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New Conditions for Univalence of Confluent Hypergeometric Function

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Abstract: Since in many particular cases checking directly the conditions from the definitions of starlikeness or convexity of a function can be difficult, in this paper we use the theory of differential subordination and in particular the method of admissible functions in order to determine conditions of starlikeness and convexity for the confluent (Kummer) hypergeometric function of the first kind. Having in mind the results obtained by Miller and Mocanu in 1990 who used $a, c \in \mathbb{R}$, for the confluent (Kummer) hypergeometric function, in this investigation a and c complex numbers are used and two criteria for univalence of the investigated function are stated. An example is also included in order to show the relevance of the original results of the paper.

Keywords: differential subordination; analytic function; starlike function; convex function; univalent function; dominant; best dominant; confluent hypergeometric function

1. Introduction

The interest in hypergeometric functions and their applications in complex analysis was renewed by their use in the proof of Bieberbach's conjecture given by L. de Branges [1]. Before that, only Merkes and Scott did research on starlikeness of certain Gaussian hypergeometric functions in 1961 [2] and this paper was preceded by only a few articles in the literature dealing with the relationship between hypergeometric functions and univalent function theory. The articles published by E. Kreyszig and J. Todd in 1959–1960, who investigated the univalence of the error function $\operatorname{Erf}(z)$ [3] and of the function $\exp(z^2) \cdot \operatorname{Erf}(z)$ [4], can be named with respect to this topic.

After the proof of Bieberbach's conjecture appeared, the research relating the theory of univalent functions and the theory of special functions started to develop and many of the results obtained are related to hypergeometric functions.

The starting point of the present paper is one of the first papers to study this relation, published in 1990 by S.S. Miller and P.T. Mocanu [5]. The method of differential subordinations was employed there to investigate univalence, starlikeness and convexity of certain hypergeometric functions. The results obtained in this paper are different from those obtained earlier by St. Ruscheweyh and V. Singh [6] in 1986 when the order of starlikeness of certain hypergeometric functions was investigated. Moreover, Miller and Mocanu's outcomes are different from those of Merkes and Scott [2].

The particular topic of confluent (Kummer) hypergeometric functions emerged as part of the study of the relationship between hypergeometric functions and univalent function theory. This notion is studied from many different points of view. In [7] univalence and convexity properties for confluent hypergeometric functions are given, in [8] convolutions involving hypergeometric series are used in order to state starlikeness properties, in [9] close-to-convexity properties of Gaussian hypergeometric functions are studied. In a more recent paper [10] the extension of the hypergeometric and confluent hypergeometric functions is achieved by introducing an extra parameter and its relationship with the hypergeometric and confluent hypergeometric functions is studied. To name some of the very recent papers on this topic, in [11] the relation of Kummer confluent hypergeometric



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function with some special classes of univalent functions is highlighted, in [12] Kummer's function is connected to Nevanlinna theory and in [13] the Mittag-Leffler function is connected with confluent hypergeometric function.

The confluent (Kummer) hypergeometric function of the first kind is defined as:

Definition 1 ([14], p. 5). Let a and c be complex numbers with $c \neq 0, -1, -2, \dots$ and consider

$$\phi(a, c; z) = {}_1F_1(a, c; z) = 1 + \frac{a}{c} \cdot \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots, z \in U. \quad (1)$$

This function is called confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$ and satisfies Kummer's differential equation:

$$z \cdot w(z) + [c - z] \cdot w'(z) - a \cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2) \dots (d+k-1) \text{ and } (d)_0 = 1$$

then (1) can be written in the form

$$\phi(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!}. \quad (2)$$

In their paper, Miller and Mocanu [5] have determined conditions on a and c real numbers such that function ϕ to be univalent in U . We consider now a and c complex numbers and use the method of differential subordinations to obtain conditions on them such that function ϕ to be univalent in U and by determining such conditions, we state criteria for univalence of confluent (Kummer) hypergeometric function.

The well-known definitions and notations familiar to the field of complex analysis are used. The unit disc of the complex plane is denoted by U . $H(U)$ stands for the class of analytic functions in the unit disc and the classical definition for class A_n is applied, being known that it contains all functions from class $H(U)$ which have the specific form $f(z) = z + a_{n+1}z^{n+1} + \dots$ with $z \in U$ and A_1 written simply A . All the functions in class A which are univalent in U form the class denoted by S . In particular, the functions in class A who have the property that $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ are called starlike functions and their class is denoted by S^* and those who have the property $\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0$ represent the class of convex functions denoted by K .

For n a positive integer and a a complex number, the class $H[a, n]$ is defined as consisting of all functions from class $H(U)$ who have the serial development $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, with $H_0 = H[0, 1]$.

In this paper, all the classical notions of the method of differential subordinations or the admissible functions method are used just as they were introduced by Miller and Mocanu in [15,16] and some aspects developed in [14] are also referred to.

Definition 2 ([14], p. 4, [17], p. 36). Let f and F be members of $H(U)$. The function f is said to be subordinate to F , written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w , analytic in U , with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Definition 3 ([14], p. 4). Let $\psi(r, s, t; z) : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ and let h be univalent in U . If p is analytic and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \quad (3)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination or more simply a dominant, if $p \prec q$ for all p satisfying (3). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (3) is said to be the best dominant of (3). (Note that the best dominant is unique up to a rotation of U).

A lemma from the theory of differential subordinations is required in order to prove the original results contained in the next section.

Lemma 1 ([14], Th. 3.4h, p. 132). Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either

- (i) h in convex, or
- (ii) Q is starlike.

In addition, assume that

$$(iii) \quad \operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\frac{\theta[q(z)]}{\phi[q(z)]} + \frac{zQ(z)}{Q(z)} \right] > 0.$$

If p is analytic in U , with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zq(z) \cdot \phi[q(z)] = h(z)$$

then $p \prec q$ and q is the best dominant.

2. Results

The confluent (Kummer) hypergeometric function was studied in [5] for $a, c \in \mathbb{R}$, $c \neq 0, -1, -2, \dots$. In this section of the paper, the results of the study of this interesting function extended for the case when $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ are presented.

Theorem 1. Let q be convex in U with $q(0) = 1$, $\operatorname{Re} q(z) > 0$, and let $\theta(w) = w^2 + w$ and $\phi(w) = w$, $w \neq 0$, be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set

$$Q(z) = zq'(z)\phi[q(z)] \text{ and } h(z) = \theta[q(z)] + Q(z), \quad z \in U.$$

If $\phi(a, c; z)$, $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, given by (1) satisfies the differential subordination

$$\phi^2(a, c; z) + \phi(a, c; z) + z\phi'(a, c; z) \cdot \phi(a, c; z) \prec h(z), \quad (4)$$

then

$$\phi(a, c; z) \prec q(z), \quad z \in U,$$

and q is the best dominant.

Proof. Since q is a convex function it is known to be univalent and satisfies the condition

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in U. \quad (5)$$

According to the theorem of analytical characterization of convexity ([18], Theorem 4.2.1, p. 50), we have that

$$\operatorname{Re} \frac{zq'(z)}{q(z)} > 0, \quad (6)$$

meaning that q is a starlike function.

Let

$$p(z) = \phi(a, c; z) = {}_1F_1(a, c; z) = 1 + \frac{a}{c} \cdot \frac{z}{1!} + \dots + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots \quad (7)$$

We have $p(0) = 1$, $p \in \mathcal{H}[1, 1]$ and p is analytic. By differentiating (7) we get

$$p'(z) = \phi'(a, c; z), \quad z \in U. \quad (8)$$

Using (7) and (8), the differential subordination (4) becomes

$$p^2(z) + p(z) + zp'(z) \cdot p(z) \prec h(z). \quad (9)$$

In order to prove the theorem, we use Lemma 1. For this, we must show that the necessary conditions (ii) and (iii) are satisfied.

We check the conditions from the hypothesis of Lemma 1.

From the hypothesis of Theorem 1, for $w = q(z)$, we have $\theta[q(z)] = q^2(z) + q(z)$ and $\varphi[q(z)] = q(z)$, which give the equality:

$$Q(z) = zq'(z) \cdot \varphi[q(z)] = zq'(z) \cdot q(z). \quad (10)$$

Differentiating (10) and doing some simple calculations, the following relation is obtained:

$$\begin{aligned} \frac{zQ'(z)}{Q(z)} &= \frac{zq'(z) \cdot q(z) + z^2q''(z) \cdot q(z) + z^2q'(z) \cdot q'(z)}{zq'(z) \cdot q(z)} \\ &= 1 + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)}, \quad z \in U. \end{aligned} \quad (11)$$

Using (5) and (6) in (11), we get

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] + \operatorname{Re} \frac{zq'(z)}{q(z)} > 0, \quad z \in U,$$

hence the function Q is starlike and this means that condition (ii) in Lemma 1 is satisfied. For $w = q(z)$, we have:

$$h(z) = q^2(z) + q(z) + zq'(z) \cdot q(z).$$

By differentiating this equality we obtain

$$h'(z) = 2q(z) \cdot q'(z) + q'(z) + q'(z) \cdot q(z) + zq''(z) \cdot q(z) + z[q'(z)]^2$$

and

$$\begin{aligned} \frac{zh'(z)}{Q(z)} &= \frac{3zq'(z) \cdot q(z) + zq'(z) + z^2q''(z) \cdot q(z) + z^2[q'(z)]^2}{zq'(z) \cdot q(z)} \\ &= 3 + \frac{1}{q(z)} + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)} \\ &= 2 + \frac{1}{q(z)} + 1 + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)}. \end{aligned}$$

Knowing from the hypothesis that $\operatorname{Re} q(z) > 0$ and combining this with relations (5) and (6) we can write:

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = 2 + \operatorname{Re} \frac{1}{q(z)} + \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] + \operatorname{Re} \frac{zq'(z)}{q(z)} > 0$$

and we conclude that relation (iii) in Lemma 1 is satisfied.

We can now apply Lemma 1 and the differential subordination (9) implies

$$p(z) \prec q(z), z \in U. \quad (12)$$

Using (7) and (12) we write

$$\phi(a, c; z) \prec q(z), z \in U,$$

and q is the best dominant. $\square$

Remark 1. For $q(z) = \frac{1+z}{1-z}$, convex in U , from Theorem 1 we obtain the following corollary.

Corollary 1. Let $q(z) = \frac{1+z}{1-z}$, be convex in U , with $q(0) = 1$, $\operatorname{Re} q(z) > 0$, $z \in U$, and let $\theta : \mathbb{C} \rightarrow \mathbb{C}$, $\theta(w) = w^2 + w$ and $\varphi : \mathbb{C} \rightarrow \mathbb{C}$, $\varphi(w) = w$ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$, $w \in q(U)$. Set

$$Q(z) = zq'(z) \cdot \varphi[q(z)] = \frac{2z(1+z)}{(1-z)^3},$$

and

$$h(z) = \theta[q(z)] + Q(z) = \frac{1+z}{1-z} + \left(\frac{1+z}{1-z}\right)^2 + \frac{2z(1+z)}{(1-z)^3}, z \in U.$$

If $\phi(a, c; z)$, $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ given by (1) satisfies the differential subordination

$$\phi^2(a, c; z) + \phi(a, c; z) + z\phi'(a, c; z) \cdot \phi(a, c; z) \prec h(z) = \frac{1+z}{1-z} + \left(\frac{1+z}{1-z}\right)^2 + \frac{2z(1+z)}{(1-z)^3}, z \in U \quad (13)$$

then

$$\phi(a, c; z) \prec q(z) = \frac{1+z}{1-z}, z \in U, \quad \text{i.e.,} \quad \operatorname{Re} \phi(a, c; z) > 0, z \in U$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

Proof. From relation (12) in the proof of Theorem 1 we have

$$\phi(a, c; z) \prec q(z) = \frac{1+z}{1-z}, z \in U. \quad (14)$$

Since $q(z) = \frac{1+z}{1-z}$ is a convex function, the differential subordination (14) is equivalent to

$$\operatorname{Re} \phi(a, c; z) > \operatorname{Re} q(z) > 0, z \in U$$

and $q = \frac{1+z}{1-z}$ is the best dominant. $\square$

Remark 2. The result obtained in Corollary 1 was proved by Miller and Mocanu ([5], Lemma 1), for a and c real numbers satisfying of the conditions:

- (i) $a > 0$ and $c \geq a$, or
- (ii) $a \leq 0$ and $c \geq 1 + (1 + a^2)^{\frac{1}{2}}$, for $z \in U$.

For $q(z) = \frac{1}{1-z}$, $z \in U$, a convex function in U , from Theorem 1, we obtain the following:

Corollary 2. Let $q(z) = \frac{1}{1-z}$, be convex in U , with $q(0) = 1$. $\operatorname{Re} q(z) > \frac{1}{2}$, $z \in U$ and let $\theta : \mathbb{C} \rightarrow \mathbb{C}$, $\theta(w) = w^2 + w$ and $\phi : \mathbb{C} \rightarrow \mathbb{C}$, $\phi(w) = w$ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, $w \in q(U)$. Set

$$Q(z) = zq'(z) \cdot \phi[q(z)] = \frac{z}{(1-z)^3},$$

$$h(z) = \theta[q(z)] + Q(z) = \frac{1}{(1-z)^2} + \frac{1}{1-z} + \frac{z}{(1-z)^3}, \quad z \in U.$$

If $\phi(a, c; z)$, $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, given by (1) satisfies the differential subordination

$$\phi^2(a, c; z) + \phi(a, c; z) + z\phi'(a, c; z) \cdot \phi(a, c; z) \prec h(z) = \frac{1}{(1-z)^2} + \frac{1}{1-z} + \frac{z}{(1-z)^3}, \quad z \in U, \quad (15)$$

then

$$\phi(a, c; z) \prec q(z) = \frac{1}{1-z}, \quad \text{i.e., } \operatorname{Re} \phi(a, c; z) > \frac{1}{2}, \quad z \in U$$

and $q(z) = \frac{1}{1-z}$ is the best dominant.

Theorem 2. Let q a convex function in U and $\gamma \in \mathbb{C} - \{0\}$ with $\operatorname{Re} \gamma > 0$. Let $\theta : \mathbb{C} \rightarrow \mathbb{C}$, $\theta(w) = w$ and $\phi : \mathbb{C} \rightarrow \mathbb{C}$, $\phi(w) = \gamma \in \mathbb{C} - \{0\}$ be analytic in domain D , containing $q(U)$, $\phi(w) \neq 0$, when $w \in q(U)$. Set

$$Q(z) = zq'(z) \cdot \phi[q(z)] \text{ and } h(z) = \theta[q(z)] + Q(z), \quad z \in U.$$

If p is analytic in U , with $p(0) = q(0)$ and satisfies the differential subordination

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec h(z), \quad z \in U, \quad (16)$$

then $p(z) \prec \frac{1}{1-z} = q(z)$, $z \in U$ and q is the best dominant.

Proof. From the hypothesis of Theorem 2, for $w = q(z) \in q(U)$, we obtain $\phi[q(z)] = \gamma$, $\operatorname{Re} \gamma > 0$, $\gamma \in \mathbb{C} - \{0\}$, and hence

$$Q(z) = zq'(z) \cdot \gamma, \quad z \in U. \quad (17)$$

We now prove that function Q is starlike. Differentiating (17), and after short calculation, we obtain

$$\frac{zQ'(z)}{Q(z)} = \frac{zq'(z) \cdot \gamma + z^2q''(z) \cdot \gamma}{zq'(z) \cdot \gamma} = 1 + \frac{zq''(z)}{q'(z)}, \quad z \in U.$$

The hypothesis of theorem states that q is a convex function, hence

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in U, \quad (18)$$

which means that function Q is starlike. We now have condition (ii) in Lemma 1 satisfied. For $w = q(z)$, $\theta[q(z)] = q(z)$, $\phi[q(z)] = \gamma$, we have

$$h(z) = q(z) + zq'(z) \cdot \gamma, \quad z \in U. \quad (19)$$

Differentiating (19) and after short calculation we obtain

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zq'(z) + zq'(z) \cdot \gamma + z^2q''(z) \cdot \gamma}{zq'(z) \cdot \gamma} = \operatorname{Re} \frac{1}{\gamma} + \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in U, \quad (20)$$

since $\operatorname{Re} \gamma > 0$ and q is convex.

We now conclude that condition (iii) in Lemma 1 is satisfied. Using Lemma 1, we obtain

$$p(z) \prec q(z), z \in U. \quad (21)$$

Since function q is a univalent solution of the equation

$$h(z) = q(z) + zq'(z) \cdot \gamma,$$

q is the best dominant subordinant of the differential subordination (16). $\square$

Remark 3. If in Theorem 2, $p(z) = \frac{c}{a}\phi'(a, c; z)$, $a \in \mathbb{C}^*$ and $q(z) = \frac{1-z}{1+z}$, $z \in U$, we deduce the following univalence criterion:

Corollary 3. Let $q(z) = \frac{1-z}{1+z}$, be a convex function in U with $q(0) = 1$. Let $\theta : \mathbb{C} \rightarrow \mathbb{C}$, $\theta(w) = w$, and $\varphi : \mathbb{C} \rightarrow \mathbb{C}$, $\varphi(w) = 1$, be analytic in domain D containing $q(U)$. Set

$$Q(z) = zq'(z) \cdot 1 = \frac{-2z}{(1+z)^2}$$

and

$$h(z) = \theta[q(z)] + zq'(z) \cdot 1 = q(z) + zq'(z) = \frac{1-z}{1+z} - \frac{2z}{(1+z)^2}, z \in U.$$

If $\frac{c}{a}\phi'(a, c; z)$, $a \in \mathbb{C}^*$ with $p(0) = q(0) = 1$, satisfies the differential subordination

$$\frac{c}{a}\phi'(a, c; z) + z\frac{c}{a}\phi''(a, c; z) \prec h(z) = \frac{1-z}{1+z} - \frac{2z}{(1+z)^2} \quad (22)$$

then

$$\frac{c}{a}\phi'(a, c; z) \prec \frac{1-z}{1+z}, \quad \text{i.e.,} \quad \operatorname{Re} \frac{c}{a}\phi'(a, c; z) > 0, z \in U,$$

and hence $\phi(a, c; z)$ is univalent in U .

Proof. Using relation (21) from the proof of Theorem 2, we get

$$\frac{c}{a}\phi'(a, c; z) \prec q(z) = \frac{1-z}{1+z}, z \in U. \quad (23)$$

Since $q(z) = \frac{1-z}{1+z}$, $z \in U$, is a convex function in U , the differential subordination (23) is equivalent to

$$\operatorname{Re} \frac{c}{a}\phi'(a, c; z) > \operatorname{Re} \frac{1-z}{1+z} > 0, z \in U.$$

$\square$

Remark 4. The result stated in Corollary 3 was obtained by Miller and Mocanu ([5], Theorem 1) for $a \neq 0$, and c real numbers satisfying one of the following conditions:

- (i) $a > -1$ and $c \geq a$, or
- (ii) $a \leq -1$ and $c \geq \sqrt{1 + (a+1)^2}$.

For $a \in \mathbb{C}^*$, $p(z) = 1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)}$, $p(0) = 1$, $p \in \mathcal{H}[1, 1]$ and $q(z) = \frac{1-z}{1+z}$, we obtain from Theorem 2 the following univalence criterion:

Corollary 4. Let $q(z) = \frac{1-z}{1+z}$ be a convex function in U , with $q(0) = 1$. Let $\theta : \mathbb{C} \rightarrow \mathbb{C}$, $\theta(w) = w$, $\varphi : \mathbb{C} \rightarrow \mathbb{C}$, $\varphi(w) = 1$, $w \in q(U)$. Set

$$Q(z) = zq'(z) \cdot 1 = \frac{-2z}{(1+z)^2},$$

$$h(z) = \theta[q(z)] + zq'(z) = \frac{1-z}{1+z} - \frac{2z}{(1+z)^2}.$$

If $1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)}$, $a \in \mathbb{C}^*$, $c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, $z \in U$, satisfies the differential subordination

$$1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} + z \left[1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} \right]' \prec h(z) = \frac{1-z}{1+z} - \frac{2z}{(1+z)^2}, \quad z \in U, \quad (24)$$

then

$$1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} \prec \frac{1-z}{1+z} = q(z), \quad \text{i.e.,} \quad \operatorname{Re} \left[1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} \right] > 0, \quad z \in U,$$

hence $\phi(a, c; z)$ is a convex function in U .

Proof. Using relation (21), from the proof of Theorem 2, we have

$$1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} \prec q(z) = \frac{1-z}{1+z}, \quad z \in U. \quad (25)$$

Since $q(z) = \frac{1-z}{1+z}$, $z \in U$, is convex in U , relation (25) is equivalent to

$$\operatorname{Re} \left[1 + \frac{z\phi''(a, c; z)}{\phi'(a, c; z)} \right] > 0, \quad z \in U,$$

hence $\phi(a, c; z)$ is convex in U . $\square$

Remark 5. This result was obtained by Miller and Mocanu ([5], Theorem 2), for $a \neq 0$ and c real numbers satisfying $c > N(a)$, where

$$N(a) = \begin{cases} |a| + 1/2 & \text{if } |a| \geq 1/3 \\ 3a^2/2 + 2/3 & \text{if } |a| \leq 1/3. \end{cases}$$

Theorem 3. If $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ and $\phi(a, c; z)$ is the hypergeometric function given by (1), then $z\phi(a, c; z)$ is starlike in U .

Proof. It is easily seen that

$$(a-1)z\phi(a, c; z) = (c-1)z\phi'(a-1, c-1; z) \text{ for } a \neq 1. \quad (26)$$

From (26), we have

$$\frac{c-1}{a-1} z \cdot \phi'(a-1, c-1; z) = z \cdot \phi(a, c; z), \quad a \neq 1. \quad (27)$$

Let

$$g(z) = \frac{c-1}{a-1} z \cdot \phi(a-1, c-1; z), \quad z \in U, \quad a \neq 1. \quad (28)$$

Differentiating (28) and after short calculation we obtain

$$\frac{zq'(z)}{g(z)} = 1 + \frac{z\phi''(a-1, c-1; z)}{\phi'(a-1, c-1; z)}, z \in U, a \neq 1.$$

Since $\phi(a-1, c-1; z)$ is convex in U , from Corollary 4 we have

$$\operatorname{Re} \frac{zq'(z)}{g(z)} = \operatorname{Re} \left[1 + \frac{z\phi''(a-1, c-1; z)}{\phi'(a-1, c-1; z)} \right] > 0, z \in U, a \neq 1. \quad (29)$$

From (29), we conclude $g(z) = z \cdot \phi(a, c; z)$, $a \neq 1$ is starlike in U .

For $a = 1$, $\phi(1, c; z) = 1 + \frac{1}{c} \cdot \frac{z}{1!} + \frac{1 \cdot 2}{c(c+1)} \cdot \frac{z^2}{2!} + \dots$ and from Corollary 4, we have that $\phi(1, c; z)$ is convex. Hence $z \cdot \phi(a, c; z)$ is starlike for $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ $\square$

Remark 6. This result was obtained by Miller and Mocanu ([5], Corollary 2.1), for $c \geq 1 + N(a-1)$, where

$$N(a) = \begin{cases} |a| + 1/2 & \text{if } |a| \geq 1/3 \\ 3a^2/2 + 2/3 & \text{if } |a| \leq 1/3. \end{cases}$$

Example 1. For $a = 1 - i$, $c = 1 + i$,

$$F(1 - i, 1 + i; z) = 1 + \frac{1 - i}{1 + i} z = 1 - iz.$$

The function $zF(1 - i, 1 + i; z) = z - iz^2$ is starlike.

Let $g(z) = zF(1 - i, 1 + i; z) = z - iz^2$. We have $g'(z) = 1 - 2iz$. We evaluate

$$\begin{aligned} \operatorname{Re} \frac{zg'(z)}{g(z)} &= \operatorname{Re} \frac{z(1 - 2iz)}{z(1 - iz)} = \operatorname{Re} \frac{1 - 2iz}{1 - iz} = \operatorname{Re} \frac{1 - iz - iz}{1 - iz} = \operatorname{Re} \left(1 - \frac{iz}{1 - iz} \right) \\ &= \operatorname{Re} \left[1 - \frac{i(\cos \alpha + i \sin \alpha)}{1 - i(\cos \alpha + i \sin \alpha)} \right] = \operatorname{Re} \left[1 - \frac{-\sin \alpha + i \cos \alpha}{1 + \sin \alpha - i \cos \alpha} \right] \\ &= \operatorname{Re} \left[1 + \frac{(\sin \alpha - i \cos \alpha)(1 + \sin \alpha + i \cos \alpha)}{(1 + \sin \alpha)^2 + \cos^2 \alpha} \right] \\ &= \operatorname{Re} \left[1 + \frac{\sin \alpha(1 + \sin \alpha) + \cos^2 \alpha}{1 + 2 \sin \alpha + 1} \right] = 1 + \frac{\sin \alpha + 1}{2(1 + \sin \alpha)} \\ &= 1 + \frac{1}{2} = \frac{3}{2} > 0, z \in U. \end{aligned}$$

Hence, $z \cdot F(1 - i, 1 + i; z)$ is starlike.

3. Discussion

Inspired by the research of Miller and Mocanu [5] on confluent (Kummer) hypergeometric function considering a and c real numbers, the present paper contains results obtained using a and c complex numbers. Subordination results are stated in three theorems and two corollaries indicating also the relation between present results and those previously obtained by Miller and Mocanu. Using Theorem 2 for special choices of the functions involved, two criteria are derived such that confluent (Kummer) hypergeometric function to be univalent. An example is also given to illustrate how the results presented in the theorems can be used and to help inspiring researchers in obtaining new, original and interesting results considering particular values for complex numbers a and c . The subordination results contained in the theorems and corollaries could give ideas for obtaining new outcomes from the study done considering particular functions.

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