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Axisymmetric Large Deflection Elastic Analysis of Hollow Annular Membranes under Transverse Uniform Loading

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Abstract: The anticipated use of a hollow linearly elastic annular membrane for designing elastic shells has provided an impetus for this paper to investigate the large deflection geometrically nonlinear phenomena of such a hollow linearly elastic annular membrane under transverse uniform loads. The so-called hollow annular membranes differ from the traditional annular membranes available in the literature only in that the former has the inner edge attached to a movable but weightless rigid concentric circular ring while the latter has the inner edge attached to a movable but weightless rigid concentric circular plate. The hollow annular membranes remove the transverse uniform loads distributed on “circular plate” due to the use of “circular ring” and result in a reduction in elastic response. In this paper, the large deflection geometrically nonlinear problem of an initially flat, peripherally fixed, linearly elastic, transversely uniformly loaded hollow annular membrane is formulated, the problem formulated is solved by using power series method, and its closed-form solution is presented for the first time. The convergence and effectiveness of the closed-form solution presented are investigated numerically. A comparison between closed-form solutions for hollow and traditional annular membranes under the same conditions is conducted, to reveal the difference in elastic response, as well as the influence of different closed-form solutions on the anticipated use for designing elastic shells.

Keywords: annular membrane; uniform transverse loading; structural analysis; large deflection; power series method; closed-form solution

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1. Introduction

The so-called circular or annular membrane problem is well known, that is, the problem of axisymmetric deformation and deflection of an initially flat circular or annular membrane subjected to uniform transverse (lateral) or normal loading. In the existing literature, however, attention is mainly focused on the circular membrane problem, while the closed-form solutions for the annular membrane problem are available in a few cases.

H. Hencky was the first scholar to deal with analytically the circular membrane problem, who used the power series method to solve the system of equations of large deflection of thin plates with vanishing bending stiffness [1]. This system of equations was, originally by A. Föppl, derived from the classical Föppl–von Kármán equations of large deflection of thin plates [2–4]. Therefore, the circular membrane problem is also called Föppl–Hencky membrane problem or simply Hencky problem, but H. Hencky originally dealt with only the case of a stress-free circular membrane subjected to uniform transverse loading [1]. A computational error in the power series solution which was presented by H. Hencky in 1915 [1] was subsequently corrected by Chien in 1948 [5] and Alekseev in 1953 [6], respectively. This solution is usually called Hencky solution, which is often cited in some studies of related issues [7–21]. The closed-form solutions of the circular membrane problem under normal loading are available in a few cases in the

existing literature [22–24]. Fichter [22] dealt analytically with the problem of axisymmetric deformation and deflection of a stress-free circular membrane under gas pressure (uniform normal loading), where the horizontal component of the applied gas pressure was included. This horizontal component is an extra component in comparison with the Hencky solution and was neglected in Campbell's solution [23]. Shi et al. [24] presented the closed-form solution for the problem of axisymmetric deformation and deflection of a prestressed circular membrane under gas pressure.

S. A. Alekseev was the first scholar to deal with analytically the annular membrane problem, who considered an annular membrane structure containing a movable rigid concentric circular thin plate on which a transverse concentrated force acts, i.e., an annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate loaded by a transverse concentrated force at its center [25]. The problem which was originally dealt with by Alekseev [25] is actually the problem of axisymmetric deformation and deflection of an initially flat, peripherally fixed annular membrane subjected to the transverse loads distributed linearly uniformly along its movable but indeformable inner edge, but Alekseev [25] failed to give a complete solution to the problem he was considering. The closed-form solution presented by Alekseev [25] is valid only when the Poisson's ratio of membrane is less than $1/3$. Sun et al. [26] presented the complete closed-form solution of the Alekseev's problem, which is suitable for the case where the Poisson's ratio of membrane may be less or greater than $1/3$, or equal to $1/3$. Yang et al. [27] extended the closed-form solution presented by Sun et al. [26] to include the case of an annular membrane with pre-stress. Lian et al. [28] dealt with analytically the problem of axisymmetric deformation and deflection of the classic Alekseev's annular membrane structure subjected to uniform transverse loading, where both the entire annular membrane and the movable rigid concentric circular thin plate are subjected to uniformly distributed transverse loads. The closed-form solution presented by Lian et al. was incorporated into improvement of a capacitive pressure sensor [28].

In this paper, attention is focused on the problem of axisymmetric deformation and deflection of a hollow annular membrane structure subjected to uniform transverse loading, an initially flat, linearly elastic, transversely uniformly loaded annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge attached to a movable but weightless rigid concentric ring. The anticipated use of the closed-form solution of this problem for elastic shell design has provided an impetus for us to investigate this large deflection phenomenon. We are carrying out a research on what is called an elastic conjugate shell of revolution, which was funded by the National Natural Science Foundation of China (Grant No. 11772072). The so-called elastic conjugate shells of revolution are considered to have an optimal relationship between structural deformation and structural stress state and thus have a strong stable bearing capacity. The closed-form solution of this problem can be used to design such an elastic conjugate shell of revolution, a hollow annular conjugate shell of revolution that bears uniformly distributed transverse loads.

Essential to the design of such a hollow annular conjugate shell of revolution is the ability to obtain the shape and stress distribution of the hollow annular membrane structure under uniformly distributed transverse loads. Therefore, the main task of this paper is to give the closed-form solutions of deflection and stress of the problem under consideration, which are not yet available in the existing literature. In the following section, the problem of axisymmetric deformation and deflection of the hollow annular membrane structure subjected to uniform transverse loading is formulated and is, by using power series method, solved. In Section 3, some important issues are discussed, such as the convergence and effectiveness of the closed-form solution presented, and the difference between the closed-form solutions presented in this paper and in Lian et al. [28] which are both suitable for uniform transverse loading. Concluding remarks are given in Section 4.

2. Membrane Equations and Solutions

Let us consider a rotationally symmetric, linearly elastic, initially flat annular membrane with Poisson's ratio ν , Young's modulus of elasticity E , inner radius b , outer radius a , and thickness h , whose outer edge is tightly fixed to an immovable clamping device and inner edge is tightly clamped by a movable but weightless rigid concentric circular ring, resulting in an immovable and indeformable outer edge and a movable but indeformable inner edge attached to a movable but weightless rigid concentric circular ring. A uniformly distributed transverse load q is quasi-statically applied onto the annular membrane, resulting in an out-of-plane displacement (deflection) of the annular membrane, as shown in Figure 1, where a cylindrical coordinate system (r, φ, w) is introduced, whose coordinate origin is placed in the centroid of the geometric intermediate plane of the initially flat annular membrane and its polar coordinate plane (r, φ) is located in the plane in which the geometric intermediate plane is located, o denotes the coordinate origin of the coordinate system (r, φ, w) , r denotes the radial coordinate, φ denotes the angle coordinate (but it is not represented in Figure 1), and w denotes the axial coordinate whose positive direction corresponds to the deflecting direction of the annular membrane.

Figure 1. Sketch of the hollow annular membrane under uniformly distributed transverse loads q , where the dash-dot line represents the geometric intermediate plane of the initially flat hollow annular membrane and there is no load q in the hollow area ($0 \leq r \leq b$).

A free body of a piece of annular membrane with radius r ($b \leq r \leq a$) is removed from the central portion of the deflected hollow annular membrane, to study the static problem of equilibrium of this free body under the joint action of the external active force $(\pi r^2 - \pi b^2)q$ produced by the transverse loads q and the reactive force $2\pi r\sigma_r h$ produced by the membrane force $\sigma_r h$ acted on the boundary r , as shown in Figure 2, where σ_r denotes the radial stress and θ denotes the slope angle of the deflected annular membrane.

Figure 2. A free body of radius $b \leq r \leq a$ removed from the deflected hollow annular membrane.

Therefore, the so-called out-of-plane equilibrium equation can be obtained from the equilibrium condition that the resultant force in the transverse (vertical) direction is equal to zero, that is

$$2\pi r\sigma_r h \sin \theta = (\pi r^2 - \pi b^2)q. \quad (1)$$

If the transverse displacement at r is denoted by $w(r)$, then

$$\sin \theta \approx \tan \theta = -\frac{dw}{dr}. \quad (2)$$

Substituting Equation (2) into Equation (1), the out-of-plane equilibrium equation can be written as

$$2r\sigma_r h \left(-\frac{dw}{dr}\right) = (r^2 - b^2)q. \quad (3)$$

In the horizontal direction parallel to the initially flat annular membrane, there are the actions of the radial membrane force $\sigma_r h$ and the circumferential membrane force $\sigma_t h$, where σ_t denotes circumferential stress. Therefore, the so-called in-plane equilibrium equation may be written as [26]

$$\frac{d}{dr}(r\sigma_r h) - \sigma_t h = 0. \quad (4)$$

If the radial strain, circumferential strain and radial displacement are denoted by e_r , e_t , and $u(r)$, respectively, then the relationship between displacement and strain, the usually so-called geometric equations, may be written as

$$e_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (5)$$

and

$$e_t = \frac{u}{r}. \quad (6)$$

Equation (5) is a second-order geometrically nonlinear equation. Moreover, the relationship between strain and stress, the so-called physical equations, is still assumed to satisfy linear elasticity

$$\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_t) \quad (7)$$

and

$$\sigma_t = \frac{E}{1-\nu^2}(e_t + \nu e_r). \quad (8)$$

Substituting Equations (5) and (6) into Equations (7) and (8) yields

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] \quad (9)$$

and

$$\sigma_t = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right]. \quad (10)$$

Eliminating dw/dx from Equations (9) and (10) and further using Equation (4) yield

$$\frac{u}{r} = \frac{1}{Eh}(\sigma_t h - \nu\sigma_r h) = \frac{1}{Eh}\left[\frac{d}{dr}(r\sigma_r h) - \nu\sigma_r h\right]. \quad (11)$$

After substituting the u of Equation (11) into Equation (9), then the usually so-called consistency equation can be written as

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\sigma_r)\right] + \frac{E}{2}\left(\frac{dw}{dr}\right)^2 = 0. \quad (12)$$

Equations (3), (4) and (12) are three equations for the solutions of σ_r , σ_t and w .

Due to the movable but indeformable inner edge at $r = b$ and the immovable and indeformable outer edge at $r = a$, the boundary conditions for solving Equations (3), (4) and (12) are

$$e_t = \frac{u}{r} = 0 \text{ at } r = b, \quad (13)$$

$$e_t = \frac{u}{r} = 0 \text{ at } r = a \quad (14)$$

and

$$w = 0 \text{ at } r = a. \quad (15)$$

Let us introduce the following nondimensional variables

$$Q = \frac{aq}{Eh}, W = \frac{w}{a}, S_r = \frac{\sigma_r}{E}, S_t = \frac{\sigma_t}{E}, \alpha = \frac{b}{a}, x = \frac{r}{a}, \quad (16)$$

and transform Equations (3), (4), (11)–(15) into

$$2xS_r\left(-\frac{dW}{dx}\right) = (x^2 - \alpha^2)Q, \quad (17)$$

$$\frac{d(xS_r)}{dx} - S_t = 0, \quad (18)$$

$$\frac{u}{r} = (1 - \nu)S_r + x\frac{dS_r}{dx}, \quad (19)$$

$$x^2\frac{d^2S_r}{dx^2} + 3x\frac{dS_r}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 = 0, \quad (20)$$

$$(1 - \nu)S_r + x\frac{dS_r}{dx} = 0 \text{ at } x = \alpha, \quad (21)$$

$$(1 - \nu)S_r + x\frac{dS_r}{dx} = 0 \text{ at } x = 1 \quad (22)$$

and

$$W = 0 \text{ at } x = 1. \quad (23)$$

Eliminating dW/dx from Equations (17) and (20), we can obtain an equation which contains only S_r

$$x^2\frac{d^2S_r}{dx^2} + 3x\frac{dS_r}{dx} + \frac{(x^2 - \alpha^2)^2Q^2}{8x^2S_r^2} = 0. \quad (24)$$

In view of the physical phenomenon that the values of stress, strain and displacement are finite within $\alpha \leq x \leq 1$, we can expand S_r and W into the power series of the $x - \beta$, i.e., letting

$$S_r = \sum_{i=0}^{\infty} c_i(x - \beta)^i \quad (25)$$

and

$$W = \sum_{i=0}^{\infty} d_i(x - \beta)^i, \quad (26)$$

where $\beta = (1 + \alpha)/2$. For convenience we introduce $X = x - \beta$, then Equations (17), (24)–(26) can be transformed into

$$2(X + \beta)S_r\left(-\frac{dW}{dX}\right) = [(X + \beta)^2 - \alpha^2]Q, \quad (27)$$

$$(X + \beta)^2\frac{d^2S_r}{dX^2} + 3(X + \beta)\frac{dS_r}{dX} + \frac{[(X + \beta)^2 - \alpha^2]^2Q^2}{8(X + \beta)^2S_r^2} = 0, \quad (28)$$

$$S_r = \sum_{i=0}^{\infty} c_i X^i \quad (29)$$

and

$$W = \sum_{i=0}^{\infty} d_i X^i. \quad (30)$$

Substituting Equation (29) into Equation (28) and letting the sums of all coefficients of the same powers of the X be equal to zero yield a system of equations for determining the recursion for the coefficients c_i . The solution to this system of equations shows that the coefficients c_i ($i = 2, 3, 4, \dots$) can be expressed into the polynomial functions with regard to the first two coefficients c_0 and c_1 (see Appendix A). Further, by substituting Equation (29) (where c_i should be expressed by c_0 and c_1 , see Appendix A) and Equation (30) into Equation (27), the coefficients d_i ($i = 1, 2, 3, \dots$) can also be expressed into c_0 and c_1 (see Appendix B).

The remaining three coefficients d_0 , c_0 and c_1 are the so-called undetermined constants, which depend on the specific problem addressed and can be determined by using the boundary conditions Equations (21)–(23) as follows. Substituting Equation (25) (where c_i should be expressed by c_0 and c_1 , see Appendix A) into the boundary conditions Equations (21) and (22) yield

$$(1 - \nu) \sum_{i=0}^{\infty} c_i (\alpha - \beta)^i + \alpha \sum_{i=1}^{\infty} i c_i (\alpha - \beta)^{i-1} = 0 \quad (31)$$

and

$$(1 - \nu) \sum_{i=0}^{\infty} c_i (1 - \beta)^i + \sum_{i=1}^{\infty} i c_i (1 - \beta)^{i-1} = 0. \quad (32)$$

Obviously, Equations (31) and (32) contain only c_0 and c_1 . Therefore, the values of c_0 and c_1 can be determined by simultaneously solving Equations (31) and (32), and the expression of S_r can thus be determined. Further, substituting Equation (26) (where d_i should be expressed by c_0 and c_1 , see Appendix B) into the boundary condition Equation (23) yields

$$d_0 = - \sum_{i=1}^{\infty} d_i (1 - \beta)^i. \quad (33)$$

Therefore, with the known c_0 and c_1 the value of d_0 can be determined by Equation (33), and the expression of W can also be determined. As for the expression of S_t , with the known expression of S_r , it can easily be determined by Equation (18).

The problem under consideration is thus solved analytically, and its closed-form solutions of stress and deflection are given.

3. Results and Discussion

Let us first discuss the convergence of the closed-form solution obtained in Section 2. It may be seen, from the derivation in Section 2, that the closed-form solutions of S_r and W are obtained by using the power series method and the power series coefficients c_i ($i = 2, 3, 4, \dots$) and d_i ($i = 1, 2, 3, \dots$) are expressed into the polynomial in the undetermined constants c_0 and c_1 . Therefore, due to these somewhat intractable expressions of the coefficients b_i and c_i (see Appendices A and B), we have to prove the convergence of the special solutions of stress and deflection, rather than that of their general solutions. To this end, we consider a numerical example of an annular membrane with thickness $h = 0.2$ mm, inner radius $b = 10$ mm, outer radius $a = 20$ mm, Young's modulus of elasticity $E = 7.84$ MPa and Poisson's ratio $\nu = 0.47$ under the action of the uniformly distributed transverse loads $q = 0.0003$ MPa. Substituting the values of a, b, h, E, ν, q into Equation (16) yields $Q = 0.00382653$ and $\alpha = 1/2$ ($\beta = (1 + \alpha)/2 = 3/4$).

For convenience of operation, the infinite power series in Equations (31)–(33) have to be truncated to n terms, that is

$$(1 - \nu) \sum_{i=0}^n c_i (\alpha - \beta)^i + \alpha \sum_{i=1}^n i c_i (\alpha - \beta)^{i-1} = 0, \quad (34)$$

$$(1 - \nu) \sum_{i=0}^n c_i (1 - \beta)^i + \sum_{i=1}^n i c_i (1 - \beta)^{i-1} = 0 \quad (35)$$

and

$$d_0 = - \sum_{i=1}^n d_i (1 - \beta)^i. \quad (36)$$

For a given value of the parameter n in Equations (34)–(36), Equations (34) and (35) are used to determine the specific numerical values of the undetermined constants c_0 and c_1 , and with the known c_0 and c_1 the value of d_0 can further be determined by Equation (36). We start calculating the numerical values of the undetermined constants c_0 , c_1 and d_0 from $n = 2$. The results of numerical value calculation of the undetermined constants c_0 , c_1 and d_0 are listed in Table 1, and the variation of c_0 , c_1 and d_0 with n are shown in Figures 3–5.

Table 1. The results of numerical value calculation of c_0 , c_1 and d_0 .

n	c_0	c_1	d_0
2	0.00684094	−0.00615869	0.04503607
3	0.01084290	−0.00559109	0.02746185
4	0.00811219	−0.00401906	0.03697619
5	0.00861287	−0.00392109	0.03464763
6	0.00832182	−0.00376782	0.03588948
7	0.00839685	−0.00375182	0.03553708
8	0.00835166	−0.00372857	0.03573562
9	0.00836447	−0.00372574	0.03567455
10	0.00835660	−0.00372169	0.03570937
11	0.00835878	−0.00372120	0.03569895
12	0.00835750	−0.00372054	0.03570462
13	0.00835783	−0.00372046	0.03570301
14	0.00835764	−0.00372036	0.03570385
15	0.00835769	−0.00372035	0.03570362

Figure 3. Variation of c_0 with n , where the blue refers to the case when n takes even and the red refers to the case when n takes odd.

Figure 4. Variation of c_1 with n , where the blue refers to the case when n takes even and the red refers to the case when n takes odd.

Figure 5. Variation of d_0 with n , where the blue refers to the case when n takes even and the red refers to the case when n takes odd.

It may be seen, from Figures 3–5, that the data sequence for c_0 , c_1 and d_0 converge rapidly to their theoretical values (their exact values) as the parameter n is increased. Therefore, the undetermined constants c_0 , c_1 and d_0 can finally take 0.00835769, -0.00372035 and 0.03570362, respectively (the values at $n = 15$, see Table 1). For examining the convergence of the special solutions of stress and deflection at both ends of the closed interval of the independent variable, i.e., at $x = 1$ or $r = a$ and at $x = \alpha$ or $r = b$, the numerical values of $c_i(1 - \beta)^i$, $c_i(\alpha - \beta)^i$, $d_i(1 - \beta)^i$ and $d_i(\alpha - \beta)^i$ are calculated by using $\alpha = 1/2$, $\beta = 3/4$ and the values of c_0 and c_1 at $n = 15$, which are listed in Tables 2 and 3. The variation of $c_i(1 - \beta)^i$, $c_i(\alpha - \beta)^i$, $d_i(1 - \beta)^i$ and $d_i(\alpha - \beta)^i$ with i are shown in Figures 6–9, showing that the special solutions of stress and deflection converge very well.

Figure 6. Variation of the $c_i(1 - \beta)^i$ at $n = 15$ with i , where $\beta = 3/4$.

Table 2. The calculated values of $c_i(1 - \beta)^i$ and $c_i(\alpha - \beta)^i$ at $n = 15$, where $\alpha = 1/2$ and $\beta = 3/4$.

i	$c_i(1 - \beta)^i$	$c_i(\alpha - \beta)^i$
0	0.00835769	0.00835769
1	−0.00093009	0.00093009
2	0.00021232	0.00021232
3	−0.00023105	0.00023105
4	$7.92290026 \times 10^{-5}$	$7.92290026 \times 10^{-5}$
5	$-2.96216622 \times 10^{-5}$	$2.96216622 \times 10^{-5}$
6	$9.62752906 \times 10^{-6}$	$9.62752906 \times 10^{-6}$
7	$-3.42910780 \times 10^{-6}$	$3.42910780 \times 10^{-6}$
8	$1.22656512 \times 10^{-6}$	$1.22656512 \times 10^{-6}$
9	$-4.64693627 \times 10^{-7}$	$4.64693627 \times 10^{-7}$
10	$1.74308977 \times 10^{-7}$	$1.74308977 \times 10^{-7}$
11	$-6.52196918 \times 10^{-8}$	$6.52196918 \times 10^{-8}$
12	$2.37687733 \times 10^{-8}$	$2.37687733 \times 10^{-8}$
13	$-8.55038019 \times 10^{-9}$	$8.55038019 \times 10^{-9}$
14	$3.03425389 \times 10^{-9}$	$3.03425389 \times 10^{-9}$
15	$-1.07537902 \times 10^{-9}$	$1.07537902 \times 10^{-9}$

Table 3. The calculated values of $d_i(1 - \beta)^i$ and $d_i(\alpha - \beta)^i$ at $n = 15$, where $\alpha = 1/2$ and $\beta = 3/4$.

i	$d_i(1 - \beta)^i$	$d_i(\alpha - \beta)^i$
0	0.03570362	0.03570362
1	−0.02384612	0.02384612
2	−0.01166018	−0.01166018
3	$4.34075197 \times 10^{-5}$	$-4.34075197 \times 10^{-5}$
4	$-1.89718510 \times 10^{-4}$	$-1.89718510 \times 10^{-4}$
5	$-5.41777244 \times 10^{-5}$	$5.41777244 \times 10^{-5}$
6	$8.46365300 \times 10^{-6}$	$8.46365300 \times 10^{-6}$
7	$-5.52803048 \times 10^{-6}$	$5.52803048 \times 10^{-6}$
8	$3.65886050 \times 10^{-7}$	$3.65886050 \times 10^{-7}$
9	$-7.99963631 \times 10^{-8}$	$7.99963631 \times 10^{-8}$
10	$-6.10745278 \times 10^{-8}$	$-6.10745278 \times 10^{-8}$
11	$7.58270646 \times 10^{-9}$	$-7.58270646 \times 10^{-9}$
12	$-1.53622173 \times 10^{-9}$	$-1.53622173 \times 10^{-9}$
13	$-1.68207783 \times 10^{-9}$	$1.68207783 \times 10^{-9}$
14	$6.15674878 \times 10^{-10}$	$6.15674878 \times 10^{-10}$
15	$-2.18144847 \times 10^{-10}$	$2.18144847 \times 10^{-10}$

Figure 7. Variation of the $c_i(\alpha - \beta)^i$ at $n = 15$ with i , where $\alpha = 1/2$ and $\beta = 3/4$.

Figure 8. Variation of the $d_i(1 - \beta)^i$ at $n = 15$ with i , where $\beta = 3/4$.

Figure 9. Variation of the $d_i(\alpha - \beta)^i$ at $n = 15$ with i , where $\alpha = 1/2$ and $\beta = 3/4$.

Now, let us address the validity of the closed-form solution obtained in Section 2, which can be proved by examining whether it, under the same action of the uniformly distributed transverse loads $q = 0.0003$ MPa, gets closer and closer to the Hencky solution as the inner radius b of the annular membrane gradually goes down. Obviously, a circular membrane can approximately be regarded as an annular membrane with an inner radius $b \rightarrow 0$. Therefore, the closed-form solution obtained in Section 2 should get closer and closer to the Hencky solution (which is for circular membrane problem) as the inner radius b of the annular membrane gradually goes to zero. Figure 10 shows the trend that the closed-form solution obtained in Section 2 approaches the Hencky solution with the decrease of the inner radius b of the annular membrane, which to some extent shows that the closed-form solution obtained in Section 2 is basically reliable.

Figure 10. Diagram of the geometry of an annular membrane with a decreasing inner radius ($b = 15, 10, 5$ and 2 mm) gradually approaching the geometry of a circular membrane ($b = 0$) under the same action of the uniformly distributed transverse loads $q = 0.0003$ MPa.

Lian et al. [28] presented the closed-form solution of the traditional annular membrane structure, the problem of axisymmetric deformation and deflection of an annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate, where both the annular membrane and the movable but weightless rigid concentric circular thin plate are subjected to the uniformly distributed transverse loads q . The closed-form solution given in Section 2 is suitable for the new hollow annular membrane structure, the problem of axisymmetric deformation and deflection of an annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge attached to a movable but weightless rigid concentric circular ring, where the uniformly distributed transverse loads q is applied only onto the annular membrane. Now, let us compare the closed-form solution presented in this paper with the closed-form solution presented in Lian et al. [28].

The difference between the two closed-form solutions is shown in Figures 11 and 12, where the solid lines represent the hollow annular membrane structure and are calculated by using the closed-form solution presented in this paper, the dash-dot lines represent the traditional annular membrane structure and are calculated by using the closed-form solution presented in Lian et al. [28]. Additionally, the two annular membranes have the same outer radius $a = 20$ mm and the same inner radius $b = 10$ mm.

Figure 11. Geometry of two annular membranes with the same inner and outer radii in hollow and traditional annular membrane structures, where the solid line represents the hollow annular membrane structure addressed in this paper and is calculated by using the closed-form solution presented by this paper, the dash-dotted line represents the traditional annular membrane structure addressed in Alekseev [25] and is calculated by using the closed-form solution presented by Lian et al. [28], and the two different annular membrane structures are subjected to the same uniformly distributed transverse loads $q = 0.0003$ MPa, while the two same annular membranes exhibit the different maximum deflections $w_m = 0.9545$ mm and 1.2644 mm.

Figure 11 shows the difference when the hollow and traditional annular membrane structures are subjected to the same uniformly distributed transverse loads $q = 0.0003$ MPa while the two same annular membranes exhibit the different maximum deflections $w_m = 0.9545$ mm and 1.2644 mm. This is because the annular membrane considered in this paper is subjected to the transverse action of $\pi(a^2 - b^2)q$, while the annular membrane considered in Lian et al. [28] is, due to its movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate, subjected to the transverse action of $\pi a^2 q$, resulting in the different maximum deflections $w_m = 0.9545$ mm and 1.2644 mm. According to the so-called conjugate theory of elastic shells that we are working on (the research funded by the National Natural Science Foundation of China, Grant No. 11772072), the two closed-form solutions presented in Lian et al. [28] (the traditional annular membrane structure) and in this paper (the hollow annular membrane structures) can be used to design two kinds of elastic conjugate shells of revolution, respec-

tively. The closed-form solution of the traditional annular membrane structures can be used to design an elastic conjugate shell of revolution with a movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate. The closed-form solution of the hollow annular membrane structures can be used to design an elastic conjugate shell of revolution with a movable but indeformable inner edge attached to a movable but weightless rigid concentric circular ring. Therefore, the difference in maximum deflections shown in Figure 11 reminds us that the above-mentioned two kinds of elastic conjugate shells of revolution should have different geometry and the maximum height (the rise of shells of revolution, the distance between the base plane (the plane on which the bottom of the shells lies) and apex of the shells), if they have the same inner and outer radii and are subjected to the same uniformly distributed transverse loads q .

Figure 12. Geometry of two annular membranes with the same inner and outer radii in hollow and traditional annular membrane structures, where the solid line represents the hollow annular membrane structure and is calculated by using the closed-form solution presented by this paper, the dash-dotted line represents the traditional annular membrane structure and is calculated by using the closed-form solution presented by Lian et al. [28], and the two different annular membrane structures are, respectively, subjected to the uniformly distributed transverse loads $q = 0.0003$ MPa and 0.00012963 MPa, while the two same annular membranes exhibit the same maximum deflection $w_m = 0.9545$ mm.

Figure 12 shows the difference when the hollow and traditional annular membrane structures are subjected to the different uniformly distributed transverse loads $q = 0.0003$ MPa and 0.00012963 MPa while the two same annular membranes exhibit the same maximum deflection $w_m = 0.9545$ mm. However, it may also be seen from Figure 12 that the two annular membranes exhibit the different deflection curve shapes. This to some extent shows that the above-mentioned two kinds of elastic conjugate shells of revolution should have different geometry, if they have the same inner radius, outer radius and maximum height (the rise of shells of revolution).

4. Concluding Remarks

In this paper, we deal analytically with the geometrically nonlinear problem of axisymmetric deformation and deflection of a new annular membrane structure subjected to transverse uniform loading. The new structure is an initially flat, linearly elastic, transversely uniformly loaded hollow annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge attached to a movable but weightless rigid concentric circular ring. The closed-form solution of the new hollow annular membrane structure is presented. The numerical examples conducted show that the closed-form solution presented has satisfactory convergence and is, taking the well-known Hencky solution as the criterion, reliable.

The traditional annular membrane structure is an annular membrane with an immovable and indeformable outer edge and a movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate. In comparison with the traditional annular membrane structure, the new hollow annular membrane structure lacks the load $\pi b^2 q$, which is uniformly distributed on the circular thin plate of the traditional annular membrane structure. The load $\pi b^2 q$ is actually acting on the inner edge of the traditional annular membrane structure and transferred to the annular membrane through its inner edge. Consequently, under the conditions of the same loads q and the same inner and outer radii of annular membrane, the traditional annular membrane structure is actually subjected to a greater external action force ($\pi a^2 q$) than the new hollow annular membrane structure ($\pi(a^2 - b^2)q$), exhibiting a greater maximum deflection. Therefore, the closed-form solution suitable for the traditional annular membrane structure does not work with the hollow annular membrane structure. Obviously, the use of the closed-form solution of the traditional annular membrane structure for the hollow annular membrane structure will cause a calculation result larger than the result calculated by using the closed-form solution of the hollow annular membrane structure.

According to the so-called conjugate theory of elastic shells that we are working on (the research funded by the National Natural Science Foundation of China, Grant No. 11772072), the two closed-form solutions of the traditional and hollow annular membrane structures can be used to design two kinds of elastic conjugate shells of revolution, respectively. The closed-form solution of the traditional annular membrane structures can be used to design an elastic conjugate shell of revolution with a movable but indeformable inner edge connected to a movable but weightless rigid concentric circular thin plate. The closed-form solution of the hollow annular membrane structures can be used to design an elastic conjugate shell of revolution with a movable but indeformable inner edge attached to a movable but weightless rigid concentric circular ring. Therefore, from the conducted comparison between the two closed-form solutions for the traditional and hollow annular membrane structures, the following conclusions can be drawn. If the above-mentioned two kinds of elastic conjugate shells of revolution have the same inner and outer radii, they should have different geometries and maximum heights (the rise of shells of revolution, the distance between the base plane (the plane on which the bottom of the shells lies) and apex of the shells) if subjected to the same transverse loads q , and should have different geometries if they have the same maximum height.

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Appendix A

$$c_2 = -\frac{1}{16\beta^4 c_0^2} (Q^2 \alpha^4 - 2Q^2 \alpha^2 \beta^2 + Q^2 \beta^4 + 24\beta^3 c_0^2 c_1),$$

$$c_3 = \frac{1}{48c_0^3 \beta^5} (2Q^2 \alpha^4 \beta c_1 - 4Q^2 \alpha^2 \beta^3 c_1 + 2Q^2 \beta^5 c_1 + 7Q^2 \alpha^4 c_0 - 10Q^2 \alpha^2 \beta^2 c_0 + 3Q^2 \beta^4 c_0 + 96\beta^3 c_0^3 c_1),$$

$$c_4 = -\frac{1}{768\beta^8 c_0^5} (24Q^2 \alpha^4 \beta^4 c_0 c_1^2 - 48Q^2 \alpha^2 \beta^6 c_0 c_1^2 + 24Q^2 \beta^8 c_0 c_1^2 + Q^4 \alpha^8 - 4Q^4 \alpha^6 \beta^2 + 6Q^4 \alpha^4 \beta^4 - 4Q^4 \alpha^2 \beta^6 + Q^4 \beta^8 + 112Q^2 \alpha^4 \beta^3 c_0^2 c_1 - 160Q^2 \alpha^2 \beta^5 c_0^2 c_1 + 48Q^2 \beta^7 c_0^2 c_1 + 188Q^2 \alpha^4 \beta^2 c_0^3 - 216Q^2 \alpha^2 \beta^4 c_0^3 + 60Q^2 \beta^6 c_0^3 + 1920\beta^5 c_0^5 c_1)$$

$$c_5 = \frac{1}{3840\beta^9 c_0^6} (96Q^2 \alpha^4 \beta^5 c_0 c_1^3 - 192Q^2 \alpha^2 \beta^7 c_0 c_1^3 + 96Q^2 \beta^9 c_0 c_1^3 + 11Q^4 \alpha^8 \beta c_1 - 44Q^4 \alpha^6 \beta^3 c_1 + 66Q^4 \alpha^4 \beta^5 c_1 - 44Q^4 \alpha^2 \beta^7 c_1 + 11Q^4 \beta^9 c_1 + 576Q^2 \alpha^4 \beta^4 c_0^2 c_1^2 - 864Q^2 \alpha^2 \beta^6 c_0^2 c_1^2 + 288Q^2 \beta^8 c_0^2 c_1^2 + 22Q^4 \alpha^8 c_0 - 72Q^4 \alpha^6 \beta^2 c_0 + 84Q^4 \alpha^4 \beta^4 c_0 - 40Q^4 \alpha^2 \beta^6 c_0 + 6Q^4 \beta^8 c_0 + 1272Q^2 \alpha^4 \beta^3 c_0^3 c_1 - 1392Q^2 \alpha^2 \beta^5 c_0^3 c_1 + 312Q^2 \beta^7 c_0^3 c_1 + 1368Q^2 \alpha^4 \beta^2 c_0^4 - 1344Q^2 \alpha^2 \beta^4 c_0^4 + 360Q^2 \beta^6 c_0^4 + 11520\beta^5 c_0^6 c_1)$$

$$c_6 = -\frac{1}{184320\beta^{12} c_0^8} (3840Q^2 \alpha^4 \beta^8 c_0^2 c_1^4 - 7680Q^2 \alpha^2 \beta^{10} c_0^2 c_1^4 + 3840Q^2 \beta^{12} c_0^2 c_1^4 + 816Q^4 \alpha^8 \beta^4 c_0 c_1^2 - 3264Q^4 \alpha^6 \beta^6 c_0 c_1^2 + 4896Q^4 \alpha^4 \beta^8 c_0 c_1^2 - 3264Q^4 \alpha^2 \beta^{10} c_0 c_1^2 + 816Q^4 \beta^{12} c_0 c_1^2 + 28416Q^2 \alpha^4 \beta^7 c_0^3 c_1^3 - 44544Q^2 \alpha^2 \beta^9 c_0^3 c_1^3 + 16128Q^2 \beta^{11} c_0^3 c_1^3 + 11Q^6 \alpha^{12} - 66Q^6 \alpha^{10} \beta^2 + 165Q^6 \alpha^8 \beta^4 - 220Q^6 \alpha^6 \beta^6 + 165Q^6 \alpha^4 \beta^8 - 66Q^6 \alpha^2 \beta^{10} + 11Q^6 \beta^{12} + 3000Q^4 \alpha^8 \beta^3 c_0^2 c_1 - 10080Q^4 \alpha^6 \beta^5 c_0^2 c_1 + 12240Q^4 \alpha^4 \beta^7 c_0^2 c_1 - 6240Q^4 \alpha^2 \beta^9 c_0^2 c_1 + 1080Q^4 \beta^{11} c_0^2 c_1 + 81216Q^2 \alpha^4 \beta^6 c_0^4 c_1^2 - 95616Q^2 \alpha^2 \beta^8 c_0^4 c_1^2 + 23616Q^2 \beta^{10} c_0^4 c_1^2 + 2856Q^4 \alpha^8 \beta^2 c_0^3 - 7968Q^4 \alpha^6 \beta^4 c_0^3 + 7728Q^4 \alpha^4 \beta^6 c_0^3 - 2976Q^4 \alpha^2 \beta^8 c_0^3 + 360Q^4 \beta^{10} c_0^3 + 113472Q^2 \alpha^4 \beta^5 c_0^5 c_1 - 99456Q^2 \alpha^2 \beta^7 c_0^5 c_1 + 18240Q^2 \beta^9 c_0^5 c_1 + 88128Q^2 \alpha^4 \beta^4 c_0^6 - 76800Q^2 \alpha^2 \beta^6 c_0^6 + 20160Q^2 \beta^8 c_0^6 + 645120\beta^7 c_0^8 c_1)$$

$$c_7 = -\frac{1}{42\beta^4 c_0^2} (60\beta^4 c_0 c_1 c_6 + 44\beta^4 c_0 c_2 c_5 + 36\beta^4 c_0 c_3 c_4 + 20\beta^4 c_1^2 c_5 + 28\beta^4 c_1 c_2 c_4 + 12\beta^4 c_1 c_3^2 + 10\beta^4 c_2^2 c_3 + 138\beta^3 c_0^2 c_6 + 196\beta^3 c_0 c_1 c_5 + 148\beta^3 c_0 c_2 c_4 + 66\beta^3 c_0 c_3^2 + 66\beta^3 c_1^2 c_4 + 100\beta^3 c_1 c_2 c_3 + 14\beta^3 c_2^3 + 165\beta^2 c_0^2 c_5 + 234\beta^2 c_0 c_1 c_4 + 186\beta^2 c_0 c_2 c_3 + 81\beta^2 c_1^2 c_3 + 69\beta^2 c_1 c_2^2 + 84\beta c_0^2 c_4 + 120\beta c_0 c_1 c_3 + 52\beta c_0 c_2^2 + 44\beta c_1^2 c_2 + 15c_0^2 c_3 + 22c_0 c_1 c_2 + 3c_1^3)$$

$$c_8 = -\frac{1}{56\beta^4 c_0^2} (84\beta^4 c_0 c_1 c_7 + 64\beta^4 c_0 c_2 c_6 + 52\beta^4 c_0 c_3 c_5 + 24\beta^4 c_0 c_4^2 + 30\beta^4 c_1^2 c_6 + 44\beta^4 c_1 c_2 c_5 + 36\beta^4 c_1 c_3 c_4 + 16\beta^4 c_2^2 c_4 + 14\beta^4 c_2 c_3^2 + 189\beta^3 c_0^2 c_7 + 282\beta^3 c_0 c_1 c_6 + 218\beta^3 c_0 c_2 c_5 + 186\beta^3 c_0 c_3 c_4 + 101\beta^3 c_1^2 c_5 + 154\beta^3 c_1 c_2 c_4 + 69\beta^3 c_1 c_3^2 + 61\beta^3 c_2^2 c_3 + 234\beta^2 c_0^2 c_6 + 348\beta^2 c_0 c_1 c_5 + 276\beta^2 c_0 c_2 c_4 + 126\beta^2 c_0 c_3^2 + 126\beta^2 c_1^2 c_4 + 204\beta^2 c_1 c_2 c_3 + 30\beta^2 c_2^3 + 125\beta c_0^2 c_5 + 186\beta c_0 c_1 c_4 + 154\beta c_0 c_2 c_3 + 69\beta c_1^2 c_3 + 61\beta c_1 c_2^2 + 24c_0^2 c_4 + 36c_0 c_1 c_3 + 16c_0 c_2^2 + 14c_1^2 c_2)$$

$$c_9 = -\frac{1}{72\beta^4 c_0^2} (112\beta^4 c_0 c_1 c_8 + 88\beta^4 c_0 c_2 c_7 + 72\beta^4 c_0 c_3 c_6 + 64\beta^4 c_0 c_4 c_5 + 42\beta^4 c_1^2 c_7 + 64\beta^4 c_1 c_2 c_6 + 52\beta^4 c_1 c_3 c_5 + 24\beta^4 c_1 c_4^2 + 24\beta^4 c_2^2 c_5 + 40\beta^4 c_2 c_3 c_4 + 6\beta^4 c_3^3 + 248\beta^3 c_0^2 c_8 + 384\beta^3 c_0 c_1 c_7 + 304\beta^3 c_0 c_2 c_6 + 256\beta^3 c_0 c_3 c_5 + 120\beta^3 c_0 c_4^2 + 144\beta^3 c_1^2 c_6 + 224\beta^3 c_1 c_2 c_5 + 192\beta^3 c_1 c_3 c_4 + 88\beta^3 c_2^2 c_4 + 80\beta^3 c_2 c_3^2 + 315\beta^2 c_0^2 c_7 + 486\beta^2 c_0 c_1 c_6 + 390\beta^2 c_0 c_2 c_5 + 342\beta^2 c_0 c_3 c_4 + 183\beta^2 c_1^2 c_5 + 294\beta^2 c_1 c_2 c_4 + 135\beta^2 c_1 c_3^2 + 123\beta^2 c_2^2 c_3 + 174\beta c_0^2 c_6 + 268\beta c_0 c_1 c_5 + 220\beta c_0 c_2 c_4 + 102\beta c_0 c_3^2 + 102\beta c_1^2 c_4 + 172\beta c_1 c_2 c_3 + 26\beta c_2^3 + 35c_0^2 c_5 + 54c_0 c_1 c_4 + 46c_0 c_2 c_3 + 21c_1^2 c_3 + 19c_1 c_2^2)$$

$$c_{10} = -\frac{1}{90\beta^4 c_0^2} (144\beta^4 c_0 c_1 c_9 + 116\beta^4 c_0 c_2 c_8 + 96\beta^4 c_0 c_3 c_7 + 84\beta^4 c_0 c_4 c_6 + 40\beta^4 c_0 c_5^2 + 56\beta^4 c_1^2 c_8 + 88\beta^4 c_1 c_2 c_7 + 72\beta^4 c_1 c_3 c_6 + 64\beta^4 c_1 c_4 c_5 + 34\beta^4 c_2^2 c_6 + 56\beta^4 c_2 c_3 c_5 + 26\beta^4 c_2 c_4^2 + 24\beta^4 c_3^2 c_4 + 315\beta^3 c_0^2 c_9 + 502\beta^3 c_0 c_1 c_8 + 406\beta^3 c_0 c_2 c_7 + 342\beta^3 c_0 c_3 c_6 + 310\beta^3 c_0 c_4 c_5 + 195\beta^3 c_1^2 c_7 + 310\beta^3 c_1 c_2 c_6 + 262\beta^3 c_1 c_3 c_5 + 123\beta^3 c_1 c_4^2 + 123\beta^3 c_2^2 c_5 + 214\beta^3 c_2 c_3 c_4 + 33\beta^3 c_3^3 + 408\beta^2 c_0^2 c_8 + 648\beta^2 c_0 c_1 c_7 + 528\beta^2 c_0 c_2 c_6 + 456\beta^2 c_0 c_3 c_5 + 216\beta^2 c_0 c_4^2 + 252\beta^2 c_1^2 c_6 + 408\beta^2 c_1 c_2 c_5 + 360\beta^2 c_1 c_3 c_4 + 168\beta^2 c_2^2 c_4 + 156\beta^2 c_2 c_3^2 + 231\beta c_0^2 c_7 + 366\beta c_0 c_1 c_6 + 302\beta c_0 c_2 c_5 + 270\beta c_0 c_3 c_4 + 143\beta c_1^2 c_5 + 238\beta c_1 c_2 c_4 + 111\beta c_1 c_3^2 + 103\beta c_2^2 c_3 + 48c_0^2 c_6 + 76c_0 c_1 c_5 + 64c_0 c_2 c_4 + 30c_0 c_3^2 + 30c_1^2 c_4 + 52c_1 c_2 c_3 + 8c_2^3)$$

$$c_{11} = -\frac{1}{110\beta^4 c_0^2} (180\beta^4 c_0 c_1 c_{10} + 148\beta^4 c_0 c_2 c_9 + 124\beta^4 c_0 c_3 c_8 + 108\beta^4 c_0 c_4 c_7 + 100\beta^4 c_0 c_5 c_6 + 72\beta^4 c_1^2 c_9 + 116\beta^4 c_1 c_2 c_8 + 96\beta^4 c_1 c_3 c_7 + 84\beta^4 c_1 c_4 c_6 + 40\beta^4 c_1 c_5^2 + 46\beta^4 c_2^2 c_7 + 76\beta^4 c_2 c_3 c_6 + 68\beta^4 c_2 c_4 c_5 + 32\beta^4 c_3^2 c_5 + 30\beta^4 c_3 c_4^2 + 390\beta^3 c_0^2 c_{10} + 636\beta^3 c_0 c_1 c_9 + 524\beta^3 c_0 c_2 c_8 + 444\beta^3 c_0 c_3 c_7 + 396\beta^3 c_0 c_4 c_6 + 190\beta^3 c_0 c_5^2 + 254\beta^3 c_1^2 c_8 + 412\beta^3 c_1 c_2 c_7 + 348\beta^3 c_1 c_3 c_6 + 316\beta^3 c_1 c_4 c_5 + 166\beta^3 c_2^2 c_6 + 284\beta^3 c_2 c_3 c_5 + 134\beta^3 c_2 c_4^2 + 126\beta^3 c_3^2 c_4 + 513\beta^2 c_0^2 c_9 + 834\beta^2 c_0 c_1 c_8 + 690\beta^2 c_0 c_2 c_7 + 594\beta^2 c_0 c_3 c_6 + 546\beta^2 c_0 c_4 c_5 + 333\beta^2 c_1^2 c_7 + 546\beta^2 c_1 c_2 c_6 + 474\beta^2 c_1 c_3 c_5 + 225\beta^2 c_1 c_4^2 + 225\beta^2 c_2^2 c_5 + 402\beta^2 c_2 c_3 c_4 + 63\beta^2 c_3^3 + 296\beta c_0^2 c_8 + 480\beta c_0 c_1 c_7 + 400\beta c_0 c_2 c_6 + 352\beta c_0 c_3 c_5 + 168\beta c_0 c_4^2 + 192\beta c_1^2 c_6 + 320\beta c_1 c_2 c_5 + 288\beta c_1 c_3 c_4 + 136\beta c_2^2 c_4 + 128\beta c_2 c_3^2 + 63c_0^2 c_7 + 102c_0 c_1 c_6 + 86c_0 c_2 c_5 + 78c_0 c_3 c_4 + 41c_1^2 c_5 + 70c_1 c_2 c_4 + 33c_1 c_3^2 + 31c_2^2 c_3)$$

$$c_{12} = -\frac{1}{132\beta^4 c_0^2} (220\beta^4 c_0 c_1 c_{11} + 184\beta^4 c_0 c_2 c_{10} + 156\beta^4 c_0 c_3 c_9 + 136\beta^4 c_0 c_4 c_8 + 124\beta^4 c_0 c_5 c_7 + 60\beta^4 c_0 c_6^2 + 90\beta^4 c_1^2 c_{10} + 148\beta^4 c_1 c_2 c_9 + 124\beta^4 c_1 c_3 c_8 + 108\beta^4 c_1 c_4 c_7 + 100\beta^4 c_1 c_5 c_6 + 60\beta^4 c_2^2 c_8 + 100\beta^4 c_2 c_3 c_7 + 88\beta^4 c_2 c_4 c_6 + 42\beta^4 c_2 c_5^2 + 42\beta^4 c_3^2 c_6 + 76\beta^4 c_3 c_4 c_5 + 12\beta^4 c_4^3 + 473\beta^3 c_0^2 c_{11} + 786\beta^3 c_0 c_1 c_{10} + 658\beta^3 c_0 c_2 c_9 + 562\beta^3 c_0 c_3 c_8 + 498\beta^3 c_0 c_4 c_7 + 466\beta^3 c_0 c_5 c_6 + 321\beta^3 c_1^2 c_9 + 530\beta^3 c_1 c_2 c_8 + 450\beta^3 c_1 c_3 c_7 + 402\beta^3 c_1 c_4 c_6 + 193\beta^3 c_1 c_5^2 + 217\beta^3 c_2^2 c_7 + 370\beta^3 c_2 c_3 c_6 + 338\beta^3 c_2 c_4 c_5 + 161\beta^3 c_3^2 c_5 + 153\beta^3 c_3 c_4^2 + 630\beta^2 c_0^2 c_{10} + 1044\beta^2 c_0 c_1 c_9 + 876\beta^2 c_0 c_2 c_8 + 756\beta^2 c_0 c_3 c_7 + 684\beta^2 c_0 c_4 c_6 + 330\beta^2 c_0 c_5^2 + 426\beta^2 c_1^2 c_8 + 708\beta^2 c_1 c_2 c_7 + 612\beta^2 c_1 c_3 c_6 + 564\beta^2 c_1 c_4 c_5 + 294\beta^2 c_2^2 c_6 + 516\beta^2 c_2 c_3 c_5 + 246\beta^2 c_2 c_4^2 + 234\beta^2 c_3^2 c_4 + 369\beta c_0^2 c_9 + 610\beta c_0 c_1 c_8 + 514\beta c_0 c_2 c_7 + 450\beta c_0 c_3 c_6 + 418\beta c_0 c_4 c_5 + 249\beta c_1^2 c_7 + 418\beta c_1 c_2 c_6 + 370\beta c_1 c_3 c_5 + 177\beta c_1 c_4^2 + 177\beta c_2^2 c_5 + 322\beta c_2 c_3 c_4 + 51\beta c_3^3 + 80c_0^2 c_8 + 132c_0 c_1 c_7 + 112c_0 c_2 c_6 + 100c_0 c_3 c_5 + 48c_0 c_4^2 + 54c_1^2 c_6 + 92c_1 c_2 c_5 + 84c_1 c_3 c_4 + 40c_2^2 c_4 + 38c_2 c_3^2)$$

$$c_{13} = -\frac{1}{156\beta^4 c_0^2} (264\beta^4 c_0 c_1 c_{12} + 224\beta^4 c_0 c_2 c_{11} + 192\beta^4 c_0 c_3 c_{10} + 168\beta^4 c_0 c_4 c_9 + 152\beta^4 c_0 c_5 c_8 + 144\beta^4 c_0 c_6 c_7 + 110\beta^4 c_1^2 c_{11} + 184\beta^4 c_1 c_2 c_{10} + 156\beta^4 c_1 c_3 c_9 + 136\beta^4 c_1 c_4 c_8 + 124\beta^4 c_1 c_5 c_7 + 60\beta^4 c_1 c_6^2 + 76\beta^4 c_2^2 c_9 + 128\beta^4 c_2 c_3 c_8 + 112\beta^4 c_2 c_4 c_7 + 104\beta^4 c_2 c_5 c_6 + 54\beta^4 c_3^2 c_7 + 96\beta^4 c_3 c_4 c_6 + 46\beta^4 c_3 c_5^2 + 44\beta^4 c_4^2 c_5 + 564\beta^3 c_0^2 c_{12} + 952\beta^3 c_0 c_1 c_{11} + 808\beta^3 c_0 c_2 c_{10} + 696\beta^3 c_0 c_3 c_9 + 616\beta^3 c_0 c_4 c_8 + 568\beta^3 c_0 c_5 c_7 + 276\beta^3 c_0 c_6^2 + 396\beta^3 c_1^2 c_{10} + 664\beta^3 c_1 c_2 c_9 + 568\beta^3 c_1 c_3 c_8 + 504\beta^3 c_1 c_4 c_7 + 472\beta^3 c_1 c_5 c_6 + 276\beta^3 c_2^2 c_8 + 472\beta^3 c_2 c_3 c_7 + 424\beta^3 c_2 c_4 c_6 + 204\beta^3 c_2 c_5^2 + 204\beta^3 c_3^2 c_6 + 376\beta^3 c_3 c_4 c_5 + 60\beta^3 c_4^3 + 759\beta^2 c_0^2 c_{11} + 1278\beta^2 c_0 c_1 c_{10} + 1086\beta^2 c_0 c_2 c_9 + 942\beta^2 c_0 c_3 c_8 + 846\beta^2 c_0 c_4 c_7 + 798\beta^2 c_0 c_5 c_6 + 531\beta^2 c_1^2 c_9 + 894\beta^2 c_1 c_2 c_8 + 774\beta^2 c_1 c_3 c_7 + 702\beta^2 c_1 c_4 c_6 + 339\beta^2 c_1 c_5^2 + 375\beta^2 c_2^2 c_7 + 654\beta^2 c_2 c_3 c_6 + 606\beta^2 c_2 c_4 c_5 + 291\beta^2 c_3^2 c_5 + 279\beta^2 c_3 c_4^2 + 450\beta c_0^2 c_{10} + 756\beta c_0 c_1 c_9 + 644\beta c_0 c_2 c_8)$$

$$\begin{aligned}
& +564\beta c_0 c_3 c_7 + 516\beta c_0 c_4 c_6 + 250\beta c_0 c_5^2 + 314\beta c_1^2 c_8 + 532\beta c_1 c_2 c_7 \\
& + 468\beta c_1 c_3 c_6 + 436\beta c_1 c_4 c_5 + 226\beta c_2^2 c_6 + 404\beta c_2 c_3 c_5 + 194\beta c_2 c_4^2 \\
& + 186\beta c_3^2 c_4 + 99c_0^2 c_9 + 166c_0 c_1 c_8 + 142c_0 c_2 c_7 + 126c_0 c_3 c_6 + 118c_0 c_4 c_5 \\
& + 69c_1^2 c_7 + 118c_1 c_2 c_6 + 106c_1 c_3 c_5 + 51c_1 c_4^2 + 51c_2^2 c_5 + 94c_2 c_3 c_4 + 15c_3^3),
\end{aligned}$$

$$\begin{aligned}
c_{14} = & -\frac{1}{182\beta^4 c_0^2} (312\beta^4 c_0 c_1 c_{13} + 268\beta^4 c_0 c_2 c_{12} + 232\beta^4 c_0 c_3 c_{11} + 204\beta^4 c_0 c_4 c_{10} \\
& + 184\beta^4 c_0 c_5 c_9 + 172\beta^4 c_0 c_6 c_8 + 84\beta^4 c_0 c_7^2 + 132\beta^4 c_1^2 c_{12} + 224\beta^4 c_1 c_2 c_{11} \\
& + 192\beta^4 c_1 c_3 c_{10} + 168\beta^4 c_1 c_4 c_9 + 152\beta^4 c_1 c_5 c_8 + 144\beta^4 c_1 c_6 c_7 + 94\beta^4 c_2^2 c_{10} \\
& + 160\beta^4 c_2 c_3 c_9 + 140\beta^4 c_2 c_4 c_8 + 128\beta^4 c_2 c_5 c_7 + 62\beta^4 c_2 c_6^2 + 68\beta^4 c_3^2 c_8 \\
& + 120\beta^4 c_3 c_4 c_7 + 112\beta^4 c_3 c_5 c_6 + 54\beta^4 c_4^2 c_6 + 52\beta^4 c_4 c_5^2 + 663\beta^3 c_0^2 c_{13} \\
& + 1134\beta^3 c_0 c_1 c_{12} + 974\beta^3 c_0 c_2 c_{11} + 846\beta^3 c_0 c_3 c_{10} + 750\beta^3 c_0 c_4 c_9 + 686\beta^3 c_0 c_5 c_8 \\
& + 654\beta^3 c_0 c_6 c_7 + 479\beta^3 c_1^2 c_{11} + 814\beta^3 c_1 c_2 c_{10} + 702\beta^3 c_1 c_3 c_9 + 622\beta^3 c_1 c_4 c_8 \\
& + 574\beta^3 c_1 c_5 c_7 + 279\beta^3 c_1 c_6^2 + 343\beta^3 c_2^2 c_9 + 590\beta^3 c_2 c_3 c_8 + 526\beta^3 c_2 c_4 c_7 \\
& + 494\beta^3 c_2 c_5 c_6 + 255\beta^3 c_3^2 c_7 + 462\beta^3 c_3 c_4 c_6 + 223\beta^3 c_3 c_5^2 + 215\beta^3 c_4^2 c_5 \\
& + 900\beta^2 c_0^2 c_{12} + 1536\beta^2 c_0 c_1 c_{11} + 1320\beta^2 c_0 c_2 c_{10} + 1152\beta^2 c_0 c_3 c_9 \\
& + 1032\beta^2 c_0 c_4 c_8 + 960\beta^2 c_0 c_5 c_7 + 468\beta^2 c_0 c_6^2 + 648\beta^2 c_1^2 c_{10} + 1104\beta^2 c_1 c_2 c_9 \\
& + 960\beta^2 c_1 c_3 c_8 + 864\beta^2 c_1 c_4 c_7 + 816\beta^2 c_1 c_5 c_6 + 468\beta^2 c_2^2 c_8 + 816\beta^2 c_2 c_3 c_7 \\
& + 744\beta^2 c_2 c_4 c_6 + 360\beta^2 c_2 c_5^2 + 360\beta^2 c_3^2 c_6 + 672\beta^2 c_3 c_4 c_5 + 108\beta^2 c_4^3 \\
& + 539\beta c_0^2 c_{11} + 918\beta c_0 c_1 c_{10} + 790\beta c_0 c_2 c_9 + 694\beta c_0 c_3 c_8 + 630\beta c_0 c_4 c_7 \\
& + 598\beta c_0 c_5 c_6 + 387\beta c_1^2 c_9 + 662\beta c_1 c_2 c_8 + 582\beta c_1 c_3 c_7 + 534\beta c_1 c_4 c_6 \\
& + 259\beta c_1 c_5^2 + 283\beta c_2^2 c_7 + 502\beta c_2 c_3 c_6 + 470\beta c_2 c_4 c_5 + 227\beta c_3^2 c_5 \\
& + 219\beta c_3 c_4^2 + 120c_0^2 c_{10} + 204c_0 c_1 c_9 + 176c_0 c_2 c_8 + 156c_0 c_3 c_7 + 144c_0 c_4 c_6 \\
& + 70c_0 c_5^2 + 86c_1^2 c_8 + 148c_1 c_2 c_7 + 132c_1 c_3 c_6 + 124c_1 c_4 c_5 + 64c_2^2 c_6 \\
& + 116c_2 c_3 c_5 + 56c_2 c_4^2 + 54c_3^2 c_4),
\end{aligned}$$

$$\begin{aligned}
c_{15} = & -\frac{1}{210\beta^4 c_0^2} (364\beta^4 c_0 c_1 c_{14} + 316\beta^4 c_0 c_2 c_{13} + 276\beta^4 c_0 c_3 c_{12} + 244\beta^4 c_0 c_4 c_{11} \\
& + 220\beta^4 c_0 c_5 c_{10} + 204\beta^4 c_0 c_6 c_9 + 196\beta^4 c_0 c_7 c_8 + 156\beta^4 c_1^2 c_{13} + 268\beta^4 c_1 c_2 c_{12} \\
& + 232\beta^4 c_1 c_3 c_{11} + 204\beta^4 c_1 c_4 c_{10} + 184\beta^4 c_1 c_5 c_9 + 172\beta^4 c_1 c_6 c_8 + 84\beta^4 c_1 c_7^2 \\
& + 114\beta^4 c_2^2 c_{11} + 196\beta^4 c_2 c_3 c_{10} + 172\beta^4 c_2 c_4 c_9 + 156\beta^4 c_2 c_5 c_8 + 148\beta^4 c_2 c_6 c_7 \\
& + 84\beta^4 c_3^2 c_9 + 148\beta^4 c_3 c_4 c_8 + 136\beta^4 c_3 c_5 c_7 + 66\beta^4 c_3 c_6^2 + 66\beta^4 c_4^2 c_7 \\
& + 124\beta^4 c_4 c_5 c_6 + 20\beta^4 c_5^3 + 770\beta^3 c_0^2 c_{14} + 1332\beta^3 c_0 c_1 c_{13} + 1156\beta^3 c_0 c_2 c_{12} \\
& + 1012\beta^3 c_0 c_3 c_{11} + 900\beta^3 c_0 c_4 c_{10} + 820\beta^3 c_0 c_5 c_9 + 772\beta^3 c_0 c_6 c_8 + 378\beta^3 c_0 c_7^2 \\
& + 570\beta^3 c_1^2 c_{12} + 980\beta^3 c_1 c_2 c_{11} + 852\beta^3 c_1 c_3 c_{10} + 756\beta^3 c_1 c_4 c_9 + 692\beta^3 c_1 c_5 c_8 \\
& + 660\beta^3 c_1 c_6 c_7 + 418\beta^3 c_2^2 c_{10} + 724\beta^3 c_2 c_3 c_9 + 644\beta^3 c_2 c_4 c_8 + 596\beta^3 c_2 c_5 c_7 \\
& + 290\beta^3 c_2 c_6^2 + 314\beta^3 c_3^2 c_8 + 564\beta^3 c_3 c_4 c_7 + 532\beta^3 c_3 c_5 c_6 + 258\beta^3 c_4^2 c_6 \\
& + 250\beta^3 c_4 c_5^2 + 1053\beta^2 c_0^2 c_{13} + 1818\beta^2 c_0 c_1 c_{12} + 1578\beta^2 c_0 c_2 c_{11} \\
& + 1386\beta^2 c_0 c_3 c_{10} + 1242\beta^2 c_0 c_4 c_9 + 1146\beta^2 c_0 c_5 c_8 + 1098\beta^2 c_0 c_6 c_7 \\
& + 777\beta^2 c_1^2 c_{11} + 1338\beta^2 c_1 c_2 c_{10} + 1170\beta^2 c_1 c_3 c_9 + 1050\beta^2 c_1 c_4 c_8 \\
& + 978\beta^2 c_1 c_5 c_7 + 477\beta^2 c_1 c_6^2 + 573\beta^2 c_2^2 c_9 + 1002\beta^2 c_2 c_3 c_8 + 906\beta^2 c_2 c_4 c_7 \\
& + 858\beta^2 c_2 c_5 c_6 + 441\beta^2 c_3^2 c_7 + 810\beta^2 c_3 c_4 c_6 + 393\beta^2 c_3 c_5^2 + 381\beta^2 c_4^2 c_5 \\
& + 636\beta c_0^2 c_{12} + 1096\beta c_0 c_1 c_{11} + 952\beta c_0 c_2 c_{10} + 840\beta c_0 c_3 c_9 + 760\beta c_0 c_4 c_8 \\
& + 712\beta c_0 c_5 c_7 + 348\beta c_0 c_6^2 + 468\beta c_1^2 c_{10} + 808\beta c_1 c_2 c_9 + 712\beta c_1 c_3 c_8 \\
& + 648\beta c_1 c_4 c_7 + 616\beta c_1 c_5 c_6 + 348\beta c_2^2 c_8 + 616\beta c_2 c_3 c_7 + 568\beta c_2 c_4 c_6 \\
& + 276\beta c_2 c_5^2 + 276\beta c_3^2 c_6 + 520\beta c_3 c_4 c_5 + 84\beta c_4^3 + 143c_0^2 c_{11} + 246c_0 c_1 c_{10} \\
& + 214c_0 c_2 c_9 + 190c_0 c_3 c_8 + 174c_0 c_4 c_7 + 166c_0 c_5 c_6 + 105c_1^2 c_9 + 182c_1 c_2 c_8 \\
& + 162c_1 c_3 c_7 + 150c_1 c_4 c_6 + 73c_1 c_5^2 + 79c_2^2 c_7 + 142c_2 c_3 c_6 + 134c_2 c_4 c_5 \\
& + 65c_3^2 c_5 + 63c_3 c_4^2)
\end{aligned}$$

Appendix B

$$d_1 = \frac{Q}{2\beta c_0} (a^2 - \beta^2),$$

$$d_2 = -\frac{Q}{4\beta^2 c_0^2} (a^2 \beta c_1 - \beta^3 c_1 + a^2 c_0 + \beta^2 c_0),$$

$$d_3 = \frac{Q}{96\beta^5 c_0^4} (16a^2 \beta^4 c_0 c_1^2 - 16\beta^6 c_0 c_1^2 + Q^2 a^6 - 3Q^2 a^4 \beta^2 + 3Q^2 a^2 \beta^4 - Q^2 \beta^6 + 40a^2 \beta^3 c_0^2 c_1 - 8\beta^5 c_0^2 c_1 + 16a^2 \beta^2 c_0^3),$$

$$d_4 = -\frac{Q}{192\beta^6 c_0^5} (24a^2 \beta^5 c_0 c_1^3 - 24\beta^7 c_0 c_1^3 + 4Q^2 a^6 \beta c_1 - 12Q^2 a^4 \beta^3 c_1 + 12Q^2 a^2 \beta^5 c_1 - 4Q^2 \beta^7 c_1 + 96a^2 \beta^4 c_0^2 c_1^2 - 48\beta^6 c_0^2 c_1^2 + 5Q^2 a^6 c_0 - 10Q^2 a^4 \beta^2 c_0 + 5Q^2 a^2 \beta^4 c_0 + 108a^2 \beta^3 c_0^3 c_1 - 12\beta^5 c_0^3 c_1 + 24a^2 \beta^2 c_0^4),$$

$$d_5 = \frac{Q}{1920\beta^9 c_0^7} (192a^2 \beta^8 c_0^2 c_1^4 - 192\beta^{10} c_0^2 c_1^4 + 58Q^2 a^6 \beta^4 c_0 c_1^2 - 174Q^2 a^4 \beta^6 c_0 c_1^2 + 174Q^2 a^2 \beta^8 c_0 c_1^2 - 58Q^2 \beta^{10} c_0 c_1^2 + 1056a^2 \beta^7 c_0^3 c_1^3 - 672\beta^9 c_0^3 c_1^3 + Q^4 a^{10} - 5Q^4 a^8 \beta^2 + 10Q^4 a^6 \beta^4 - 10Q^4 a^4 \beta^6 + 5Q^4 a^2 \beta^8 - Q^4 \beta^{10} + 152Q^2 a^6 \beta^3 c_0^2 c_1 - 344Q^2 a^4 \beta^5 c_0^2 c_1 + 232Q^2 a^2 \beta^7 c_0^2 c_1 - 40Q^2 \beta^9 c_0^2 c_1 + 1968a^2 \beta^6 c_0^4 c_1^2 - 624\beta^8 c_0^4 c_1^2 + 87Q^2 a^6 \beta^2 c_0^3 - 137Q^2 a^4 \beta^4 c_0^3 + 53Q^2 a^2 \beta^6 c_0^3 - 3Q^2 \beta^8 c_0^3 + 1344a^2 \beta^5 c_0^5 c_1 - 96\beta^7 c_0^5 c_1 + 192a^2 \beta^4 c_0^6),$$

$$d_6 = -\frac{Q}{23040\beta^{10} c_0^8} (1920a^2 \beta^9 c_0^2 c_1^5 - 1920\beta^{11} c_0^2 c_1^5 + 888Q^2 a^6 \beta^5 c_0 c_1^3 - 2664Q^2 a^4 \beta^7 c_0 c_1^3 + 2664Q^2 a^2 \beta^9 c_0 c_1^3 - 888Q^2 \beta^{11} c_0 c_1^3 + 13440a^2 \beta^8 c_0^3 c_1^4 - 9600\beta^{10} c_0^3 c_1^4 + 43Q^4 a^{10} \beta c_1 - 215Q^4 a^8 \beta^3 c_1 + 430Q^4 a^6 \beta^5 c_1 - 430Q^4 a^4 \beta^7 c_1 + 215Q^4 a^2 \beta^9 c_1 - 43Q^4 \beta^{11} c_1 + 3588Q^2 a^6 \beta^4 c_0^2 c_1^2 - 8660Q^2 a^4 \beta^6 c_0^2 c_1^2 + 6556Q^2 a^2 \beta^8 c_0^2 c_1^2 - 1484Q^2 \beta^{10} c_0^2 c_1^2 + 35040a^2 \beta^7 c_0^4 c_1^3 - 15840\beta^9 c_0^4 c_1^3 + 56Q^4 a^{10} c_0 - 232Q^4 a^8 \beta^2 c_0 + 368Q^4 a^6 \beta^4 c_0 - 272Q^4 a^4 \beta^6 c_0 + 88Q^4 a^2 \beta^8 c_0 - 8Q^4 \beta^{10} c_0 + 4416Q^2 a^6 \beta^3 c_0^3 c_1 - 8192Q^2 a^4 \beta^5 c_0^3 c_1 + 4352Q^2 a^2 \beta^7 c_0^3 c_1 - 576Q^2 \beta^9 c_0^3 c_1 + 40800a^2 \beta^6 c_0^5 c_1^2 - 9120\beta^8 c_0^5 c_1^2 + 1554Q^2 a^6 \beta^2 c_0^4 - 2066Q^2 a^4 \beta^4 c_0^4 + 702Q^2 a^2 \beta^6 c_0^4 - 30Q^2 \beta^8 c_0^4 + 19200a^2 \beta^5 c_0^6 c_1 - 960\beta^7 c_0^6 c_1 + 1920a^2 \beta^4 c_0^7),$$

$$d_7 = -\frac{1}{7\beta c_0} (6\beta c_1 d_6 + 5\beta c_2 d_5 + 4\beta c_3 d_4 + 3\beta c_4 d_3 + 2\beta c_5 d_2 + \beta c_6 d_1 + 6c_0 d_6 + 5c_1 d_5 + 4c_2 d_4 + 3c_3 d_3 + 2c_4 d_2 + c_5 d_1),$$

$$d_8 = -\frac{1}{8\beta c_0} (7\beta c_1 d_7 + 6\beta c_2 d_6 + 5\beta c_3 d_5 + 4\beta c_4 d_4 + 3\beta c_5 d_3 + 2\beta c_6 d_2 + \beta c_7 d_1 + 7c_0 d_7 + 6c_1 d_6 + 5c_2 d_5 + 4c_3 d_4 + 3c_4 d_3 + 2c_5 d_2 + c_6 d_1),$$

$$d_9 = -\frac{1}{9\beta c_0} (8\beta c_1 d_8 + 7\beta c_2 d_7 + 6\beta c_3 d_6 + 5\beta c_4 d_5 + 4\beta c_5 d_4 + 3\beta c_6 d_3 + 2\beta c_7 d_2 + \beta c_8 d_1 + 8c_0 d_8 + 7c_1 d_7 + 6c_2 d_6 + 5c_3 d_5 + 4c_4 d_4 + 3c_5 d_3 + 2c_6 d_2 + c_7 d_1),$$

$$d_{10} = -\frac{1}{10\beta c_0} (9\beta c_1 d_9 + 8\beta c_2 d_8 + 7\beta c_3 d_7 + 6\beta c_4 d_6 + 5\beta c_5 d_5 + 4\beta c_6 d_4 + 3\beta c_7 d_3 + 2\beta c_8 d_2 + \beta c_9 d_1 + 9c_0 d_9 + 8c_1 d_8 + 7c_2 d_7 + 6c_3 d_6 + 5c_4 d_5 + 4c_5 d_4 + 3c_6 d_3 + 2c_7 d_2 + c_8 d_1),$$

$$d_{11} = -\frac{1}{11\beta c_0} (10\beta c_1 d_{10} + 9\beta c_2 d_9 + 8\beta c_3 d_8 + 7\beta c_4 d_7 + 6\beta c_5 d_6 + 5\beta c_6 d_5 + 4\beta c_7 d_4 + 3\beta c_8 d_3 + 2\beta c_9 d_2 + \beta c_{10} d_1 + 10c_0 d_{10} + 9c_1 d_9 + 8c_2 d_8 + 7c_3 d_7 + 6c_4 d_6 + 5c_5 d_5 + 4c_6 d_4 + 3c_7 d_3 + 2c_8 d_2 + c_9 d_1),$$

$$\begin{aligned}
d_{12} &= -\frac{1}{12\beta c_0}(11\beta c_1 d_{11} + 10\beta c_2 d_{10} + 9\beta c_3 d_9 + 8\beta c_4 d_8 + 7\beta c_5 d_7 + 6\beta c_6 d_6 \\
&\quad + 5\beta c_7 d_5 + 4\beta c_8 d_4 + 3\beta c_9 d_3 + 2\beta c_{10} d_2 + \beta c_{11} d_1 + 11c_0 d_{11} + 10c_1 d_{10} + 9c_2 d_9 \\
&\quad + 8c_3 d_8 + 7c_4 d_7 + 6c_5 d_6 + 5c_6 d_5 + 4c_7 d_4 + 3c_8 d_3 + 2c_9 d_2 + c_{10} d_1) \\
d_{13} &= -\frac{1}{13\beta c_0}(12\beta c_1 d_{12} + 11\beta c_2 d_{11} + 10\beta c_3 d_{10} + 9\beta c_4 d_9 + 8\beta c_5 d_8 \\
&\quad + 7\beta c_6 d_7 + 6\beta c_7 d_6 + 5\beta c_8 d_5 + 4\beta c_9 d_4 + 3\beta c_{10} d_3 + 2\beta c_{11} d_2 + \beta c_{12} d_1 + 12c_0 d_{12} + 11c_1 d_{11} \\
&\quad + 10c_2 d_{10} + 9c_3 d_9 + 8c_4 d_8 + 7c_5 d_7 + 6c_6 d_6 + 5c_7 d_5 + 4c_8 d_4 + 3c_9 d_3 + 2c_{10} d_2 + c_{11} d_1) \\
d_{14} &= -\frac{1}{14\beta c_0}(143\beta c_1 d_{13} + 132\beta c_2 d_{12} + 11\beta c_3 d_{11} + 10\beta c_4 d_{10} + 9\beta c_5 d_9 + 8\beta c_6 d_8 + 7\beta c_7 d_7 \\
&\quad + 6\beta c_8 d_6 + 5\beta c_9 d_5 + 4\beta c_{10} d_4 + 3\beta c_{11} d_3 + 2\beta c_{12} d_2 + \beta c_{13} d_1 + 13c_0 d_{13} + 12c_1 d_{12} + 11c_2 d_{11} \\
&\quad + 10c_3 d_{10} + 9c_4 d_9 + 8c_5 d_8 + 7c_6 d_7 + 6c_7 d_6 + 5c_8 d_5 + 4c_9 d_4 + 3c_{10} d_3 + 2c_{11} d_2 + c_{12} d_1) \\
d_{15} &= -\frac{1}{15\beta c_0}(14\beta c_1 d_{14} + 13\beta c_2 d_{13} + 12\beta c_3 d_{12} + 11\beta c_4 d_{11} + 10\beta c_5 d_{10} + 9\beta c_6 d_9 \\
&\quad + 8\beta c_7 d_8 + 7\beta c_8 d_7 + 6\beta c_9 d_6 + 5\beta c_{10} d_5 + 4\beta c_{11} d_4 + 3\beta c_{12} d_3 + 2\beta c_{13} d_2 + \beta c_{14} d_1 \\
&\quad + 14c_0 d_{14} + 13c_1 d_{13} + 12c_2 d_{12} + 11c_3 d_{11} \\
&\quad + 10c_4 d_{10} + 9c_5 d_9 + 8c_6 d_8 + 7c_7 d_7 + 6c_8 d_6 + 5c_9 d_5 + 4c_{10} d_4 + 3c_{11} d_3 + 2c_{12} d_2 + c_{13} d_1)
\end{aligned}$$

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