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On the Solutions of the b-Family of Novikov Equation

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Abstract: In this paper, we study the symmetric travelling wave solutions of the b-family of the Novikov equation. We show that the b-family of the Novikov equation can provide symmetric travelling wave solutions, such as peakon, kink and smooth soliton solutions. In particular, the single peakon, two-peakon, stationary kink, anti-kink, two-kink, two-anti-kink, bell-shape soliton and hat-shape soliton solutions are presented in an explicit formula.

Keywords: the b-family of Novikov equation; peakon; kink; soliton solutions

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1. Introduction

The b-family of the Camassa–Holm equation

$$m_t + um_x + bu_x m = 0,\ m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant and $u(x,t)$ is fluid velocity. Equation (1) was first proposed by Holm and Stanley in studying the exchange of stability in the dynamics of solitary waves under changes in the nonlinear balance in a $1+1$ evolutionary PDE related to shallow water waves and turbulence [1,2]. In the case of $b \neq 0$, peakon solutions of Equation (1) were discussed in [1,2]. In the case of $b = 0$, Xia and Qiao showed that Equation (1) provides N-kink, bell-shape and hat-shape solitary solutions [3]. For $b = 2$, Equation (1) becomes the well-known Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \tag{2}$$

which was originally implied in Fokas and Fuchssteiner in [4], but became well-known when Camassa and Holm [5] derived it as a model for the unidirectional propagation of shallow water over a flat bottom. The CH equation was found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [4–6]. The famous feature of the CH equation is that it provides peaked soliton (peakon) solutions [4,5], which present an essential feature of the travelling waves of largest amplitude [7–9]. For $b = 3$, Equation (1) becomes the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}, \tag{3}$$

which can be regarded as another model for nonlinear shallow water dynamics with peakons [10,11]. The integrability of the DP equation was shown by constructing a Lax pair, and deriving two infinite sequences of conservation laws in [12].

In this paper, we are concerned with the b-family of the Novikov equation

$$m_t + u^2 m_x + buu_x m = 0, \quad m = u - u_{xx}, \tag{4}$$

where b is an arbitrary constant. It is easy to see that the b-family of the Novikov Equation (4) has nonlinear terms that are cubic, rather than quadratic, of the b-family of CH Equation (1). The Cauchy problem of the b-family of the Novikov Equation (4) was studied in [13].

For $b = 3$, Equation (4) becomes the Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}, \tag{5}$$

which was discovered by Vladimir Novikov [14] in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity. In [15,16], it was shown that the Novikov equation provides peakon solutions such as the CH and DP equations. Additionally, the Novikov Equation (5) has a Lax pair in matrix form and a bi-Hamiltonian structure. Moreover, it has infinitely many conserved quantities.

The purpose of this paper is to investigate the solutions of the b-family of the Novikov Equation (4) in the case of $b \neq 0$ and $b = 0$. We will show that Equation (4) possesses symmetric travelling wave solutions, such as peakon, kink and smooth soliton solutions. In particular, the single peakon, two-peakon, stationary kink, anti-kink, two-kink, two-anti-kink, bell-shape soliton and hat-shape soliton solutions are presented in an explicit formula and plotted.

The rest of this paper is organized as follows. In Section 2, we derive the N-peakon solutions in the case of $b \neq 0$. In Section 3, we discuss the N-kink and smooth soliton solutions in the case of $b = 0$.

2. Peakon Solutions

In this section, we derive the N-peakon solutions in the case of $b \neq 0$. We assume the N-peakon solution as the form

$$u = \sum_{j=1}^{N} p_j(t) \mathrm{e}^{-|x - q_j(t)|}, \tag{6}$$

where $p_j(t)$ and $q_j(t)$ are to be determined. The derivatives of (6) do not exist at $x = q_j(t)$, thus (6) can not satisfy Equation (4) in a classical sense. However, in the distribution, we have

$$u_x = -\sum_{j=1}^{N} p_j(t)\, \mathrm{sgn}(x - q_j(t)) \mathrm{e}^{-|x - q_j(t)|}, \tag{7}$$

$$m = 2\sum_{j=1}^{N} p_j(t) \delta(x - q_j(t)), \tag{8}$$

$$m_t = 2\sum_{j=1}^{N} p_{j,t} \delta(x - q_j(t)) - 2\sum_{j=1}^{N} p_j q_{j,t} \delta'(x - q_j(t)), \tag{9}$$

$$m_x = 2\sum_{j=1}^{N} p_j(t) \delta'(x - q_j(t)). \tag{10}$$

Substituting (6)–(10) into (4) and integrating against the test function with compact support, we obtain that $p_j(t)$ and $q_j(t)$ evolve according to the dynamical system:

$$\begin{cases} q_{j,t} = \left(\sum\limits_{i=1}^{N} p_i \mathrm{e}^{-|q_j - q_i|} \right)^2, & 1 \le j \le N, \\ p_{j,t} = (b-2) p_j \left(\sum\limits_{i=1}^{N} p_i \mathrm{e}^{-|q_j - q_i|} \right) \left(\sum\limits_{i=1}^{N} p_i\, \mathrm{sgn}(q_j - q_i) \mathrm{e}^{-|q_j - q_i|} \right), & 1 \le j \le N. \end{cases} \tag{11}$$

For $N = 1$, (11) is reduced to

$$\begin{cases} q_{1,t} = p_1^2, \\ p_{1,t} = 0. \end{cases}$$

Thus, the single peakon solution (See Figure 1) is

$$u = \pm\sqrt{c}\mathrm{e}^{-|x-ct|}, \quad c > 0. \tag{12}$$

For $N = 2$, (11) is reduced to

$$\begin{cases} q_{1,t} = \left(p_1 + p_2\mathrm{e}^{-|q_1-q_2|}\right)^2, \\ q_{2,t} = \left(p_1\mathrm{e}^{-|q_2-q_1|} + p_2\right)^2, \\ p_{1,t} = (b-2)p_1p_2\left(p_1 + p_2\mathrm{e}^{-|q_1-q_2|}\right)\operatorname{sgn}(q_1-q_2)\mathrm{e}^{-|q_1-q_2|}, \\ p_{2,t} = (b-2)p_1p_2\left(p_2 + p_1\mathrm{e}^{-|q_1-q_2|}\right)\operatorname{sgn}(q_2-q_1)\mathrm{e}^{-|q_2-q_1|}. \end{cases} \tag{13}$$

Solving (13), we have

$$\begin{cases} q_1(t) - q_2(t) = C, \\ p_1(t) = -p_2(t) = -\dfrac{1}{\sqrt{2bt\mathrm{e}^{-2C} - 2bt\mathrm{e}^{-C} - 4t\mathrm{e}^{-2C} + 4t\mathrm{e}^{-C}}}. \end{cases} \tag{14}$$

In particular, for $C = 1, q_2(t) = t, b = 1$, the solution (See Figure 2) becomes

$$u(x,t) = -\frac{1}{\sqrt{2t\mathrm{e}^{-1} - 2t\mathrm{e}^{-2}}}\mathrm{e}^{-|x-t-1|} + \frac{1}{\sqrt{2t\mathrm{e}^{-1} - 2t\mathrm{e}^{-2}}}\mathrm{e}^{-|x-t|}. \tag{15}$$

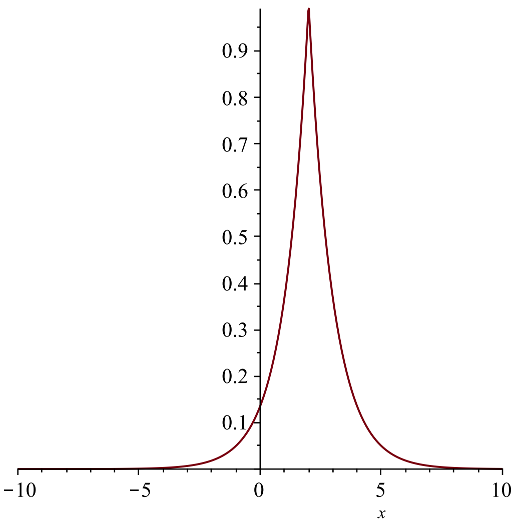

Figure 1. The positive single peakon solution determined by (12) with $c = 1$ at time $t = 2$.

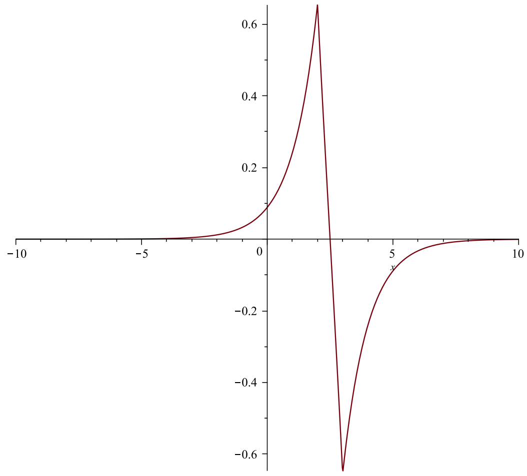

Figure 2. The two-peakon solution (15) at time $t = 2$.

3. Kink and Smooth Soliton Solutions

In this section, we discuss the N-kink and smooth soliton solutions in the case of $b = 0$, namely

$$m_t + u^2 m_x = 0, \quad m = u - u_{xx}. \tag{16}$$

We suppose that the N-kink solution as the form

$$u = \sum_{j=1}^{N} c_j \operatorname{sgn}(x - q_j(t)) \left(\mathrm{e}^{-|x - q_j(t)|} - 1 \right), \tag{17}$$

where c_j are arbitrary constants and $q_j(t)$ are to be determined. The derivatives of (17) do not exist at $x = q_j(t)$, thus (17) can not satisfy Equation (4) in a classical sense. However, in the distribution, we have

$$u_x = -\sum_{j=1}^{N} c_j \mathrm{e}^{-|x - q_j(t)|}, \tag{18}$$

$$m_t = 2 \sum_{j=1}^{N} c_j q_{j,t} \delta(x - q_j(t)), \tag{19}$$

$$m_x = -2 \sum_{j=1}^{N} c_j \delta(x - q_j(t)). \tag{20}$$

Substituting (17)–(20) into (16) and integrating against the test function with compact support, we obtain that $q_j(t)$ evolves according to the dynamical system:

$$q_{j,t} = \left(\sum_{i=1}^{N} c_i \operatorname{sgn}(q_j - q_i) \left(\mathrm{e}^{-|q_j - q_i|} - 1 \right) \right)^2, \; 1 \le j \le N. \tag{21}$$

For $N = 1$, we have $q_{1,t} = 0$, which yields $q_1 = c$, where c is an arbitrary constant. Thus the single kink solution (See Figures 3 and 4) is stationary and it reads

$$u = c_1 \operatorname{sgn}(x - c) \left(\mathrm{e}^{-|x - c|} - 1 \right). \tag{22}$$

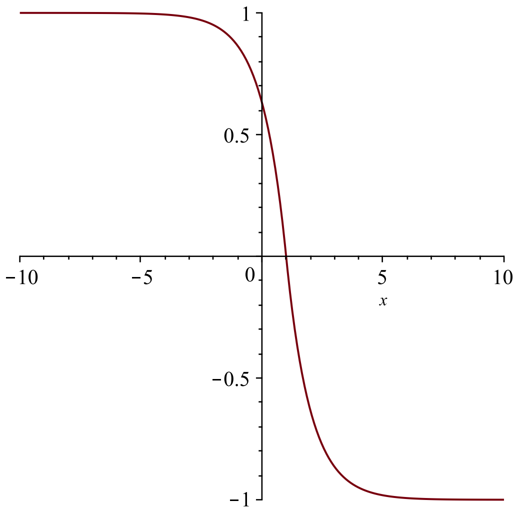

Figure 3. The stationary kink solution determined by (22) with $c_1 = c = 1$.

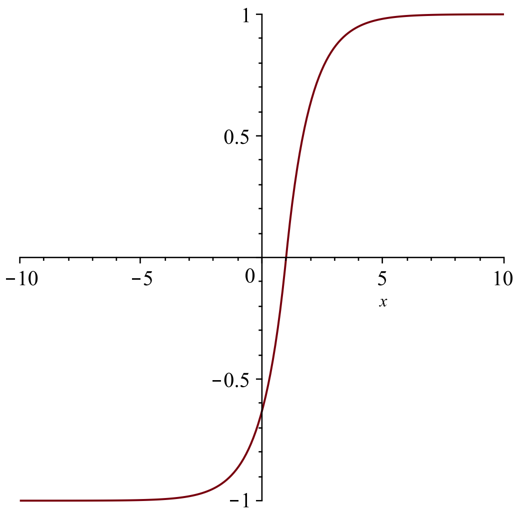

Figure 4. The stationary anti-kink solution determined by (22) with $c_1 = -1, c = 1$.

For $N = 2$, (21) is reduced to

$$\begin{cases} q_{1,t} = \left[c_2 \operatorname{sgn}(q_1 - q_2)\left(\mathrm{e}^{-|q_1-q_2|} - 1\right)\right]^2, \\ q_{2,t} = \left[c_1 \operatorname{sgn}(q_2 - q_1)\left(\mathrm{e}^{-|q_2-q_1|} - 1\right)\right]^2. \end{cases} \tag{23}$$

If $c_1^2 = c_2^2$, we obtain

$$\begin{cases} q_1(t) = \left[c_1 \operatorname{sgn}(C_1)(\mathrm{e}^{-|C_1|} - 1)\right]^2 t, \\ q_2(t) = q_1(t) - C_1, \end{cases} \tag{24}$$

where C_1 is an arbitrary constant. The solution (See Figures 5 and 6) becomes

$$u(x,t) = c_1 \operatorname{sgn}(x - q_1(t))\left(\mathrm{e}^{-|x-q_1(t)|} - 1\right) + c_2 \operatorname{sgn}(x - q_2(t))(\mathrm{e}^{-|x-q_2(t)|} - 1), \tag{25}$$

where q_1 and q_2 are given by (24).

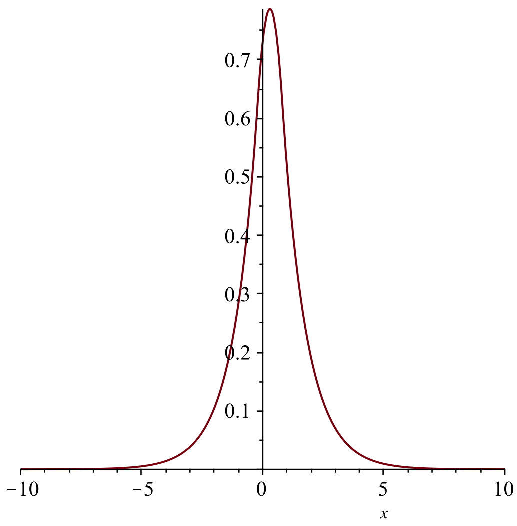

Figure 5. The bell-shape solution determined by (25) with $c_1 = C_1 = 1, c_2 = -1$ at time $t = 2$.

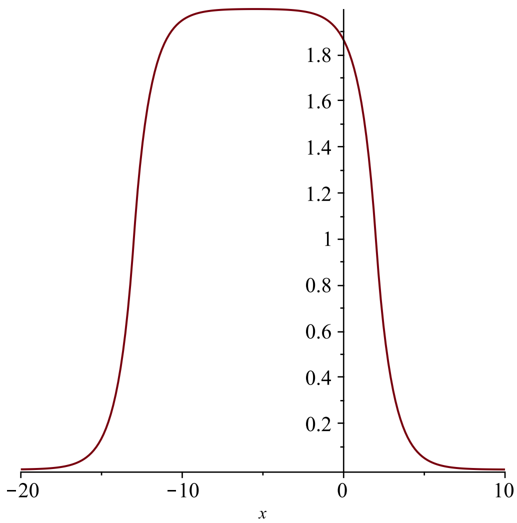

Figure 6. The hat-shape solution determined by (25) with $c_1 = 1, c_2 = -1, C_1 = 15$ at time $t = 2$.

If $c_1^2 \neq c_2^2$, we obtain

$$q_1(t) - q_2(t) = \ln\left(\frac{1 + \mathrm{LambertW}(\mathrm{e}^{(c_1^2 - c_2^2)t})}{\mathrm{LambertW}(\mathrm{e}^{(c_1^2 - c_2^2)t})}\right) \triangleq g(t). \tag{26}$$

In particular, for $q_1(t) = \frac{1}{2}g(t)$ and $q_2(t) = -\frac{1}{2}g(t)$, the solution (See Figures 7 and 8) becomes

$$u(x,t) = c_1 \operatorname{sgn}(x - \frac{1}{2}g(t))\left(\mathrm{e}^{-|x - \frac{1}{2}g(t)|} - 1\right) + c_2 \operatorname{sgn}(x + \frac{1}{2}g(t))\left(\mathrm{e}^{-|x + \frac{1}{2}g(t)|} - 1\right). \tag{27}$$

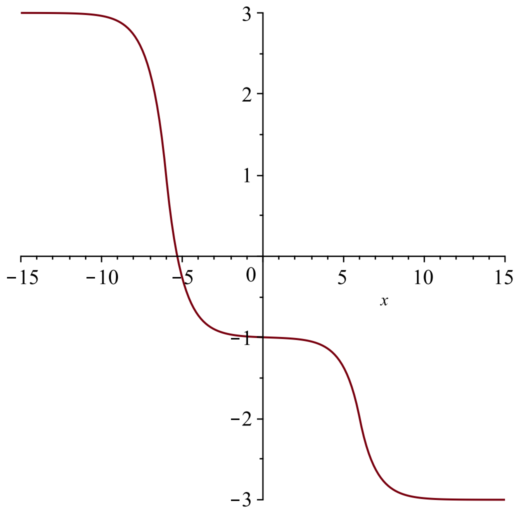

Figure 7. The two kink solution determined by (27) with $c_1 = 2$, $c_2 = 1$ at time $t = 4$.

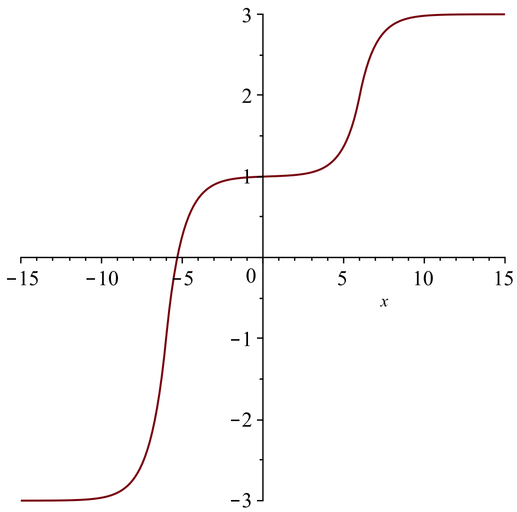

Figure 8. The two anti-kink solution determined by (27) with $c_1 = -2$, $c_2 = -1$ at time $t = 4$

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