_n\dots m_1)_2}(x)\, dx &= -\int\limits_\Omega u_n^{(k_n\dots k_1)_2}(x) u_n^{(m_n\dots m_1)_2}(x)\, dx. \end{aligned} \tag{24}$$

This immediately implies the orthogonality:

$$\int\limits_\Omega u_n^{(k_n\dots k_1)_2}(x) u_n^{(m_n\dots m_1)_2}(x)\, dx = 0.$$

The theorem is proved. □

Corollary 5. *If $H(x)$ is a harmonic polynomial, then the polynomials $H^{(k_n\dots k_1)_2}(x)$ for different k are orthogonal on $\partial\Omega$ and therefore these polynomials are linearly independent.*

Proof. Indeed, for $k \neq m$, similarly to (24), by Lemma 4.1 from [37], we obtain:

$$\begin{aligned}\int\limits_{\partial\Omega} H^{(k_n\dots k_1)_2}(x) H^{(m_n\dots m_1)_2}(x)\, ds &= \int\limits_{\partial\Omega} H^{(k_n\dots k_1)_2}(S_i x) H^{(m_n\dots m_1)_2}(S_i x)\, ds \\ = (-1)^{k_i+m_i} \int\limits_{\partial\Omega} H^{(k_n\dots k_1)_2}(x) H^{(m_n\dots m_1)_2}(x)\, ds &= -\int\limits_{\partial\Omega} H^{(k_n\dots k_1)_2}(x) H^{(m_n\dots m_1)_2}(x)\, ds,\end{aligned}$$

whence the assertion of the corollary follows. □

Remark 1. *If we denote:*

$$U_n(x) = \left(u_n^i(x)\right)^T_{i=0,\dots,2^n-1}, \quad W_n(x) = \left(w_\mu(S_n^{i_n} \dots S_1^{i_1} x)\right)^T_{i=0,\dots,2^n-1},$$
$$\mathbf{V}_n = \left(\mathbf{a}_n^i\right)^T_{i=0,\dots,2^n-1} = \left((-1)^{i\otimes j}\right)_{i,j=0,\dots,2^n-1},$$

then equalities (23) can be written in the matrix form $U_n = \mathbf{V}_n W_n$, where the matrix $\mathbf{V}_n$ is symmetric and orthogonal.

Indeed the symmetry of $\mathbf{V}_n$ follows from the equality $(-1)^{i\otimes j} = (-1)^{j\otimes i}$ and the orthogonality is proved in Theorem 3.

Example 5. *For $n = 2$, according to Example 3, the matrix $\mathbf{V}_2$ has the form:*

$$\mathbf{V}_2 = \left(\mathbf{a}_2^i\right)^T_{i=0,\dots,3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It is seen that the matrix $\mathbf{V}_2$ is symmetric and orthogonal.

Now, we transform the results of Theorem 6 and investigate the completeness of the eigenfunctions of Problem S.

Theorem 7. *Let $\mu_n^k \neq 0$, $k = 0, \ldots, 2^n - 1$. Then the system of eigenfunctions of the Dirichlet problem (1) and (2) is complete in $L_2(\Omega)$ and has the form:*

$$u_n^{\mu,m,k,j}(x) = \frac{1}{|x|^{l/2-1}} J_{m+l/2-1}(\sqrt{\mu}|x|) H_m^{(k_n\ldots k_1)_2,j}(x/|x|), \tag{25}$$

where $J_\nu(t)$ is the Bessel function of the first kind, $\sqrt{\mu}$ is a root of the Bessel function $J_{m+l/2-1}(t)$, $\left\{H_m^{(k_n\ldots k_1)_2,j}(\xi) : j = 1, \ldots, j_k\right\}$ is a system of orthogonal on $\partial\Omega$ homogeneous harmonic polynomials of degree m and generalized parity $k = (k_n \ldots k_1)_2$. The eigenvalues of problem S are $\lambda_{\mu,k} = \mu \sum\limits_{i=0}^{2^n-1} (-1)^{k\otimes i} a_i$.

Proof. Since the eigenfunctions of problem (16) and (17) have the form (see, for example, Refs. [38,39]):

$$w_{(\mu,m,j)}(x) = \frac{1}{|x|^{l/2-1}} J_{m+l/2-1}(\sqrt{\mu}|x|) H_m^j\left(\frac{x}{|x|}\right), \tag{26}$$

where $\left\{H_m^j(x) : j = 1, \ldots, h_m\right\}$, $h_m = \frac{2m+l-2}{l-2}\binom{m+l-3}{l-3}$ $(l > 2)$ is the system of homogeneous harmonic polynomials of degree m orthogonal on $\partial\Omega$ (see, for example, Ref. [40]) and $|x| = |S_i x|$, then the expansion (23) rather refers to homogeneous harmonic polynomials $H_m^j(x)$. We decompose the entire space of homogeneous harmonic polynomials of degree m into the sum of subspaces of the same "generalized parity" $(k_n \ldots k_1)_2$ (see equality (19)). This is possible due to the proof in Corollary 5, orthogonality on $\partial\Omega$ of harmonic polynomials of different "generalized parity" k, and then in each subspace we choose a complete system $\left\{H_m^{(k_n\ldots k_1)_2,j}(x) : j = 1, \ldots, j_k\right\}$ of homogeneous harmonic polynomials orthogonal on $\partial\Omega$. Note that for some k it is possible $j_k = 0$, that is, for such k components $H_m^{(k_n\ldots k_1)_2,j}(x)$ are missing (see Example 4). Taking into account the notations of Lemma 2 and adding the "generalized parity" index k, we obtain the functions (25):

$$\begin{aligned} u_n^{\mu,m,k,j}(x) &= \sum_{(i_n\ldots i_1)_2=0}^{(1\ldots1)_2} \frac{(-1)^{k\otimes i}}{|S_n^{i_n}\ldots S_1^{i_1}x|^{m+l/2-1}} J_{m+l/2-1}\left(\sqrt{\mu}|S_n^{i_n}\ldots S_1^{i_1}x|\right) H_m^{k,j}\left(S_n^{i_n}\ldots S_1^{i_1}x\right) \\ &= \frac{1}{|x|^{l/2-1}} J_{m+l/2-1}(\sqrt{\mu}|x|) H_m^{(k_n\ldots k_1)_2,j}(x/|x|). \end{aligned}$$

In Theorem 6 it is shown that the functions $u_n^{\mu,m,k,j}(x)$ are orthogonal for fixed μ and m. Moreover, since the Bessel functions $J_{m+l/2-1}(\sqrt{\mu}t)$ are orthogonal in $L_2((0,1);t)$ for each fixed $m \in \mathbb{N}_0$ and different μ, and the polynomials $\left\{H_m^{(k_n\ldots k_1)_2,j}(x)\right\}$ are orthogonal in $L_2(\partial\Omega)$ for different (m,k,j), then the functions $u_n^{\mu,m,k,j}(x)$ from (25) are orthogonal in $L_2(\Omega)$. Indeed, for different (μ, m, k, j) we have the equality:

$$\begin{aligned} &\int_\Omega u_n^{\mu_1,m_1,k_1,j_1}(x) u_n^{\mu_2,m_2,k_2,j_2}(x)\,dx \\ &\qquad = \int_0^1 \rho J_{m_1+l/2-1}(\sqrt{\mu}\rho) J_{m_2+l/2-1}(\sqrt{\mu}\rho) d\rho \cdot \int_{\partial\Omega} H_{m_1}^{k_1,j_1}(\xi) H_{m_2}^{k_2,j_2}(\xi)\,ds_\xi = 0. \end{aligned}$$

For $\mu_1 \neq \mu_2$ and $m_1 = m_2$, due to the properties of the Bessel functions, the first factor is zero. If $m_1 \neq m_2$, by the property of harmonic polynomials, the second factor from the right is zero. If $m_1 = m_2$ and $\mu_1 = \mu_2$, then for $(k_1, j_1) \neq (k_2, j_2)$ the second factor from the right, is zero by the construction of the polynomials $H_m^{k,j}(x)$ and in view of Corollary 5.

The constructed system of functions (25) is complete in $L_2(\Omega) = L_2((0,1) \times \partial\Omega)$ by Lemma 2 from [41] (p. 33): the system $\{J_{m+l/2-1}(\sqrt{\mu}\rho) : J_{m+l/2-1}(\sqrt{\mu}) = 0\}$ is orthogonal and complete in $L_2((0,1);t)$ for each m, and the system $\left\{H_m^{k,j}(\xi)\right\}$ is orthogonal and complete in $L_2(\partial\Omega)$ for different $\{m, k, j\}$. The theorem is proved. □

Example 6. *Let $l = 2$, $n = 2$, $S_1x = (-x_1, x_2)$, $S_2x = (x_1, -x_2)$ then problem S has the form:*

$$a_0\Delta u(x_1, x_2) + a_1\Delta u(-x_1, x_2) + a_2\Delta u(x_1, -x_2) + a_3\Delta u(-x_1, -x_2) + \lambda u(x) = 0, \ x \in \Omega,$$
$$u(x) = 0, \ x \in \partial\Omega.$$

Let us find the eigenfunctions of the problem (1) and (2) using Example 4. The eigenfunctions of the Dirichlet problem (16) and (17) in the polar coordinate system are determined according to equality (26) (see also [41]) (p. 392) in the form:

$$w_{(\mu,m,0)}(x) = J_m(\sqrt{\mu}r)\cos m\varphi, \quad w_{(\mu,m,1)}(x) = J_m(\sqrt{\mu}r)\sin m\varphi, \ m \in \mathbb{N}_0,$$

where $\sqrt{\mu}$ is a positive root of the Bessel function $J_m(t)$:

$$J_m(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+m)!j!}\left(\frac{t}{2}\right)^{2j+m}.$$

Using Formula (25), we write:

$$u_2^{\mu,m,k,j}(x) = J_m(\sqrt{\mu}|x|)H_m^{(k_2k_1)_2,j}(x/|x|), \ j = 1, \dots, j_k.$$

According to Example 4, for m even $j_0 = j_2 = 1$, $j_1 = j_3 = 0$ and for m odd $j_0 = j_2 = 0$, $j_1 = j_3 = 1$. Therefore, taking into account (13), we write:

$$\begin{aligned}
u_2^{\mu,2m,0,1}(x) &= J_{2m}(\sqrt{\mu}r)\cos 2m\varphi, && \lambda_{\mu,k} = \mu(a_0 + a_1 + a_2 + a_3)\\
u_2^{\mu,2m+1,2,1}(x) &= J_{2m+1}(\sqrt{\mu}r)\sin(2m+1)\varphi, && \lambda_{\mu,k} = \mu(a_0 + a_1 - a_2 - a_3)\\
u_2^{\mu,2m+1,1,1}(x) &= J_{2m+1}(\sqrt{\mu}r)\cos(2m+1)\varphi, && \lambda_{\mu,k} = \mu(a_0 - a_1 + a_2 - a_3)\\
u_2^{\mu,2m,3,1}(x) &= J_{2m}(\sqrt{\mu}r)\sin 2m\varphi, && \lambda_{\mu,k} = \mu(a_0 - a_1 - a_2 + a_3),
\end{aligned}$$

where $\sqrt{\mu}$ is a root of the corresponding Bessel function and $m \in \mathbb{N}_0$. The obtained functions are complete in $L_2(\Omega)$.

5. Conclusions

Summarizing the investigation carried out, we note that due to the properties of the special form matrices A_n from the equality (4), studied in Theorems 1–3, we managed in Theorem 5, Theorem 6, and then in Theorem 7 to write out the complete system of eigenfunctions and eigenvalues of the nonlocal problem S. If we consider possible further applications of the proposed method, we note that a similar method can be used to study the eigenfunctions and eigenvalues of the Neumann and Robin boundary value problems in a ball. Moreover, we hope that the proposed method also allows for a given nonlocal Laplace operator to investigate the spectral problem in l-dimensional parallelepiped and to find an explicit form of the eigenfunctions and eigenvalues of the Dirichlet and Neumann boundary value problems, as well as for problems with periodic conditions. Described problems are the subject of further work and we are going to consider them in our next articles.

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